A NEW NUMERICAL TECHNIQUE FOR SOLVING FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We propose conformable Adomian decomposition method (CADM) for fractional partial differential equations (FPDEs). This method is a new Adomian decomposition method (ADM) based on conformable derivative operator (CDO) to solve FPDEs. At the same time, conformable reduced differential transform method (CRDTM) for FPDEs is briefly given and a numerical comparison is made between this method and the newly introduced CADM. In applied science, CADM can be used as an alternative method to obtain approximate and analytical solutions for FPDEs as CRDTM. In this study, linear and non-linear three problems are solved by these two methods. In these methods, the obtained solutions take the form of a convergent series with easily computable algorithms. For the applications, the obtained results by these methods are compared to each other and with the exact solutions. When applied to FPDEs, it is seem that CADM approach produces easy, fast and reliable solutions as CRDTM.

1. INTRODUCTION

Fractional differential equations have a substantial contributions in fields,e.g. optics, biology, physics, chemistry, mathematics, fluids mechanics, applied mathematics, and engineering [18, 26, 40–42]. We recall that finding an analytical solutions to these problems is not always possible [9–11, 20, 24, 27–29, 37–39]. As a result, it becomes crucial to manage these problems appropriately and solve them or develop the required solutions. ADM, which is introduced [4–6] in the 1980's, is one of the important mathematical methods used to solve many problems in real world. Since then, a number of studies have been conducted on ADM such as linear and non-linear, homogeneous and non-homogeneous operator equations which including fractional or non-fractional ODEs, PDEs, integral equations, integro-differential equations, etc. (see [12, 13, 15, 16, 25, 30, 32–36] and references therein). A new derivative called CDO was suggested [1, 7, 22]. By the help of it, the behaviors of many problems were investigated and some solutions techniques were applied

[1–3, 7, 8, 14, 17, 19, 21, 23, 31]. This new subject gives academicians an opportunity to study further in many engineering, physical and applied mathematics problems.

The aim of this study is to introduce CADM by using CDO and ADM for the first time in the literature. This method can be used to solve many linear and nonlinear FPDEs. We will briefly mentioned CRDTM to compare our CADM with it. The problems will be solved both by the CRDTM and the first proposed CADM. The obtained solutions by these methods will be compared. Thus, in section 2, we present some basic definitions and important properties of CDO. Next, in section 3, we propose CADM. In sections 4, we introduce CRDTM to compare with our method. In section 5, we give applications of CADM and CRDTM. We give the conclusion in the final section.

2. BASIC DEFINITIONS

Definition 1. Given a function $f_1 : [0, \infty) \to \mathbb{R}$. Then the CDO of f_1 order α is defined by [1, 7, 22]:

$$(T_\alpha f_1)(t) = \lim_{\varepsilon \to 0} \frac{f_1(t + \varepsilon t^{1-\alpha}) - f_1(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1]$.

Lemma 1 ([1, 7, 22]). *Let f_1, g_1 be α and β-differentiable at a point $t > 0$ for α. Then*
(i) $T_\alpha(af_1 + bg_1) = a(T_\alpha f_1) + b(T_\alpha g_1)$ for all $a, b \in \mathbb{R}$ and $\alpha \in (0, 1]$,
(ii) $T_\alpha(f_1(t)) = 0$, for constant function $f_1(t) = \lambda$, $\alpha \in (0, 1]$,
(iii) $T_\alpha(f_1 g_1) = f_1(T_\alpha g_1) + g_1(T_\alpha f_1)$, $\alpha \in (0, 1]$,
(iv) $T_\alpha(f_1/g_1) = \dfrac{g_1(T_\alpha f_1) - f_1(T_\alpha g_1)}{g_1^2}$, $\alpha \in (0, 1]$,
(v) If f_1 is n times differentiable at t, then $T_\alpha(f_1(t)) = t^{\lceil \alpha \rceil - \alpha} f_1^{(\lceil \alpha \rceil)}(t)$, $\alpha \in (n, n+1]$. Where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

Lemma 2. *[1] Suppose that f_1 is infinitely α-differentiable function for $\alpha \in (0, 1]$ at a neighborhood of a point t_0. Then f_1 has the conformable power series expansion:*

$$f_1(t) = \sum_{k=0}^{\infty} \frac{\left(_t T_\alpha^{(k)} f_1\right)(t_0)(t - t_0)^{\alpha k}}{\alpha^k k!}, \quad t_0 < t < t_0 + R^{1/\alpha}, R > 0.$$

Here $\left(T_\alpha^{(k)} f_1\right)(t_0)$ denotes the application of the conformable derivative for k times.

