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Lie symmetry analysis and conservation laws for the time fractional simplified modified Kawahara equation

Abstract: In this work, Lie symmetry analysis for the time fractional simplified modified Kawahara (SMK) equation with Riemann-Liouville (RL) derivative, is analyzed. We transform the time fractional SMK equation to nonlinear ordinary differential equation (ODE) of fractional order using its Lie point symmetries with a new dependent variable. In the reduced equation, the derivative is in the Erdelyi-Kober (EK) sense. We solve the reduced fractional ODE using a power series technique. Using Ibragimov's nonlocal conservation method to time fractional partial differential equations, we compute conservation laws (Cls) for the time fractional SMK equation. Some figures of the obtained explicit solution are presented.

Keywords: time fractional SMK, Lie symmetry, exact solutions, conservation laws

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1 Introduction

Symmetry analysis has many applications in the field of science and engineering. Lie's method is one of the global and efficient methods for investigating analytical solutions and symmetry properties of nonlinear partial differential equations (NLPDEs) [1-17]. Fractional calculus has been successfully used to explain many complex nonlinear phenomena and dynamic processes in physics, engineering, electromagnetics, viscoelasticity, and electrochemistry [18-34].

Generally, physical phenomenon might depend on its current state and on its historical states, which can be modelled successfully by applying the theory of derivatives and integrals of fractional order [35, 36]. Due to this, several analytical techniques are used to derive exact, explicit, and numerical solutions of nonlinear fractional partial differential equations (FPDEs) [30-34]. We find very few studies of symmetry analysis for FPDEs and their group properties are not plainly understood [37-41].

In other words, Cls are universally known to possess an important role in the analysis of NLPDEs from a physical viewpoint [42] . If the considered system has Cls, then its integrability will be possible [43 ,44]. Noether theorem supplies us with a strategic idea for constructing Cls of NLPDEs so long as the Noether symmetry associated with the Lagrangian is known for Euler-Lagrange equations [45]. Nevertheless, there are some techniques in the literature for obtaining the Cls of the NLPDEs, that do not have the Lagrangian [46-47].

Time fractional NLPDEs come from classical NLPDEs by replacing its time derivative with a fractional derivative. In the present work, we study Lie symmetry analysis, explicit solution using the power series technique and Ibragimov's nonlocal Cls [48] for the time fractional SMK equation given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \beta u^2 u_x + \gamma u_{xxxxx} = 0, \qquad (1)$$

in Eq. (1), $0 < \alpha \leq 1$, and β and γ are arbitrary constants, and α is the order of the fractional time derivative. If $\alpha = 1$, Eq. (1) reduces to the classical SMK equation which was considered for exact travelling wave solutions and Cls in [49-51]. Moreover, one can find more details on the construction of analytical, exact, numerical solutions, and other information for classical NLPDEs, in [52-69].

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2 Preliminaries

Consider the RL fractional derivative [70, 71] given by

$$D^\alpha f(t) = \begin{cases} \frac{d^n f}{dt^n}, & \alpha = n, \\ \frac{d}{dt^n} I^{n-\alpha} f(t), & 0 \le n - 1 < \alpha < n, \end{cases}$$

where n is a natural number and $I^\mu f(t)$ is the RL fractional integral of order μ given by

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s) ds, \ \mu > 0$$

$$I^\mu f(t) = f(t) \tag{2}$$

and $\Gamma(z)$ represents Gamma function.

Consider time-fractional PDEs as below

$$\partial_t^\alpha u = F(t, x, u, u_x, u_{xx}, u_{xxx}, \dots), \quad (0 < \alpha < 1). \tag{3}$$

