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Analysis of mixed-order Caputo fractional system with nonlocal integral boundary condition

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Abstract: This paper deals with a mixed-order Caputo fractional system with nonlocal integral boundary conditions. This study can be considered as an extension of previous studies, since the orders of the equations lie on different intervals. We discuss the existence and uniqueness of the solution using fixed point methods. We enrich the study with an example.

Key words: Caputo derivative, coupled system, boundary conditions, existence and uniqueness, fixed point theorem

1. Introduction

Fractional differential equations (FDEs) have attracted interest for a long time since they are capable of modeling different problems arising in different fields such as physics, biology, optics, and control systems [5–8, 12, 14]. Fractional derivatives have eliminated the drawbacks of integer-order derivatives on account of their nonlocal characteristics. This property gives us a chance to understand the memory effect on the system and it is especially important to understand the material properties.

Studies concerning the existence and uniqueness of the solution of FDEs with classical, nonlocal integral type and coupled boundary conditions can be found in the literature. For example, a coupled system of nonlinear Caputo–Fabrizio FDEs of order $0 < \alpha < 1$ with classical homogeneous boundary conditions (BCs) is discussed in [4]. A three-point boundary value problem (BVP) with Riemann–Liouville derivative was addressed in [1]. A system of nonlinear FDEs of orders $1 < \alpha, \beta < 2$ was investigated in [15]. A nonlinear system of coupled Riemann–Liouville FDEs of order $q - 1 < \alpha < q$ for $q \geq 3$ with coupled integral BCs was studied in [10]. A system of Caputo FDEs of order $0 < \alpha < 1$ with nonlocal integral BCs was analyzed in [11]. A similar FDE of order $1 < \alpha < 2$ with a more general source function was studied in [2]. Authors discussed a system of FDEs coupled with a new type of integral boundary conditions in [3].

In this study, we analyze the following coupled system of nonlinear FDEs with fractional integral boundary conditions:

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$$ {}^{C}\mathcal{D}^{\alpha}u(t) = f(t,u(t),v(t)), \quad 0<\alpha\leq 1, \quad 0<t<1, \tag{1.1a} $$

$$ {}^{C}\mathcal{D}^{\beta}v(t) = g(t,u(t),v(t)), \quad 1<\beta\leq 2, \quad 0<t<1, \tag{1.1b} $$

$$ u(0) = \zeta\mathcal{I}^{p}u(\mu), \quad 0<\mu<1, \tag{1.1c} $$

$$ v(0) = \delta\mathcal{I}^{q}v(\xi), \quad 0<\xi<1, \quad v(1)=\psi v(\theta), \quad 0<\theta<1, \tag{1.1d} $$

where ${}^{C}\mathcal{D}^{\alpha}u(t)$ denotes the Caputo derivative of $u(t)$ of order α and $\mathcal{I}^{p}u(\mu)$ denotes the Riemann–Liouville integral of $u(\mu)$ of order p, $f,g\in C([0,1]\times\mathbb{R}^2,\mathbb{R})$ and $p,q,\zeta,\delta,\psi\in\mathbb{R}$. This study can be considered as an extension of the previous studies, since the orders of the equations lie on different intervals, i.e. $0<\alpha\leq 1$ and $1<\beta\leq 2$. Existence of the solution of the FDE (1.1) has been justified using the Leray–Schauder alternative and then existence and uniqueness results have been proven using Banach's fixed point theorem. To the best of our knowledge, this is the first study concerning a coupled system of Caputo FDEs of mixed orders $0<\alpha\leq 1$ and $1<\beta\leq 2$ with nonlocal integral boundary conditions and justifying the existence and uniqueness of the solution.

This paper is organized as follows. In Section 2, some basic fractional differentiation/integration operators are introduced and two auxiliary lemmas are proven. In Section 3, the existence and uniqueness of the solution of the model (1.1) are shown and an example is presented.

2. Preliminary results

We define some basic fractional differentiation/integration operators and state some of their properties [13]. Then we present some auxiliary lemmas for the corresponding linear problem.