3. Conformable Adomian decomposition method

We will briefly introduce CADM for FPDEs in this section. We write the non-linear FPDEs in the standard operator form

$$L_\alpha(u(x,t)) + R(u(x,t)) + N(u(x,t)) = g(x,t) \tag{3.1}$$

where $L_\alpha = {_\alpha}T$ is a linear operator with conformable derivative of order α ($n < \alpha \le n+1$), N is a non-linear operator, R is the other part of the linear operator and $g(x,t)$ is a non-homogeneous term. If the linear operator in eq. (3.1) is applied to Lemma 1, the following equation is obtained:

$$t^{\lceil\alpha\rceil-\alpha}\frac{\partial^{\lceil\alpha\rceil}}{\partial t^{\lceil\alpha\rceil}}u(x,t) + R(u(x,t))N(u(x,t)) = g(x,t). \tag{3.2}$$

Applying $L_\alpha^{-1} = \int\limits_0^t\int\limits_0^{\xi_1}\cdots\int\limits_n^{\xi_{n-1}}\frac{1}{\xi_n^{\lceil\alpha\rceil-\alpha}}(.)\,d\xi_n d\xi_{n-1}\cdots d\xi_1$, ($n < \alpha \le n+1$) the inverse of operator, to both sides of (3.2), it is obtained as

$$L_\alpha^{-1}L_\alpha(u(x,t)) = L_\alpha^{-1}g(x,t) - L_\alpha^{-1}R(u(x,t)) - L_\alpha^{-1}N(u(x,t)). \tag{3.3}$$

The general solution of the given equation is decomposed into the sum

$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t). \tag{3.4}$$

The non-linear part $N(u)$ can be decomposed into the infinite polynomial series obtained by

$$N(u) = \sum_{n=0}^{\infty}A_n,\,(u_0,u_1,\ldots,u_n), \tag{3.5}$$

where A_n is the so-called Adomian polynomials (APs). These APs can be calculated for all types of non-linearity by the help of algorithms built by Adomian [5,6,12,15,25,32]. u and $N(u)$, respectively, is obtained as

$$u = \sum_{i=0}^{\infty}\lambda^i u_i,\, N(u) = N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right) = \sum_{i=0}^{\infty}\lambda^i A_i \tag{3.6}$$

where λ is the convenience parameter. From (3.6), APs A_n are obtained as

$$n!A_n = \frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\right]_{\lambda=0}.$$

These APs can be calculated easily with the following Maple code:

Substituting (3.4) and (3.5) into (3.3), it is obtained

$$\sum_{n=0}^{\infty}u_n = \theta + L_\alpha^{-1}g - L_\alpha^{-1}R\left(\sum_{n=0}^{\infty}u_n\right) - L_\alpha^{-1}\left(\sum_{n=0}^{\infty}A_n\right). \tag{3.7}$$

```
restart:
ord:=m:     #Order (Enter a number for m.)
NF:=N(u(x,t))   #Nonlinear Function
u[t]:=sum(u[b]*t^b,b=0..ord):
NF[t]:=subs(u(x,t)=u[t],NF):
s:=expand(NF[t],t):
dt:=unapply(s,t):
for i from 0 to ord do
A[i]:=((D@@i)(dt)(0)/i!):
print(AA[i],A[i]): #Adomian Polinoms
od:
```

where $\theta = u(x,0)$ is initial condition (IC). From (3.7), the iterates are defined by the following recursive formulas:

$$u_0 = \theta + L_\alpha^{-1} g ,$$
$$u_1 = -L_\alpha^{-1} R\, u_0 - L_\alpha^{-1} A_0 ,$$
$$\vdots$$
$$u_{n+1} = -L_\alpha^{-1} R\, u_n - L_\alpha^{-1} A_n , \; n \geq 0 .$$

(3.8)

Therefore, from (3.8), the approximate solution of (3.1) is obtained by

$$\tilde{u}_m(x,t) = \sum_{n=0}^{m} u_n(x,t).$$

(3.9)

Hence, from (3.9), the exact solution of (3.1) can be obtained as

$$u(x,t) = \lim_{m \to \infty} \tilde{u}_m(x,t).$$

4. Conformable reduced differential transform method

In this section, it is given basic definitions and properties of CRDTM for FPDEs [3].