Given a one-parameter Lie group of infinitesimal transformations of the form

$$\begin{aligned}
\bar{t} &= t + \epsilon \xi^2(t, x, u) + O(\epsilon^2), \\
\bar{x} &= x + \epsilon \xi^1(t, x, u) + O(\epsilon^2), \\
\bar{u} &= u + \epsilon \eta(t, x, u) + O(\epsilon^2), \\
\frac{\partial^\alpha \bar{u}}{\partial \bar{t}} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon \eta_\alpha^0(t, x, u) + O(\epsilon^2), \\
\frac{\partial \bar{u}}{\partial \bar{x}} &= \frac{\partial u}{\partial x} + \epsilon \eta^x(t, x, u) + O(\epsilon^2), \\
\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \epsilon \eta^{xx}(t, x, u) + O(\epsilon^2), \\
\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \epsilon \eta^{xxx}(t, x, u) + O(\epsilon^2), \\
\frac{\partial^4 \bar{u}}{\partial \bar{x}^4} &= \frac{\partial^4 u}{\partial x^4} + \epsilon \eta^{xxxx}(t, x, u) + O(\epsilon^2), \\
\frac{\partial^5 \bar{u}}{\partial \bar{x}^5} &= \frac{\partial^5 u}{\partial x^5} + \epsilon \eta^{xxxxx}(t, x, u) + O(\epsilon^2),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\eta^x &= D_x(\eta) - u_x D_x(\xi^1) - u_t D_t(\xi^2), \\
\eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\xi^1) - u_{xx} D_t(\xi^2), \\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_t(\xi^2), \\
\eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt} D_x(\xi^1) - u_{xxxx} D_t(\xi^2), \\
\eta^{xxxxx} &= D_x(\eta^{xxxx}) - u_{xxxxt} D_x(\xi^1) - u_{xxxxx} D_t(\xi^2),
\end{aligned} \tag{5}$$

In Eq. (5), D_x is the total differential operator defined by

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \dots \ .$$

The corresponding Lie algebra of symmetries consists of a set of vector fields of the form

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{6}$$

The vector field Eq. (6) is a Lie point symmetry of Eq. (3) provided

$$P^{\alpha,5} rX(\bigtriangledown)|_{\bigtriangledown=0} = 0. \tag{7}$$

Also, the invariance condition yields [72] gives

$$\xi^2(t, x, u)|_{t=0} = 0, \tag{8}$$

and the αth extended infinitesimal related to RL fractional time derivative with Eq. (8) is given by [54, 55].

$$\eta_\alpha^0 = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\xi^2)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu \tag{9}$$

$$- \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\xi^1) D_t^{\alpha-n}(u_x)$$

$$+ \sum_{n=1}^\infty \left[\binom{\alpha}{n} \frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\xi^2) \right] D_t^{\alpha-n}(u),$$

in Eq. (9),

$$\mu = \sum_{n=2}^\infty \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \tag{10}$$

$$\times [-u]^r \frac{\partial^m}{\partial t^m} [u^{k-r}] \frac{\partial^{n-m+k}}{\partial t^{n-m} \partial u^k}.$$

It is worth noting that, $\mu = 0$ if the infinitesimal η is linear in u, due to the presence of $\frac{\partial \eta^k}{\partial u^k}$, where $k \ge 2$ in Eq. (10).

Definition 2.1. *The function $u = \Theta(x, t)$ is an invariant solution of Eq. (3) corresponding to the infinitesimal generator Eq. (6) provided that*

1. $u = \Theta(x, t)$ satisfies Eq. (3).

2. $u = \Theta(x, t)$ is an invariant surface of Eq. (5), that is to say

$$\xi^2(x, t, \Theta)\Theta_t + \xi^1(x, t, \Theta)\Theta_x = \eta(x, t, \Theta).$$

3 Lie symmetries and reduction for Eq. (1)

Suppose that Eq. (1) is an invariant under Eq. (5), we have that

$$\bar{u}_t^\alpha + \beta \bar{u}^2 \bar{u}_{\bar{x}} + \gamma \bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \tag{11}$$

so that, $u = u(x, t)$ satisfies Eq. (1). Using Eq. (5) in Eq. (11), we get the invariant equation

$$\eta_\alpha^0 + (2\beta u u_x)\eta + (\beta u^2)\eta^x + \gamma\eta^{xxxxx} = 0. \qquad (12)$$