Definition 2.1 *Let $f:[0,\infty)\to\mathbb{R}$ be an at least n-times continuously differentiable function. The Caputo derivative of order α is defined as*

$$ {}^{C}\mathcal{D}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)\,d\tau, \quad n-1<\alpha<n, \quad n=[\alpha]+1, $$

where $[\alpha]$ denotes the integer part of the order α.

Definition 2.2 *The Riemann–Liouville integral of order α is defined as*

$$ \mathcal{I}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau, \quad 0<\alpha, $$

provided the integral exists.

Lemma 2.3 *Let $\alpha,\beta\geq 0$, $f\in L_1([a,b])$. Then $\mathcal{I}^{\alpha}\,\mathcal{I}^{\beta}f(t)=\mathcal{I}^{\alpha+\beta}f(t)$ and ${}^{C}\mathcal{D}^{\beta}\,\mathcal{I}^{\beta}f(t)=f(t)$ for all $t\in[a,b]$.*

Lemma 2.4 *Let $\beta>\alpha>0$, $f\in L_1([a,b])$. Then ${}^{C}\mathcal{D}^{\alpha}\,\mathcal{I}^{\beta}f(t)=\mathcal{I}^{\beta-\alpha}f(t)$ for all $t\in[a,b]$.*

We present some auxiliary lemmas before deriving the main theorems.

Lemma 2.5 *Let* $\zeta \neq \frac{\Gamma(p+1)}{\mu^p}$ *and* $h \in C([0,1],\mathbb{R})$. *Then the FDE*

$$^{C}\mathcal{D}^{\alpha}u(t) = h(t), \quad 0 < \alpha \leq 1,$$

with the boundary condition

$$u(0) = \zeta \mathcal{I}^p u(\mu) = \zeta \int_0^{\mu} \frac{(\mu-\tau)^{p-1}}{\Gamma(p)} u(\tau)\, d\tau, \quad 0 < \mu < 1,$$

has the solution

$$u(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)\, d\tau - \frac{\zeta\Gamma(p+1)}{\zeta\mu^p - \Gamma(p+1)} \mathcal{I}^{p+\alpha}h(\mu), \quad t \in [0,1].$$

Proof The proof is given in [11]. □

For ease of notation, we set

$$\Pi_1 = \frac{\Gamma(q+1)}{\Gamma(q+1)-\xi^q}, \ \Pi_2 = \frac{\xi^{q+1}}{\Gamma(q+2)-\xi^q(q+1)}, \ \Pi_3 = \frac{1}{1-\psi\theta+(1-\psi)\Pi_2}.$$

Lemma 2.6 *Let* $\Gamma(q+1) \neq \xi^q$, $\frac{1-\psi\theta}{\psi-1} \neq \frac{\xi^{q+1}}{\Gamma(q+2)-\xi^q(q+1)}$ *and* $k \in C([0,1],\mathbb{R})$. *The FDE*

$$^{C}\mathcal{D}^{\beta}v(t) = k(t), \quad 1 < \beta \leq 2, \tag{2.1a}$$

with the boundary conditions

$$v(0) = \delta \mathcal{I}^q v(\xi) = \delta \int_0^{\xi} \frac{(\xi-\tau)^{q-1}}{\Gamma(q)} v(\tau)\, d\tau, \quad 0 < \xi < 1, \tag{2.1b}$$

$$v(1) = \psi v(\theta), \quad 0 < \theta < 1, \tag{2.1c}$$

has the solution

$$\begin{aligned} v(t) &= (\Pi_1 + \Pi_3(\psi-1)\,\Pi_1\,(t+\Pi_2))\,\mathcal{I}^{q+\beta}k(\xi) - \Pi_3\,(t+\Pi_2)\int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}k(\tau)\,d\tau \\ &+ \Pi_3\,\psi\,(t+\Pi_2)\int_0^{\theta}\frac{(\theta-\tau)^{\beta-1}}{\Gamma(\beta)}k(\tau)\,d\tau + \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}k(\tau)\,d\tau, \quad t\in[0,1]. \end{aligned} \tag{2.2}$$