Definition 2. Assume $u(x,t)$ is analytic function and differentiated continuously with respect to time t and space x in the its domain. the conformable reduced differential transformed (CRDT) of $u(x,t)$ is defined as [3]

$$U_k^\alpha(x) = \frac{1}{\alpha^k k!} \Big[\big({}_t T_\alpha^{(k)} u \big) \Big]_{t=t_0}$$

where some $0 < \alpha \leq 1$, α is describing the order of CDO,
${}_t T_\alpha^{(k)} u = \underbrace{({}_t T_\alpha\, {}_t T_\alpha \cdots {}_t T_\alpha)}_{k \text{ times}} u(x,t)$ and $U_k^\alpha(x)$ is the CRDT function.

Definition 3. Let $U_k^\alpha(x)$ be the CRDT of $u(x,t)$. Inverse CRDT of $U_k^\alpha(x)$ is defined as [3]

$$u(x,t) = \sum_{k=0}^{\infty} U_k^\alpha(x)(t-t_0)^{\alpha k} = \sum_{k=0}^{\infty} \frac{1}{\alpha^k k!} \left[{}_t T_\alpha^{(k)} u \right]_{t=t_0} (t-t_0)^{\alpha k}$$

CRDT of ICs for integer order derivatives are defined as [3]

$$U_k^\alpha(x) = \begin{cases} \frac{1}{(\alpha k)!} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x,t) \right]_{t=t_0} & if \ \alpha k \in \mathbb{Z}^+ \\ 0 & if \ \alpha k \notin \mathbb{Z}^+ \end{cases} \quad for \ k = 0,1,2,...,\left(\frac{n}{\alpha} - 1 \right)$$

where n is the order of CDO of PDE.

By consideration of

$$U_0^\alpha(x) = f(x)$$

as transformation of IC

$$u(x,0) = f(x).$$

A straightforward iterative calculations gives the $U_k^\alpha(x)$ values for $k = 1,2,3,...,n$. Then the set of $\{U_k^\alpha(x)\}_{k=0}^n$ gives the approximate result as:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k^\alpha(x) t^{k\alpha},$$

where n is approximate result order. The exact solution can be obtained as:

$$u(x,t) = \lim_{n \to \infty} \tilde{u}_n(x,t)$$

The fundamental operations of CRDTM that can be deduced from Definition 2 and Definition 3 are listed in Table 1 [3].

5. NUMERICAL CONSIDERATION

To illustrate the effectiveness of the given CADM and CRDTM, three examples are considered in this section. All the results are calculated by software MAPLE.

Example 1. Firstly, consider the linear time and space fractional diffusion equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^{2\beta}}{\partial x^{2\beta}} u(x,t) \ t > 0, \ x \in R, \ 0 < \alpha, \beta \le 1 \tag{5.1}$$

with the IC

$$u(x,0) = \sin\left(\frac{x^\beta}{\beta} \right). \tag{5.2}$$

Exact result of the problem (5.1) in conformable sense is

$$u(x,t) = \sin\left(\frac{x^\beta}{\beta} \right) e^{-\frac{t^\alpha}{\alpha}}.$$

TABLE 1. Basic operations of the CRDTM [3].

Original function	Transformed function
$u(x,t)$	$U_k^\alpha(x) = \frac{1}{\alpha^k k!}\left[\left({}_t T_\alpha^{(k)} u\right)\right]_{t=t_0}$
$u(x,t) = av(x,t) \pm bw(x,t)$	$U_k^\alpha(x) = aV_k^\alpha(x) \pm bW_k^\alpha(x)$
$u(x,t) = v(x,t)w(x,t)$	$U_k^\alpha(x) = \sum\limits_{s=0}^{k} V_s^\alpha(x) W_{k-s}^\alpha(x)$
$u(x,t) = {}_t T_\alpha v(x,t)$	$U_k^\alpha(x) = \alpha(k+1) V_{k+1}^\alpha(x)$
$u(x,t) = x^m(t-t_0)^n$	$U_k^\alpha(x) = x^m \delta\left(k - \frac{n}{\alpha}\right)$