Putting the values of η_α^0, η^x and η^{xxxxx} from Eq. (5) and Eq. (9) into Eq. (12) and isolating coefficients in partial derivatives with respect to x and power of u, we get

$$\partial_t^\alpha \eta - u\partial_t^\alpha \eta_u - \beta u^2 \eta_x - \gamma\eta_{xxxxx} = 0,$$

$$\binom{\alpha}{n}\partial_t^n(\eta) - \binom{\alpha}{n+1}D_t^{n+1}(\xi^2) = 0, \quad n = 1, 2, \ldots$$

$$\xi_u^1 = \xi_u^2 = \xi_t^1 = \xi_x^2 = \eta_{uu} = 0,$$

$$5\xi_x^1 - \alpha\xi_t^2 = 0.$$

Solving these equations, we get:

$$\xi^1 = c_1 + x\alpha c_2, \quad \xi^2 = 5tc_2, \quad \eta = -2\alpha u c_2,$$

where c_1 and c_2 are arbitrary constants. Thus infinitesimal symmetry group for Eq. (1) is spanned by the two vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\alpha\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\alpha\frac{\partial}{\partial u}. \qquad (13)$$

The similarity variables for the infinitesimal generator X_2 can be obtained by solving the following equations

$$\frac{dx}{\alpha x} = \frac{dt}{5t} = -\frac{du}{2\alpha u}.$$

Solving the above equations, we get

$$z_1 = xt^{-\frac{\alpha}{5}}, \quad z_2 = ut^{\frac{2\alpha}{5}}. \qquad (14)$$

Hence, from the symmetry X_2, we get the group-invariant solution

$$u = t^{-\frac{2\alpha}{5}}f(\xi), \quad \xi = xt^{-\frac{\alpha}{5}}, \qquad (15)$$

in Eq. (15), f is an arbitrary function of ξ. Using Eq. (15), Eq. (1) is transformed to a special nonlinear ODE of fractional order.

Consider the following theorem

Theorem 3.1. *The similarity transformation Eq. (15) reduces Eq. (1) to the nonlinear ODE of fractional order as below:*

$$\left(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha}f\right)(\xi) + \beta f^2 f_\xi + \gamma f_{\xi\xi\xi\xi} = 0 \qquad (16)$$

with the EK fractional differential operator [22]

$$\left(P_\beta^{\xi^2,\alpha}f\right) = \Pi_{j=0}^{n-1}\left(\xi^2 + j - \frac{1}{\beta}\frac{d}{d\xi}\right)(K_\beta^{\xi^2+\alpha,n-\alpha}f)(\xi), \qquad (17)$$

$$n = \begin{cases} [\alpha] + 1, & \alpha \neq \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases} \qquad (18)$$

where

$$(K_\beta^{\xi^2,\alpha}f)(\xi) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\xi^2+\alpha)}f(\xi u^{\frac{1}{\beta}})du, & \alpha > 0, \\ f(\xi), & \alpha = 0, \end{cases} \qquad (19)$$

is the EK fractional integral operator [74, 75].

Proof. Let $n - 1 < \alpha < 1$, $n = 1, 2, 3,\ldots$. Based on the RL fractional derivative in Eq. (15), we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_1^t (t-s)^{n-\alpha-1}s^{\frac{\alpha}{5}}f(xs^{-(\frac{\alpha}{5})})ds\right]. \qquad (20)$$

Let $v = \frac{t}{s}$, $ds = -\frac{t}{v^2}dv$. Thus, Eq. (20) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1}v^{-(n+1-\frac{7\alpha}{5})}f(\xi v^{\frac{\alpha}{5}})dv\right]. \qquad (21)$$

Applying EK fractional integral operator Eq. (19) in Eq. (21), we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\right)(\xi)\right]. \qquad (22)$$

We simplify the right hand side of Eq. (22). Consider $\xi = xt^{-\frac{\alpha}{5}}$, $\phi \in (0, \infty)$, we acquire

$$t\frac{\partial}{\partial t}\phi(\xi) = tx(-\frac{\alpha}{5})t^{-\frac{\alpha}{5}-1}\phi'(\xi) = -\frac{\alpha}{5}\xi\frac{\partial}{\partial\xi}\phi(\xi). \qquad (23)$$

Hence,

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\right)(\xi)\right] = \qquad (24)$$

$$\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\right)(\xi)\right)\right]$$

$$=$$

$$\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{7\alpha}{5}-1}\left(n - \frac{7\alpha}{5} - \frac{\alpha}{5}\xi\frac{\partial}{\partial\xi}\right)\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\right)(\xi)\right].$$

Repeating $n - 1$ times, we have

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\right)(\xi)\right] = \qquad (25)$$

$$\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha}f\right)(\xi)\right)\right]$$

$$= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[t^{n-\frac{7\alpha}{5}-1} \left(n - \frac{7\alpha}{5} - \frac{\alpha}{5} \xi \frac{\partial}{\partial \xi} \right) \right.$$
$$\left. \times \left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} f \right)(\xi) \right].$$

.