Proof We write the solution of the model in (2.1) as

$$v(t) = d_1 t + d_2 + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} k(s)\, ds, \tag{2.3}$$

where d_1, d_2 are arbitrary real numbers. Using initial conditions, we have

$$v(0) = d_2 = \delta \mathcal{I}^q v(\xi),$$

$$v(1) = d_1 + d_2 + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} k(s)\, ds.$$

Applying the Riemann–Liouville integral of order q to v, we obtain the following equation:

$$\mathcal{I}^q v(t) = \mathcal{I}^{q+\beta}k(t) + d_1 \frac{t^{q+1}}{\Gamma(q+2)} + d_2 \frac{t^q}{\Gamma(q+1)}. \tag{2.4}$$

By taking $t = \xi$ in (2.4), we get

$$\mathcal{I}^q v(\xi) = \mathcal{I}^{q+\beta} k(\xi) + d_1 \frac{\xi^{q+1}}{\Gamma(q+2)} + d_2 \frac{\xi^q}{\Gamma(q+1)} = v(0) = d_2. \tag{2.5}$$

By taking $t = 1$ in (2.3), we get

$$d_1 + d_2 + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} k(s)\, ds = v(1) = \psi v(\theta). \tag{2.6}$$

Solving the equation (2.5–2.6) for d_1 and d_2, we obtain

$$d_1 = \Pi_3 \left((\psi - 1)\, \Pi_1\, \mathcal{I}^{q+\beta} k(\xi) - \int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)} k(\tau)\, d\tau + \psi \int_0^\theta \frac{(\theta-\tau)^{\beta-1}}{\Gamma(\beta)} k(\tau)\, d\tau \right),$$

$$d_2 = \Pi_1\, \mathcal{I}^{q+\beta} k(\xi) + d_1 \Pi_2.$$

Substituting the values of d_1 and d_2 into (2.3), the desired result (2.2) is obtained. □

3. Main results

We define

$$U = \{u(t) \ : \ u(t) \in C^1([0,1])\} \quad \text{and} \quad V = \{v(t) \ : \ v(t) \in C^1([0,1])\},$$

with the norms

$$\|u(t)\| = \max\{|u(t)|, \quad t \in [0,1]\} \quad \text{and} \quad \|v(t)\| = \max\{|v(t)|, \quad t \in [0,1]\},$$

respectively. We note that $(U, ||\cdot||)$ and $(V, ||\cdot||)$ are Banach spaces. Moreover, the product space $(U \times V, \|\cdot\|)$ is a Banach space associated to the norm $\|(u,v)\| = \|u\| + \|v\|$.

Lemma 3.1 *(Leray–Schauder alternative [9]) Let $\mathfrak{F} : S \to S$ be a completely continuous operator. Let*

$$\mathcal{S}(\mathfrak{F}) = \{x \in S : \quad x = \omega\, \mathfrak{F}(x) \quad \textit{for some } 0 < \omega < 1\}.$$

Then either the set $\mathcal{S}(\mathfrak{F})$ is unbounded, or $\mathfrak{F}$ has at least one fixed point.

By using Lemmas 2.5–2.6, we define an operator $\mathcal{F} : U \times V \to U \times V$ by

$$\mathcal{F}(u,v)(t) := (\mathcal{F}_1(u,v)(t), \mathcal{F}_2(u,v)(t)),$$

where

$$\mathcal{F}_1(u,v)(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha-1)}\, f(\tau, u(\tau), v(\tau))\, d\tau$$
$$- \frac{\zeta\Gamma(p+1)}{\zeta\mu^p - \Gamma(p+1)} \int_0^\mu \frac{(\mu-\tau)^{p+\alpha-1}}{\Gamma(p+\alpha)}\, f(\tau, u(\tau), v(\tau))\, d\tau,$$