Solution by CADM: Solve this problem by using CADM. Let $L_\alpha = T_\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ be a linear operator, then the operator form of (5.1) is as follows

$$T_\alpha u(x,t) = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} \quad t > 0, x \in R, 0 < \alpha, \beta \le 1 \tag{5.3}$$

By the help of Lemma 1, eq. (5.3) can be written as

$$t^{1-\beta} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} \quad t > 0, x \in R, 0 < \alpha, \beta \le 1. \tag{5.4}$$

If $L_\alpha^{-1} = \int\limits_0^t \frac{1}{\xi^{1-\alpha}}(.)\,d\xi$,which is the inverse of L_α, is applied to both sides of eq. (5.4), we get

$$u(x,t) = u(x,0) - L_\alpha^{-1}\left(\frac{\partial^{2\beta}}{\partial x^{2\beta}} u(x,t)\right).$$

According to (3.8) and the IC (5.2), we can write

$$
\begin{aligned}
u_0 &= \sin\left(\frac{x^\beta}{\beta}\right), \\
u_1 &= -L_\alpha^{-1}\left(\frac{\partial^2}{\partial x^2} u_0\right), \\
&\vdots \\
u_{n+1} &= -L_\alpha^{-1}\left(\frac{\partial^2}{\partial x^2} u_n\right), n \ge 0 .
\end{aligned}
\tag{5.5}
$$

From (5.5), we conclude the terms of decomposition series as:

$$u_0 = \sin\left(\frac{x^\beta}{\beta}\right), \; u_1 = -\sin\left(\frac{x^\beta}{\beta}\right)\frac{t^\alpha}{\alpha},$$

$$u_2 = \sin\left(\frac{x^\beta}{\beta}\right)\frac{t^{2\alpha}}{2\alpha^2}, \cdots, u_n = (-1)^n \sin\left(\frac{x^\beta}{\beta}\right)\frac{t^{n\alpha}}{n!\alpha^n}, \cdots \tag{5.6}$$

Thus, by using (5.6), the approximate solution of (5.1) obtained by CADM is

$$\tilde{u}_m(x,t) = \sum_{n=0}^{m} u_n(x,t) = \sum_{n=0}^{m} (-1)^n \sin\left(\frac{x^\beta}{\beta}\right)\frac{t^{n\alpha}}{n!\alpha^n}. \tag{5.7}$$

From (5.7) we obtain

$$u(x,t) = \lim_{m\to\infty} \tilde{u}_m(x,t) = \sin\left(\frac{x^\beta}{\beta}\right) e^{-\frac{t^\alpha}{\alpha}}. \tag{5.8}$$

This analytical approximate solution (5.8) is the exact solution.

Solution by CRDTM: Now solve this problem by using CRDTM. By taking the CRDT of (5.1), it can be obtained that

$$\alpha(k+1)U_{k+1}^\alpha(x) = \frac{\partial^{2\beta}}{\partial x^{2\beta}} U_k^\alpha(x) \tag{5.9}$$

where $U_k^\alpha(x)$ is the CRDT function. From the IC (5.2) we write

$$U_0^\alpha(x) = \sin\left(\frac{x^\beta}{\beta}\right) \tag{5.10}$$

Substituting (5.10) into (5.9), it can be obtained the following $U_k^\alpha(x)$ values

$$U_1^\alpha(x) = -\sin\left(\frac{x^\beta}{\beta}\right)\frac{1}{\alpha}, \; U_2^\alpha(x) = \sin\left(\frac{x^\beta}{\beta}\right)\frac{1}{2!\alpha^2},$$

$$U_3^\alpha(x) = -\sin\left(\frac{x^\beta}{\beta}\right)\frac{1}{3!\alpha^3}, \cdots, U_n^\alpha(x) = \sin\left(\frac{x^\beta}{\beta}\right)\frac{(-1)^n}{n!\alpha^n}, \cdots$$

Then, the set of values $\{U_k^\alpha(x)\}_{k=0}^{n}$ gives the following approximate result

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k^\alpha(x) t^{k\alpha} = \sum_{k=0}^{n} \sin\left(\frac{x^\beta}{\beta}\right)\frac{(-1)^k}{k!\alpha^k} t^{k\alpha}. \tag{5.11}$$

From (5.11) we obtain

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t) = \sin\left(\frac{x^\beta}{\beta}\right) e^{-\frac{t^\alpha}{\alpha}}. \tag{5.12}$$

This approximate solution (5.12) is the exact solution.

