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# On the solutions of a fractional boundary value problem

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**Abstract:** This paper is devoted to showing the existence and uniqueness of solution of a regular second-order nonlinear fractional differential equation subject to the ordinary boundary conditions. The Banach fixed point theorem is used to prove the results.

**Key words:** Fractional equation, nonlinear equation, fixed point theorems

## 1. Introduction

It is well known that nonlinear fractional differential equations are used to describe real-world problems (see [5, 11, 15] and the references therein). Very recently, the Mittag-Leffler stability approach of nonlinear fractional-order systems in the presence of the impulses was investigated [19]. The exact solutions and maximal dimension of invariant subspaces corresponding to the time fractional coupled nonlinear partial differential equations were reported in [17]. We recall that in the last few years special attention was paid to finding new methods and techniques to solve numerical solutions for nonlinear fractional ordinary fractional equations (see, for example, [16] and the references therein). An approximate solution of fractional differential equations utilizing the basic properties of artificial neural networks for function approximation was reported in [14]. The existence of solutions corresponding to the delayed nonlinear fractional functional differential equations in the presence of three-point integral boundary value conditions was recently achieved [23]. The existence and multiplicity of positive solutions for a nonlinear fractional differential equation boundary-value problem were debated in [22]. More results on boundary value problems for nonlinear fractional differential equations can be found in [1, 3] and the references therein.

We recall that classical Sturm–Liouville equations are among the important tools to get some information about the existence and uniqueness of solutions of real-world problems [4, 7]. For example, an interesting nonlinear second-order differential equation together with some boundary conditions was studied on $[0,\infty)$ by Guseinov and Yaslan [8] as follows:

$$-(p(x)y')' + q(x)y = f(x,y), \tag{1.1}$$

with the real-valued measurable functions $p$ and $q$ on the given interval.

On the other hand, classical Sturm–Liouville differential expressions can be generalized by fractional Sturm–Liouville differential expressions [6]. Fractional derivatives are defined by the fractional integrals.

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Namely, the left-sided Riemann–Liouville fractional integral is defined as

$$I_{a+}^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{y(s)}{(x-s)^{1-\alpha}} ds,$$

where $x > a$ and $\mathrm{Re}\alpha > 0$. Similarly, the right-sided Riemann–Liouville fractional integral is defined as

$$I_{b-}^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \frac{y(s)}{(s-x)^{1-\alpha}} ds,$$

where $x < a$ and $\mathrm{Re}\alpha > 0$. We denote by $\Gamma$ the gamma function. Then, for $0 < \alpha < 1$, the left-sided Riemann–Liouville derivative of $y$ is defined as

$$D_{a+}^{\alpha} y = \frac{d}{dx} I_{a^+}^{1-\alpha} y.$$

Moreover, the right-sided Riemann–Liouville derivative of $y$ is defined as

$$D_{b-}^{\alpha} y = -\frac{d}{dx} I_{b^-}^{1-\alpha} y.$$

Other derivative formulas can also be given. The left-sided Caputo derivative of order $\alpha$, $0 < \alpha < 1$, is defined as

$${}^{c}D_{a+}^{\alpha} y = D_{a+}^{\alpha}(y(x) - y(a)),$$

where $x > a$. Similarly, the right-sided Caputo derivative of order $\alpha$, $0 < \alpha < 1$, is defined as

$${}^{c}D_{b-}^{\alpha} y = D_{b-}^{\alpha}(y(x) - y(b)).$$

Then Sturm–Liouville equations can be described with left- or right-sided Riemann–Liouville derivatives of order $\alpha$ or left- or right-sided Caputo derivatives of order $\alpha$.

In this article our main aim is to prove the existence and uniqueness result for a regular fractional second-order equation of order $\alpha$, $1/2 < \alpha < 1$, similar to equation (1.1) but on the finite interval with ordinary boundary conditions. First we pass to the integral operator associated with the nonlinear problem. Then, using the Banach fixed point theorem, we find the fixed point of the operator and this gives the unique solution of the nonlinear fractional second-order problem. We should note that in the literature there are some papers on nonlinear boundary-value problems [2, 9, 10, 18, 20, 21].

## 2. Nonlinear regular fractional problem

We consider the Hilbert space $H$, which is equipped with the usual inner product

$$(y, z) = \int_{a}^{b} y\bar{z} dx.$$

The corresponding norm is defined by $\|y\| = [(y, y)]^{1/2}$.