and

$$\mathcal{F}_2(u,v)(t) = (\Pi_1 + \Pi_3(\psi-1)\,\Pi_1\,(t+\Pi_2))\int_0^{\xi} \frac{(\xi-\tau)^{q+\beta-1}}{\Gamma(q+\beta)}\, g(\tau,u(\tau),v(\tau))\,d\tau$$
$$-\,\Pi_3\,(t+\Pi_2)\int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}\, g(\tau,u(\tau),v(\tau))\,d\tau$$
$$+\,\Pi_3\,\psi\,(t+\Pi_2)\int_0^{\theta} \frac{(\theta-\tau)^{\beta-1}}{\Gamma(\beta)}\, g(\tau,u(\tau),v(\tau))\,d\tau$$
$$+\int_0^{t} \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}\, g(\tau,u(\tau),v(\tau))\,d\tau, \quad t\in[0,1].$$

For ease of notation in the next theorems, we set

$$\chi_1 = \left\{\frac{1}{\Gamma(\alpha+1)} + \frac{\mu^{p+\alpha}|\zeta|\Gamma(p+1)}{|\Gamma(p+1)-\zeta\mu^p|\Gamma(p+q+1)}\right\}, \tag{3.1a}$$

$$\chi_2 = (|\Pi_1 + \Pi_3\,(\psi-1)\,\Pi_1\,(1+\Pi_2)|\frac{\xi^{q+\beta}}{\Gamma(q+\beta+1)} + \frac{|\Pi_3(1+\Pi_2)|}{\Gamma(\beta+1)}$$
$$+\,|\Pi_3\psi(1+\Pi_2)|\frac{\theta^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+1)}), \tag{3.1b}$$

$$\chi_0 = \min\{1-(\chi_1\,\hat{f}_1 + \chi_2\,\hat{g}_1), 1-(\chi_1\,\hat{f}_2 + \chi_2\,\hat{g}_2)\}. \tag{3.1c}$$

Now we use Leray–Schauder alternative to justify the existence of the solution of the FDE (1.1).

Theorem 3.2 *Let* $\zeta \neq \frac{\Gamma(p+1)}{\mu^p}$, $\Gamma(q+1)\neq \xi^q$, $\frac{1-\psi\theta}{\psi-1} \neq \frac{\xi^{q+1}}{\Gamma(q+2)-\xi^q(q+1)}$,

$$|f(t,x_1,x_2)| \le \hat{f}_0 + \hat{f}_1|x_1| + \hat{f}_2|x_2|, \quad \textit{and} \quad |g(t,x_1,x_2)| \le \hat{g}_0 + \hat{g}_1|x_1| + \hat{g}_2|x_2|,$$

with $\hat{f}_i, \hat{g}_i \ge 0 (i=1,2)$ *and* $\hat{f}_0 > 0$, $\hat{g}_0 > 0$. *Moreover,*

$$\chi_1\,\hat{f}_1 + \chi_2\,\hat{g}_1 < 1 \quad \textit{and} \quad \chi_1\,\hat{f}_2 + \chi_2\,\hat{g}_2 < 1$$

hold where χ_1 *and* χ_2 *satisfy (3.1). Then the FDE (1.1) has at least one solution.*

Proof The continuity of f and g implies that the operator $\mathcal{F}$ is continuous. Assume that $\Omega \subset U\times V$ is a bounded set and L_f and L_g are positive real numbers with

$$|f(t,u(t),v(t))| \le L_f, \quad |g(t,u(t),v(t))| \le L_g, \quad \forall (u,v)\in\Omega.$$

First, by [11, Thm. 3.2], the following estimate holds:

$$|\mathcal{F}_1(u,v)(t)| \le L_f \left\{\frac{1}{\Gamma(\alpha+1)} + \frac{\mu^{p+\alpha}|\zeta|\Gamma(p+1)}{|\Gamma(p+1)-\zeta\mu^p|\Gamma(p+q+1)}\right\} = L_f\,\chi_1. \tag{3.2}$$