.

$$= t^{-\frac{7\alpha}{5}} \Pi_{j=0}^{n-1} \left[\left(1 - \frac{7\alpha}{5} + j - \frac{\alpha}{5} \xi \frac{\partial}{\partial \xi} \right) \right.$$
$$\left. \times \left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} f \right)(\xi) \right].$$

Applying EK fractional differential operator Eq. (17) in Eq. (25), we get

$$\frac{\partial^n u}{\partial t^n} \left[\left(t^{n-\frac{7\alpha}{5}} \left(K_{\frac{5}{\alpha}}^{1+\frac{\alpha}{5},n-\alpha} f \right)(\xi) \right) \right] = \qquad (26)$$
$$t^{-\frac{7\alpha}{5}} \left(P_{\frac{5}{\alpha}}^{1-\frac{5\alpha}{6},\alpha} f \right)(\xi)$$

Substituting Eq. (26) into Eq. (22), we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\frac{7\alpha}{5}} \left(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha} f \right)(\xi) \qquad (27)$$

Thus, Eq. (1) can be reduced into a fractional order ODE

$$\left(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha} f \right)(\xi) + \beta f^2 f_\xi + \gamma f_{\xi\xi\xi\xi} = 0 \qquad (28)$$

The proof of the theorem is completed. □

4 Conservation laws

We now construct the Cls for Eq. (1). We start with some definitions. The RL left-sided time-fractional derivative given by

$$oD_t^\alpha u = D_t^n(oI^{n-\alpha}u), \qquad (29)$$

where D_t is the total differential operator with respect to t, $n = [\alpha] + 1$, and $oI^{n-\alpha}u$ represents the left sided time-fractional integral of $n - \alpha$ order given by

$$(oI^{n-\alpha}u)(x, t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u(\theta, x)}{(t-\theta)^{1-n+\alpha}} d\theta. \qquad (30)$$

In Eq. (30), $\Gamma(z)$ represents Gamma function.

A Cls for Eq. (1) is represented as

$$D_t(C^t) + D_x(C^x) = 0, \qquad (31)$$

where $C^t = C^t(x, t, u, ...)$, $C^x = C^x(x, t, u, ...)$, and Eq. (31) holds for all solutions $u(x, t)$ of the Eq. (1).

We now apply Ibragimov method [48] for constructing the Cls of Eq. (1). Lagrangian for Eq. (1) can be presented as

$$L = v(x, t) \left(\frac{\partial^\alpha u}{\partial t^\alpha} + \beta u^2 u_x + \gamma u_{xxxxx} \right) \qquad (32)$$

where $v(x, t)$ is another dependent variable. The Euler-Lagrange operator [45, 46] is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^\star \frac{\partial}{\partial D_t^\alpha u} - D_x \frac{\partial}{\partial u_x} + D_{xx}^2 \frac{\partial}{\partial u_{xx}} - D_{xxx}^3 \frac{\partial}{\partial u_{xxx}} \qquad (33)$$
$$+ D_{xxxx}^4 \frac{\partial}{\partial u_{xxxx}} - D_{xxxxx}^5 \frac{\partial}{\partial u_{xxxxx}},$$

where $(D_t^\alpha)^\star$ is the adjoint operator of (D_t^α). The adjoint equation to Eq. (1) is given by [48]

$$\frac{\delta L}{\delta u} = 0. \qquad (34)$$

Consider two independent variables x, t and one dependent variable $u(x, t)$, we have that

$$\bar{X} + D_t(\xi^2)l + D_x(\xi^1)l = W \frac{\delta}{\delta u} + D_t N^t + D_x N^x, \qquad (35)$$

in Eq. (35), l represent the identity operator, $\frac{\delta}{\delta u}$ is the Euler-Lagrangian operator, N^t and N^x represent the Noether operation, $\bar{X}$ is defined by