Remark 1. If take $\alpha = \beta = 1$ in the problem (5.1), then Example 1 is reduced to standard diffusion equation

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) \; t > 0, \; x \in R$$

with IC

$$u(x,0) = \sin(x)$$

and our analytical approximate solutions (5.8) and (5.12) imply

$$u(x,t) = \sin(x)e^{-t}$$

and this solutions is the exact result of the standard problem in the literature.

The Aproximate solutions obtained by both CADM and CRDTM give us the exact solution.

Example 2. Secondly, let us consider the non-linear time and space fractional gas dynamics equation:

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) + \frac{1}{2} \frac{\partial^\beta}{\partial x^\beta} u^2(x,t) - u(x,t)(1 - u(x,t)) = 0 \;, \; 0 \; \alpha, \beta \leq 1 \qquad (5.13)$$

subject to IC

$$u(x,0) = e^{-\frac{x^\beta}{\beta}}. \qquad (5.14)$$

The exact solutions of (5.13) in conformable sense is

$$u(x,t) = e^{\frac{t^\alpha}{\alpha} - \frac{x^\beta}{\beta}}.$$

Solution by CADM: Solve the problem by using CADM. Let $L_\alpha = T_\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ be a linear operator, then the operator form of (5.13) is as follows

$$T_\alpha u(x,t) = -\frac{1}{2} \frac{\partial^\beta}{\partial x^\beta} u^2(x,t) + u(x,t)(1 - u(x,t)) \;, \; 0 \; \alpha, \beta \leq 1. \qquad (5.15)$$

By the help of Lemma 1, eq. (5.15) can be written as

$$t^{1-\alpha} \frac{\partial u(x,t)}{\partial t} = u(x,t) - u(x,t) \frac{\partial^\beta}{\partial x^\beta} u(x,t) - u^2(x,t), \; 0 \; \alpha, \beta \leq 1. \qquad (5.16)$$

If $L_\alpha^{-1} = \int\limits_0^t \frac{1}{\xi^{1-\alpha}} (.) \, d\xi$,which is the inverse of L_α, is applied to both sides of eq. (5.16), we get

$$u(x,t) = u(x,0) + L_\alpha^{-1}(u(x,t)) - L_\alpha^{-1}\left(u(x,t) \frac{\partial^\beta}{\partial x^\beta} u(x,t) + u^2(x,t)\right).$$

According to (3.8) and IC (5.14), we can write the following recursive relations:

$$u_0 = e^{-\frac{x^\beta}{\beta}}$$
$$u_1 = L_\alpha^{-1}(u_0) - L_\alpha^{-1}(A_0)$$
$$\vdots$$
$$u_{n+1} = L_\alpha^{-1}(u_n) - L_\alpha^{-1}(A_n), n \geq 0. \tag{5.17}$$

where A_n's are APs. By using the Maple code above, for the non-linear term

$$N(u(x)) = u(x,t)\frac{\partial}{\partial x}u(x,t) + u^2(x,t),$$

the APs can be obtain as:

$$A_0 = u_0^2 + u_0 \frac{\partial^\beta}{\partial x^\beta}u_0$$
$$A_1 = 2u_0 u_1 + u_0 \frac{\partial^\beta}{\partial x^\beta}u_1 + u_1 \frac{\partial^\beta}{\partial x^\beta}u_0$$
$$A_2 = u_1^2 + 2u_0 u_2 + u_0 \frac{\partial^\beta}{\partial x^\beta}u_2 + u_1 \frac{\partial^\beta}{\partial x^\beta}u_1 + u_2 \frac{\partial^\beta}{\partial x^\beta}u_0 \tag{5.18}$$
$$A_3 = 2u_1 u_2 + 2u_0 u_3 + u_0 \frac{\partial^\beta}{\partial x^\beta}u_3 + u_1 \frac{\partial^\beta}{\partial x^\beta}u_2 + u_2 \frac{\partial^\beta}{\partial x^\beta}u_1 + u_3 \frac{\partial^\beta}{\partial x^\beta}u_0$$
$$\vdots$$