On the other hand, we show that

$$
\begin{aligned}
&|\mathcal{F}_2(u,v)(t)|\\
&\leq \Big|\Pi_1+\Pi_1\ \Pi_3(\psi-1)\ (t+\Pi_2)\Big| \int_0^{\xi} \frac{(\xi-s)^{q+\beta-1}}{\Gamma(q+\beta)}\ |g(s,u(s),v(s))|\ ds\\
&+\Big|\Pi_3\ (t+\Pi_2)\Big| \int_0^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\ |g(s,u(s),v(s))|\ ds\\
&+\Big|\psi\ (t+\Pi_2)\ \Pi_3\Big|\ \int_0^{\theta} \frac{(\theta-s)^{\beta-1}}{\Gamma(\beta)}\ |g(s,u(s),v(s))|\ ds\\
&+\int_0^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ |g(s,u(s),v(s))|\ ds\\
&\leq L_g(\Big|\Pi_1+\Pi_3(\psi-1)\ \Pi_1\ (1+\Pi_2)\Big| \frac{\xi^{q+\beta}}{\Gamma(q+\beta+1)}+\frac{|\Pi_3(1+\Pi_2)|}{\Gamma(\beta+1)}\\
&+\Big|\Pi_3\psi(1+\Pi_2)\Big|\frac{\theta^{\beta}}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta+1)})=L_g\ \chi_2.
\end{aligned}
\tag{3.3}
$$

The estimates (3.2)–(3.3) show that $\mathcal{F}$ is uniformly bounded. Now we prove that $\mathcal{F}$ is equicontinuous. We fix $0 \leq t_1 \leq t_2 \leq 1$. By [11, Thm. 3.2], we have

$$
\begin{aligned}
&\Big|\mathcal{F}_1(u(t_2),v(t_2))-\mathcal{F}_1(u(t_1),v(t_1))\Big|\\
&=\Big|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha-1)}\ f(s,u(s),v(s))\ ds-\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha-1)}\ f(s,u(s),v(s))\ ds\Big|\\
&\leq \frac{L_f}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha}).
\end{aligned}
\tag{3.4}
$$

On the other hand, for the operator $\mathcal{F}_2(u,v)(t)$, we obtain

$$
\begin{aligned}
&\Big|\mathcal{F}_2(u(t_2),v(t_2))-\mathcal{F}_2(u(t_1),v(t_1))\Big|\\
&=\Big|\Pi_3(\psi-1)\ \Pi_1\ (t_2-t_1)\int_0^{\xi} \frac{(\xi-\tau)^{q+\beta-1}}{\Gamma(q+\beta)}\ g(\tau,u(\tau),v(\tau))\ d\tau\Big|\\
&+\Big|\Pi_3\ (t_2-t_1)\int_0^{1} \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}\ g(\tau,u(\tau),v(\tau))\ d\tau\Big|\\
&+\Big|\Pi_3\ \psi\ (t_2-t_1)\ \int_0^{\theta} \frac{(\theta-\tau)^{\beta-1}}{\Gamma(\beta)}\ g(\tau,u(\tau),v(\tau))\ d\tau\Big|\\
&+\Big|\int_0^{t_2} \frac{(t_2-\tau)^{\beta-1}}{\Gamma(\beta)}\ g(\tau,u(\tau),v(\tau))\ d\tau-\int_0^{t_1} \frac{(t_1-\tau)^{\beta-1}}{\Gamma(\beta)}\ g(\tau,u(\tau),v(\tau))\ d\tau\Big|\\
&\leq L_g\left\{\frac{t_2^{\beta}-t_1^{\beta}}{\Gamma(\beta+1)}+\Big(\frac{|\Pi_3\ \Pi_1\ (\psi-1)|}{\Gamma(q+\beta+1)}\xi^{q+\beta}+\frac{|\Pi_3|}{\Gamma(\beta+1)}(1+\psi\ \theta^{\beta}\Big)(t_2-t_1)\right\}.
\end{aligned}
\tag{3.5}
$$

Thus, $\mathcal{F}$ is completely continuous according to the inequalities (3.4)–(3.5).