$$\bar{X} = \xi^2 \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \eta_\alpha^0 \frac{\partial}{\partial D_t^\alpha u} + \eta^x \frac{\partial}{\partial u_x} + \eta^{xx} \frac{\partial}{\partial u_{xx}} \qquad (36)$$
$$+ \eta^{xxx} \frac{\partial}{\partial u_{xxx}} + \eta^{xxxx} \frac{\partial}{\partial u_{xxxx}} + \eta^{xxxxx} \frac{\partial}{\partial u_{xxxxx}},$$

and the Lie characteristic function W is given by

$$W = \eta - \xi^2 u_t - \xi^1 u_x. \qquad (37)$$

When RL time-fractional derivative is used in Eq. (1), N^t is defined by [45, 46]

$$N^t = \xi^2 l + \sum_{k=0}^{n-1} (-1)^k oD^{\alpha-1-k}(W) D_t^k \frac{\partial}{\partial oD_t^\alpha u} - (-1)^n \qquad (38)$$

$$\times J\left(W, D_t^n \frac{\partial}{\partial oD_t^\alpha u}\right).$$

With J given by

$$J(f, g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^T \frac{f(\xi^2, x)g(\mu, x)}{(\mu - \xi^2)^{\alpha+1-n}} \, d\mu dt. \qquad (39)$$

For Eq. (1), the operator N^x is given by

$$N^x = \xi^1 l + W\left(\frac{\partial}{\partial u_x} - D_x \frac{\partial}{\partial u_{xx}} + D_x^2 \frac{\partial}{\partial u_{xxx}}\right.$$
$$\left. - D_x^3 \frac{\partial}{\partial u_{xxxx}} + D_x^4 \frac{\partial}{\partial u_{xxxxx}}\right) + D_x(W)\left(\frac{\partial}{\partial u_{xx}}\right.$$
$$\left. - D_x \frac{\partial}{\partial u_{xxx}} + D_x^2 \frac{\partial}{\partial u_{xxxx}} - D_x^3 \frac{\partial}{\partial u_{xxxxx}}\right)$$
$$+ D_x^2(W)\left(\frac{\partial}{\partial u_{xxx}} - D_x \frac{\partial}{\partial u_{xxxx}} + D_x^2 \frac{\partial}{\partial u_{xxxxx}}\right)$$
$$+ D_x^3(W)\left(\frac{\partial}{\partial u_{xxxx}} - D_x \frac{\partial}{\partial u_{xxxxx}}\right)$$
$$+ D_x^4(W_i)\frac{\partial}{\partial u_{xxxxx}} \qquad (40)$$

The invariance condition for any given generator X of Eq. (1) and its solutions reads

$$(\bar{X}L + D_t(\xi^2)L + D_x(\xi^1)L)|_{Eq.(1)} = 0, \qquad (41)$$

and consequently the Cls of Eq. (1) can be written as

$$D_t(N^t L) + D_x(N^x L) = 0. \qquad (42)$$

Now, we present the Cls for Eq. (1) using the basic definitions presented above. We consider two cases corresponding to the order of α.

Case 1. When $\alpha \in (0, 1)$, with the help of Eq. (38) and Eq. (39), the components of the conserved vectors are

$$C_i^t = \xi^2 L + (-1)^0 oD_t^{\alpha-1}(W_i)D_t^0 \frac{\partial L}{\partial(oD_t^\alpha u)} - (-1)^1$$
$$\times J\left(W_i, D_t^1 \frac{\partial L}{\partial(oD_t^\alpha u)}\right)$$
$$= v_o D_t^{\alpha-1}(W_i) + J(W_i, v_t),$$

$$C_i^x = \xi^1 l + W_i\left(\frac{\partial}{\partial u_x} - D_x \frac{\partial}{\partial u_{xx}} + D_x^2 \frac{\partial}{\partial u_{xxx}} - D_x^3 \frac{\partial}{\partial u_{xxxx}}\right.$$
$$\left. + D_x^4 \frac{\partial}{\partial u_{xxxxx}}\right) + D_x(W_i)\left(\frac{\partial}{\partial u_{xx}} - D_x \frac{\partial}{\partial u_{xxx}}\right.$$

$$+ D_x^2 \frac{\partial}{\partial u_{xxxx}} - D_x^3 \frac{\partial}{\partial u_{xxxxx}}\right) + D_x^2(W_i)\left(\frac{\partial}{\partial u_{xxx}}\right.$$
$$\left. - D_x \frac{\partial}{\partial u_{xxxx}} + D_x^2 \frac{\partial}{\partial u_{xxxxx}}\right) + D_x^3(W_i)\left(\frac{\partial}{\partial u_{xxxx}}\right.$$
$$\left. - D_x \frac{\partial}{\partial u_{xxxxx}}\right) + D_x^4(W_i)\frac{\partial}{\partial u_{xxxxx}}$$