From (5.17) and (5.18), we conclude the terms of decomposition series as:

$$u_0 = e^{-\frac{x^\beta}{\beta}}, \ u_1 = e^{-\frac{x^\beta}{\beta}}\frac{t^\alpha}{\alpha}, u_2 = e^{-\frac{x^\beta}{\beta}}\frac{t^{2\alpha}}{2\alpha^2}, \cdots, u_n = e^{-\frac{x^\beta}{\beta}}\frac{t^{n\alpha}}{n!\alpha^n}, \cdots \tag{5.19}$$

Thus, From (5.19), the approximate solution of (5.13) obtained by CADM is

$$\tilde{u}_m(x,t) = \sum_{n=0}^{m} u_n(x,t) = \sum_{n=0}^{m} e^{-\frac{x^\beta}{\beta}}\frac{t^{n\alpha}}{n!\alpha^n}. \tag{5.20}$$

From (5.20) we obtain

$$u(x,t) = \lim_{m \to \infty} \tilde{u}_m(x,t) = e^{\frac{t^\alpha}{\alpha} - \frac{x^\beta}{\beta}}. \tag{5.21}$$

This analytical approximate solution (5.21) is the exact solution.

Solution by CRDTM: Now solve this problem by using CRDTM. By taking the CRDT of (5.13), it can be obtained that

$$\alpha(k+1)U_{k+1}^\alpha(x) = -\sum_{r=0}^{k} U_{k-r}^\alpha(x)\frac{\partial^\beta}{\partial x^\beta}U_r^\alpha(x) + U_k^\alpha(x) - \sum_{r=0}^{k} U_{k-r}^\alpha(x)U_r^\alpha(x) \tag{5.22}$$

where $U_k^\alpha(x)$ is the CRDT function. From the IC (5.14) we write

$$U_0^\alpha(x) = e^{-\frac{x^\beta}{\beta}} \tag{5.23}$$

Substituting (5.23) into (5.22), it can be obtained the following $U_k^\alpha(x)$ values

$$U_1^\alpha(x) = e^{-\frac{x^\beta}{\beta}}\frac{1}{\alpha}, \ \ U_2^\alpha(x) = e^{-\frac{x^\beta}{\beta}}\frac{1}{2!\alpha^2}, \cdots, U_n^\alpha(x) = e^{-\frac{x^\beta}{\beta}}\frac{1}{n!\alpha^n}, \cdots$$

Then, the set of values $\{U_k^\alpha(x)\}_{k=0}^n$ gives the following approximate result

$$\tilde{u}_n(x,t) = \sum_{k=0}^n U_k^\alpha(x)t^{k\alpha} = \sum_{k=0}^n e^{-\frac{x^\beta}{\beta}}\frac{1}{k!\alpha^k}t^{k\alpha}. \tag{5.24}$$

From (5.24) we obtain

$$u(x,t) = \lim_{n\to\infty}\tilde{u}_n(x,t) = e^{\frac{t^\alpha}{\alpha}-\frac{x^\beta}{\beta}}. \tag{5.25}$$

This approximate solution (5.25) is the exact solution.

Remark 2. If take $\alpha = \beta = 1$ in the problem (5.13), then Example 2 is reduced to standard gas dynamics equation

$$\frac{\partial}{\partial t}u(x,t) + \frac{1}{2}\frac{\partial}{\partial x}u^2(x,t) - u(x,t)(1-u(x,t))=0$$

with IC

$$u(x,0) = e^{-x}$$

our analytical approximate solutions (5.25) and (5.21) imply

$$u(x,t) = e^{t-x}$$

and this solution is the exact result of the standard problem in the literature.

The approximate solutions obtained by both CADM and CRDTM give us the existed exact solution.

Example 3. Finally, let us consider the non-linear time and space FPDE:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) + (1+u(x,t))\frac{\partial^\alpha}{\partial x^\alpha}u(x,t)=0 \ , \ 0 \ \alpha \le 1 \tag{5.26}$$

subject to IC

$$u(x,0) = \frac{x^\alpha - \alpha}{2\alpha}. \tag{5.27}$$

The exact solutions of (5.26) in conformable sense is

$$u(x,t) = \frac{x^\alpha - t^\alpha - \alpha}{t^\alpha - 2\alpha}.$$

Solution by CADM: Solve the problem by using CADM. Let $L_\alpha = T_\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ be a linear operator, then the operator form of (5.26) is as follows

$$T_\alpha u(x,t) = -(1+u(x,t))\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) \ , \ 0 \ \alpha \le 1. \tag{5.28}$$