As the last step of the proof, we will justify that

$$\mathcal{S} = \{(u,v) \in U \times V \mid (u,v) = \omega\, \mathcal{F}(u,v), 0 \leq \omega \leq 1\}$$

is bounded. For $(u,v) \in \mathcal{S}$, $(u,v) = \omega\, \mathcal{F}(u,v)$ holds. Then, for all $t \in [0,1]$, we get

$$u(t) = \omega\, \mathcal{F}_1(u,v), \quad v(t) = \omega\, \mathcal{F}_2(u,v).$$

Then, by [11, Thm. 3.2], we have

$$|u(t)| \leq \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|\zeta|\mu^{p+\alpha}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1)-\zeta\mu^p|} \right\} (\hat{f}_0 + \hat{f}_1|u(t)| + \hat{f}_2|v(t)|).$$

Now we show that

$$|v(t)| \leq \Big(|\Pi_1 + \Pi_3(\psi-1)\,\Pi_1\,(1+\Pi_2)| \frac{\xi^{q+\beta}}{\Gamma(q+\beta+1)} + \frac{|\Pi_3(1+\Pi_2)|}{\Gamma(\beta+1)}$$
$$+ |\Pi_3\psi(1+\Pi_2)| \frac{\theta^\beta}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+1)} \Big) (\hat{g}_0 + \hat{g}_1|u(t)| + \hat{g}_2|v(t)|)$$

holds. These inequalities imply that

$$\|u(t)\| \leq \chi_1\,(\hat{f}_0 + \hat{f}_1|u(t)| + \hat{f}_2|v(t)|), \quad \|v(t)\| \leq \chi_2\,(\hat{g}_0 + \hat{g}_1|u(t)| + \hat{g}_2|v(t)|).$$

Then we obtain

$$\|u(t)\| + \|v(t)\| = (\chi_1\,\hat{f}_0 + \chi_2\,\hat{g}_0) + (\chi_1\,\hat{f}_1 + \chi_2\,\hat{g}_1)\|u(t)\| + (\chi_1\,\hat{f}_2 + \chi_2\,\hat{g}_2)\|v(t)\|.$$

Thus,

$$\|(u,v)\| \leq \frac{\chi_1\,\hat{f}_0 + \chi_2\,\hat{g}_0}{\chi_0}, \quad \text{for all } t \in [0,1].$$

Now we conclude that $\mathcal{S}$ is bounded. By Theorem 3.1, there is at least one fixed point of $\mathcal{F}$, which means that the FDE (1.1) has at least one solution. □

Banach's contraction principle is used to prove the existence and uniqueness of the solution of the FDE (1.1).

Theorem 3.3 *Let $f,g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions such that*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq \tilde{f}_1|u_1 - v_1| + \tilde{f}_2|u_2 - v_2|,$$
$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \leq \tilde{g}_1|u_1 - v_1| + \tilde{g}_2|u_2 - v_2|,$$

where $\tilde{f}_i, \tilde{g}_i$ are positive constants and $u_i, v_i \in \mathbb{R}$ for $i = 1,2$ for all $t \in [0,1]$. Suppose that

$$\chi_1\,(\tilde{f}_1 + \tilde{f}_2) + \chi_2\,(\tilde{g}_1 + \tilde{g}_2) < 1.$$

Then the FDE (1.1) has a unique solution.

Proof Let $\sup_{t\in[0,1]} f(t,0,0) = S_1 < \infty$, $\sup_{t\in[0,1]} g(t,0,0) = S_2 < \infty$ with

$$\varepsilon \geq \frac{S_1\chi_1 + S_2\chi_2}{1 - \chi_1\ (\tilde{f}_1 + \tilde{f}_2) - \chi_2\ (\tilde{g}_1 + \tilde{g}_2)}.$$

For $B_\varepsilon = \{(u,v) \in U \times V : \|(u,v)\| \leq \varepsilon\}$, we show that $\mathcal{F}B_\varepsilon \subset B_\varepsilon$. By [11, Thm. 3.3], we have

$$\begin{aligned}
&|\mathcal{F}_1(u,v)(t)| \\
&\leq \left\{\frac{1}{\Gamma(\alpha+1)} + \frac{|\zeta|\mu^{p+\alpha}\Gamma(p+1)}{\Gamma(p+q+1)|\Gamma(p+1)-\zeta\mu^p|}\right\}(\tilde{f}_1\|u\| + \tilde{f}_2\|v\| + S_1) \\
&\leq \chi_1\ ((\tilde{f}_1 + \tilde{f}_2)\varepsilon + S_1), \quad (u,v) \in B_\varepsilon.
\end{aligned}$$