Let us consider the differential expression

$$\mathcal{L}y = {}^{c}D_{b-}^{\alpha}\left(p(x)\,{}^{c}D_{a+}^{\alpha}y\right),$$

where $1/2 < \alpha < 1$. Here we assume that on the interval $[a,b]$ $p$ is a real-valued and continuous function and $-\infty < a < b < \infty$.

Let $\mathcal{D}$ be a subset of $H$ such that $\mathcal{L}y$ is meaningful and $\mathcal{L}y \in H$. For $y \in \mathcal{D}$ we shall consider the following nonlinear fractional equation,

$$\mathcal{L}y = f(x,y), \tag{2.1}$$

on the interval $[a,b]$, where $f(x,y)$ is a real-valued, continuous function on $[a,b]\times\mathbb{R}$ satisfying

$$|f(x,y)| \le |h(x)| + \delta\,|y(x)| \tag{2.2}$$

with $h \in H$ and $\delta > 0$.

We should note that inequality (2.2) implies that the transformation $y \to f(x,y)$ is continuous in $H$ and maps $H$ into $H$ [8, 13].

Throughout the paper we deal with equation (2.1) together with the following boundary conditions:

$$y(a) = 0, \tag{2.3}$$

$$^{c}D_{a+}^{\alpha}y(b) = 0. \tag{2.4}$$

Following the same idea of [12] we can introduce the solution of the problems (2.1), (2.3), and (2.4) in $H$ as follows:

$$y(x) = \int_a^b G(x,t)f(t,y(t))dt,\ \ x\in[a,b], \tag{2.5}$$

where

$$G(x,t) = \begin{cases} \frac{1}{[\Gamma(\alpha)]^2}\int_a^t \frac{(x-s)^{\alpha-1}(t-s)^{\alpha-1}}{p(s)}ds, & t \le x \\ \frac{1}{[\Gamma(\alpha)]^2}\int_a^x \frac{(x-s)^{\alpha-1}(t-s)^{\alpha-1}}{p(s)}ds, & t > x \end{cases}.$$

Note that [12]

$$\int_a^b\int_a^b |G(x,t)|^2\,dxdt < \infty. \tag{2.6}$$

Using (2.2) and (2.6) we may introduce the operator $T: H \to H$ as follows:

$$Ty = \int_a^b G(x,t)f(t,y(t))dt,\ \ x\in[a,b], \tag{2.7}$$

where $y \in H$. Consequently, (2.5) and (2.7) give the following equality:

$$Ty = y.$$

## 3. Solution of the problem

In this section we introduce the main result. Before this, we shall recall the Banach fixed point theorem.

**Banach fixed point theorem.** *Let $H$ be a Banach space and $K$ be a nonempty closed subset of $H$. Assume for all $y, z \in K$ that*

$$\|Ly - Lz\| \leq \theta \|y - z\|, \; 0 < \theta < 1,$$

*where $L$ is a operator such that $L : K \to K$. Then $L$ has a unique fixed point in $K$.*

Now we can introduce the following theorem.

**Theorem 3.1.** *Assume that the following inequality holds. Let us assume that there exists a constant $M$ such that*

$$\int_a^b |f(x, y(x)) - f(x, z(x))|^2 \, dx \leq M^2 \int_a^b |y(x) - z(x)|^2 \, dx, \tag{3.1}$$

*where $M$ is a constant and $y, z \in H$. Then there is a unique solution of the nonlinear problems (2.1), (2.3), and (2.4) in $H$ provided that*

$$M \left[ \int_a^b \int_a^b |G(x,t)|^2 \, dx dt \right]^{1/2} < 1.$$

**Proof** For arbitrary $y, z \in H$ we have

$$|Ty - Tz|^2 \leq \int_a^b |G(x,t)|^2 \, dt \int_a^b |f(t, y(t)) - f(t, z(t))|^2 \, dt. \tag{3.2}$$

(3.1) and (3.2) show that

$$|Ty - Tz|^2 \leq M^2 \|y - z\|^2 \int_a^b |G(x,t)|^2 \, dt$$

for all $x$ in $[a, b]$. Consequently, for

$$\theta = M \left[ \int_a^b \int_a^b |G(x,t)|^2 \, dx dt \right]^{1/2} < 1$$

we obtain that $T$ is a contraction mapping in $H$. Thus, by the Banach fixed point theorem the proof is completed. □

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