By the help of Lemma 1, eq. (5.28) can be written as

$$t^{1-\alpha}\frac{\partial u(x,t)}{\partial t} = -\frac{\partial^\alpha}{\partial x^\alpha}u(x,t) - u(x,t)\frac{\partial^\alpha}{\partial x^\alpha}u(x,t), \ 0 \ \alpha \leq 1. \qquad (5.29)$$

If $L_\alpha^{-1} = \int\limits_0^t \frac{1}{\xi^{1-\alpha}}(.)\,d\xi$, which is the inverse of L_α, is applied to both sides of eq. (5.29), we get

$$u(x,t) = u(x,0) - L_\alpha^{-1}\left(\frac{\partial^\alpha}{\partial x^\alpha}u(x,t)\right) - L_\alpha^{-1}\left(u(x,t)\frac{\partial^\alpha}{\partial x^\alpha}u(x,t)\right).$$

According to 3.8 and IC (5.27), we can write the following recursive relations:

$$\begin{aligned}
u_0 &= \frac{x^\alpha - \alpha}{2\alpha} \\
u_1 &= L_\alpha^{-1}(u_0) - L_\alpha^{-1}(A_0) \\
&\vdots \\
u_{n+1} &= L_\alpha^{-1}(u_n) - L_\alpha^{-1}(A_n), n \geq 0.
\end{aligned} \qquad (5.30)$$

where A_n's are APs. By using the Maple code above, for the non-linear term

$$N(u(x)) = u(x,t)\frac{\partial}{\partial x}u(x,t) + u^2(x,t),$$

the APs can be obtain as:

$$\begin{aligned}
A_0 &= u_0\frac{\partial^\alpha}{\partial x^\alpha}u_0 \\
A_1 &= u_0\frac{\partial^\alpha}{\partial x^\alpha}u_1 + u_1\frac{\partial^\alpha}{\partial x^\alpha}u_0 \\
A_2 &= u_0\frac{\partial^\alpha}{\partial x^\alpha}u_2 + u_1\frac{\partial^\alpha}{\partial x^\alpha}u_1 + u_2\frac{\partial^\alpha}{\partial x^\alpha}u_0 \\
A_3 &= u_0\frac{\partial^\alpha}{\partial x^\alpha}u_3 + u_1\frac{\partial^\alpha}{\partial x^\alpha}u_2 + u_2\frac{\partial^\alpha}{\partial x^\alpha}u_1 + u_3\frac{\partial^\alpha}{\partial x^\alpha}u_0 \\
&\vdots
\end{aligned} \qquad (5.31)$$

From (5.30) and (5.31), we conclude the terms of decomposition series as:

$$u_0 = \frac{x^\alpha - \alpha}{2\alpha}, u_1 = -\frac{x^\alpha + \alpha}{(2\alpha)^2}t^\alpha, u_2 = \frac{x^\alpha + \alpha}{(2\alpha)^3}t^{2\alpha},$$

$$u_3 = -\frac{x^\alpha + \alpha}{(2\alpha)^4}t^{3\alpha}, \cdots, u_n = (-1)^n\frac{x^\alpha + \alpha}{(2\alpha)^{n+1}}t^{n\alpha}, \cdots \qquad (5.32)$$

Thus, from (5.32), the approximate solution of (5.26) obtained by CADM is

$$\tilde{u}_m(x,t) = \sum_{n=0}^m u_n(x,t) = \frac{x^\alpha - \alpha}{2\alpha} + \sum_{n=1}^m (-1)^m\frac{x^\alpha + \alpha}{(2\alpha)^{m+1}}t^{m\alpha}. \qquad (5.33)$$

Solution by CRDTM: Now solve this problem by using CRDTM. By taking the CRDT of (5.26), it can be obtained that

$$\alpha(k+1)U_{k+1}(x) = -U_k(x) - \sum_{r=0}^{k} U_{k-r}(x)\frac{\partial}{\partial x}U_r(x) \tag{5.34}$$

where $U_k^\alpha(x)$ is the CRDT function. From the IC (5.27) we write

$$U_0^\alpha(x) = \frac{x^\alpha - \alpha}{2\alpha} \tag{5.35}$$

Substituting (5.35) into (5.34), it can be obtained the following $U_k^\alpha(x)$ values

$$U_1^\alpha(x) = -\frac{x^\alpha + \alpha}{(2\alpha)^2}, \ \ U_2^\alpha(x) = \frac{x^\alpha + \alpha}{(2\alpha)^3}, \cdots, U_n^\alpha(x) = (-1)^n\frac{x^\alpha + \alpha}{(2\alpha)^{n+1}}, \cdots$$