It implies that

$$\|\mathcal{F}_1(u,v)(t)\| \leq \chi_1\ ((\tilde{f}_1 + \tilde{f}_2)\varepsilon + S_1). \tag{3.6}$$

Furthermore, we show that

$$\begin{aligned}
&|\mathcal{F}_2(u,v)(t)| \\
&\leq \frac{|\Pi_1 + \Pi_3(\psi-1)\ \Pi_1\ (t+\Pi_2)|}{\Gamma(q+\beta)} \int_0^\xi (\xi-\tau)^{q+\beta-1}\ (|g(\tau,u(\tau),v(\tau)) - g(0,u(0),v(0))| + |g(0,u(0),v(0))|)\ d\tau \\
&+ \frac{|\Pi_3\ (t+\Pi_2)|}{\Gamma(\beta)} \int_0^1 (1-\tau)^{\beta-1}\ (|g(\tau,u(\tau),v(\tau)) - g(0,u(0),v(0))| + |g(0,u(0),v(0))|)\ d\tau \\
&+ \frac{|\Pi_3\ \psi\ (t+\Pi_2)|}{\Gamma(\beta)} \int_0^\theta (\theta-\tau)^{\beta-1}\ (|g(\tau,u(\tau),v(\tau)) - g(0,u(0),v(0))| + |g(0,u(0),v(0))|)\ d\tau \\
&+ \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\ (|g(\tau,u(\tau),v(\tau)) - g(0,u(0),v(0))| + |g(0,u(0),v(0))|)\ d\tau \\
&\leq \chi_2\ ((\tilde{g}_1 + \tilde{g}_2)\varepsilon + S_2).
\end{aligned}$$

It means that

$$\|\mathcal{F}_2(u,v)(t)\| \leq \chi_2\ ((\tilde{g}_1 + \tilde{g}_2)\varepsilon + S_2). \tag{3.7}$$

By the inequalities (3.6)–(3.7), we have

$$\|\mathcal{F}(u,v)(t)\| \leq \varepsilon.$$

For $(u_2,v_2),(u_1,v_1) \in U \times V$ and for any $t \in [0,1]$, by [11, Thm. 3.3], we have

$$\|\mathcal{F}_1(u_2,v_2)(t) - \mathcal{F}_1(u_1,v_1)(t)\| \leq \chi_1\ (\tilde{f}_1 + \tilde{f}_2)(\|u_2 - u_1\| + \|v_2 - v_1\|). \tag{3.8}$$

On the other hand, we get

$$\begin{aligned}
&\left|\mathcal{F}_2(u_2,v_2)(t) - \mathcal{F}_2(u_1,v_1)(t)\right| \\
&\leq \left|\Pi_1 + \Pi_3(\psi-1)\ \Pi_1\ (t+\Pi_2)\right| \int_0^\xi \frac{(\xi-\tau)^{q+\beta-1}}{\Gamma(q+\beta)}\ |g(\tau,u_2(\tau),v_2(\tau)) - g(\tau,u_1(\tau),v_1(\tau))|\ d\tau \\
&+ \left|\Pi_3\ (t+\Pi_2)\right| \int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}\ |g(\tau,u_2(\tau),v_2(\tau)) - g(\tau,u_1(\tau),v_1(\tau))|\ d\tau \\
&+ \left|\Pi_3\ \psi\ (t+\Pi_2)\right| \int_0^\theta \frac{(\theta-\tau)^{\beta-1}}{\Gamma(\beta)}\ |g(\tau,u_2(\tau),v_2(\tau)) - g(\tau,u_1(\tau),v_1(\tau))|\ d\tau \\
&+ \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}\ |g(\tau,u_2(\tau),v_2(\tau)) - g(\tau,u_1(\tau),v_1(\tau))|\ d\tau \\
&\leq \chi_2\ (\tilde{g}_1 + \tilde{g}_2)(\|u_2 - u_1\| + \|v_2 - v_1\|).
\end{aligned}$$

Then:

$$\|\mathcal{F}_2(u_2,v_2)(t)-\mathcal{F}_2(u_1,v_1)(t)\| \leq \chi_2\,(\tilde{g}_1+\tilde{g}_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{3.9}$$

From the inequalities (3.8–3.9), we obtain

$$\begin{aligned}&\|\mathcal{F}(u_2,v_2)(t)-\mathcal{F}(u_1,v_1)(t)\|\\ &\leq [\chi_1\,(\tilde{f}_1+\tilde{f}_2)+\chi_2\,(\tilde{g}_1+\tilde{g}_2)](\|u_2-u_1\|+\|v_2-v_1\|).\end{aligned}$$

The operator $\mathcal{F}$ is a contraction operator under the condition $\chi_1\,(\tilde{f}_1+\tilde{f}_2)+\chi_2\,(\tilde{g}_1+\tilde{g}_2)<1$. By the Banach theorem, $\mathcal{F}$ has a unique fixed point and it shows that the FDE has a unique solution. □

3.1. An example

We consider the FDE

$${}^{C}\mathcal{D}^{7/10}u(t)=\frac{1}{\sqrt{25+t^2}}+e^{-3t}\frac{|u(t)|}{25(1+|u(t)|)}+t^2\frac{\sin^2(v(t))}{50},\quad 0<t<1, \tag{3.10a}$$

$${}^{C}\mathcal{D}^{10/7}v(t)=\frac{1}{\sqrt{36+t^4}}+\frac{\sin^2(u(t))}{2\sqrt{144+t^2}}+e^{-2t}\frac{|v(t)|}{72(1+|v(t)|)},\quad 0<t<1, \tag{3.10b}$$

$$u(0)=\sqrt{5}\mathcal{I}^{6/5}u(1/3), \tag{3.10c}$$

$$v(0)=\sqrt{7}\mathcal{I}^{2/5}v(1/2),\quad v(1)=\frac{v(2/5)}{3}. \tag{3.10d}$$

We note that $\frac{\Gamma(p+1)}{\mu^p}\approx 4.1176\neq\zeta=\sqrt{5}$, $\Gamma(q+1)\approx 0.8873\neq\xi^q\approx 0.7579$, $\frac{1-\psi\theta}{\psi-1}\approx -1.3\neq\frac{\xi^{q+1}}{\Gamma(q+2)-\xi^q(q+1)}\approx 2.0916$. Moreover, $\hat{f}_0=1/5$, $\hat{f}_1=1/25=\hat{f}_2$, $\hat{g}_0=1/6$, $\hat{g}_1=1/12$, $\hat{g}_2=1/72$, which lead to

$$\chi_1\,\hat{f}_1+\chi_2\,\hat{g}_1\approx 0.2332<1,\quad \chi_1\,\hat{f}_2+\chi_2\,\hat{g}_2\approx 0.0897<1.$$

Then, by Thm. (3.2), there exists a solution to the FDE (3.10). In addition,

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\leq\frac{1}{25}|u_1-u_2|+\frac{1}{25}|v_1-v_2|,$$

$$|g(t,u_1,v_1)-g(t,u_2,v_2)|\leq\frac{1}{12}|u_1-u_2|+\frac{1}{72}|v_1-v_2|,$$

where $\tilde{f}_1=\frac{1}{25}$, $\tilde{f}_2=\frac{1}{25}$, $\tilde{g}_1=\frac{1}{12}$, $\tilde{g}_2=\frac{1}{72}$, $\chi_1\approx 1.5250$, $\chi_2\approx 2.0668$, and

$$\chi_1\,(\tilde{f}_1+\tilde{f}_2)+\chi_2\,(\tilde{g}_1+\tilde{g}_2)\approx 0.0897<1.$$

Thus, by Thm. 3.3, the solution to the FDE (3.10) is unique.

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