Then, the set of values $\left\{U_k^\alpha(x)\right\}_{k=0}^{n}$ gives the following approximate result

$$\tilde{u}_m(x,t) = \sum_{k=0}^{m} U_k^\alpha(x)t^{k\alpha} = \frac{x^\alpha - \alpha}{2\alpha} + \sum_{n=1}^{m}(-1)^m\frac{x^\alpha + \alpha}{(2\alpha)^{m+1}}t^{m\alpha}. \tag{5.36}$$

Now, we compare the seventh iteration CADM and CRDTM solutions with the exact solution on the graphs for some α values. These comparisons can be seen in fig. 1 and fig. 2.
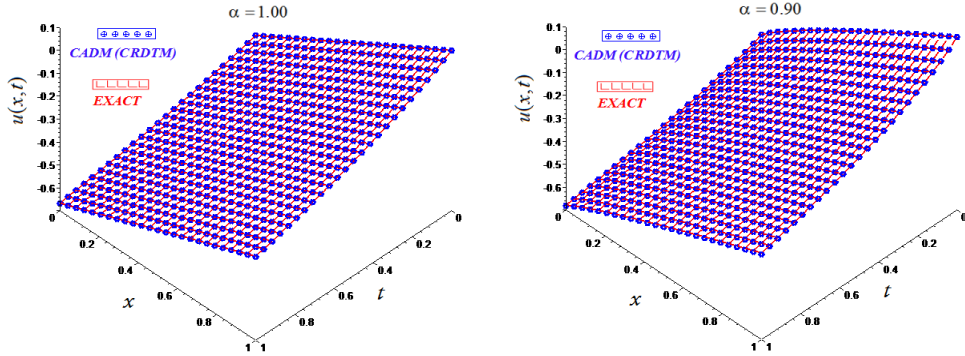

FIGURE 1. Comparison of seventh iteration approximate results of CADM (CRDTM) with the exact solutions for eq. (5.26).
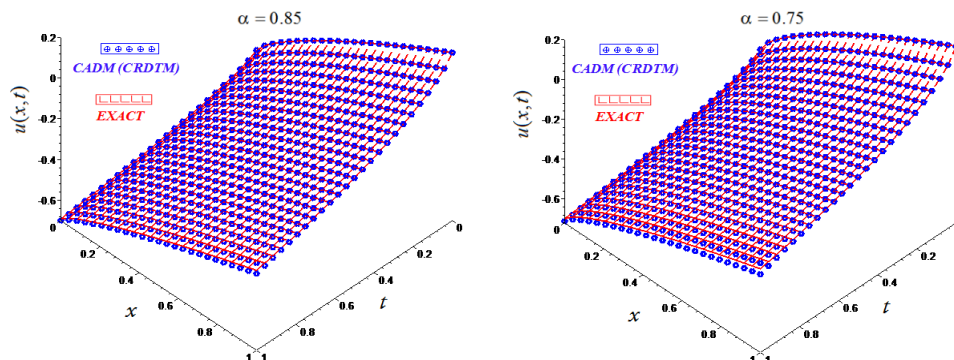
FIGURE 2. Comparison of seventh iteration approximate results of CADM (CRDTM) with the exact solutions for eq. (5.26).

6. CONCLUSION

The fundamental goal of this article is to construct the approximate solutions of FPDEs. The goal has been achieved by using CADM for the first time and it is compared with CRDTM. CADM and CRDTM are applied to different linear and non-linear conformable time and space FPDEs. And also the approximate analytical solutions obtained by CADM and CRDTM are compared to each other and with the exact solutions. CADM and CRDTM offer solutions with easily computable components as convergent series. Approximate solutions obtained by CADM are exactly same as the solutions obtained by CRDTM for time and space FPDEs. The CADM gives quantitatively reliable results as CRDTM, and also it requires less computational work than existing other methods. As a result, in recent years, FDEs emerging as models in fields such as mathematics, physics, chemistry, biology and engineering makes it necessary to investigate the methods of solutions and we hope that this study is an improvement in this direction.

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