$$= W_i\left(\beta u^2 v + D_x^4 \gamma v\right) + D_x(W_i)\left(-D_x^3 \gamma v\right)$$
$$+ D_x^2(W_i)\left(D_x^2 \gamma v\right) - \gamma v_x D_x^3(W_i) + \gamma v D_x^4(W_i),$$

where $i = 1, 2$ and the functions W_i are given by

$$W_1 = -u_x, \quad W_2 = -2u\alpha - 5tu_t - \alpha x u_x.$$

Case 2. When $\alpha \in (1, 2)$, with the help of Eq. (38) and Eq. (39), the components of the conserved vectors are

$$C_i^t = \xi^2 L + (-1)^0 oD_t^{\alpha-1}(W_i)D_t^0 \frac{\partial L}{\partial(oD_t^\alpha u)} - (-1)^1$$
$$J\left(W_i, D_t^1 \frac{\partial L}{\partial(oD_t^\alpha u)}\right) + (-1)^1 oD_t^{\alpha-2}(W_i)D_t^1$$
$$\frac{\partial L}{\partial(oD_t^\alpha u)} - (-1)^1 J\left(W_i, D_t^1 \frac{\partial L}{\partial(oD_t^\alpha u)}\right)$$
$$= v_o D_t^{\alpha-1}(W_i) + J(W_i, v_t) - v_t oD_t^{\alpha-2}(W_i) - J(W_i, v_{tt})$$

$$C_i^x = \xi^1 l + W_i\left(\frac{\partial}{\partial u_x} - D_x \frac{\partial}{\partial u_{xx}} + D_x^2 \frac{\partial}{\partial u_{xxx}} - D_x^3 \frac{\partial}{\partial u_{xxxx}}\right.$$
$$\left. + D_x^4 \frac{\partial}{\partial u_{xxxxx}}\right) + D_x(W_i)\left(\frac{\partial}{\partial u_{xx}} - D_x \frac{\partial}{\partial u_{xxx}}\right.$$
$$+ D_x^2 \frac{\partial}{\partial u_{xxxx}} - D_x^3 \frac{\partial}{\partial u_{xxxxx}}\right) + D_x^2(W_i)\left(\frac{\partial}{\partial u_{xxx}}\right.$$
$$\left. - D_x \frac{\partial}{\partial u_{xxxx}} + D_x^2 \frac{\partial}{\partial u_{xxxxx}}\right) + D_x^3(W_i)\left(\frac{\partial}{\partial u_{xxxx}}\right.$$
$$\left. - D_x \frac{\partial}{\partial u_{xxxxx}}\right) + D_x^4(W_i)\frac{\partial}{\partial u_{xxxxx}}$$

$$= W_i\left(\beta u^2 v + D_x^4 \gamma v\right) + D_x(W_i)\left(-D_x^3 \gamma v\right)$$
$$+ D_x^2(W_i)\left(D_x^2 \gamma v\right) - \gamma v_x D_x^3(W_i) + \gamma v D_x^4(W_i),$$

where $i = 1, 2$ and the functions W_i are given by

$$W_1 = -u_x, \quad W_2 = -2u\alpha - 5tu_t - \alpha x u_x.$$

5 Explicit power series solutions

Here, we investigate the exact analytic solutions via power series method [76] and symbolic computations [77] for Eq. (28). Set

$$f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \tag{43}$$

from Eq. (43), we can have

$$f' = \sum_{n=0}^{\infty} n a_n \xi^{n-1}, \tag{44}$$

$$f^{(5)} = \sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)(n-4) a_n \xi^{n-5}.$$

Substituting Eqs. (44) into Eq. (28), we obtain

$$\sum_{n=0}^{\infty} \frac{\Gamma(2 - \frac{12\alpha}{5} + \frac{n\alpha}{5})}{\Gamma(2 - \frac{7\alpha}{5} + \frac{n\alpha}{5})} a_n \xi^n + \beta \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} a_n \xi^n$$

$$\times \sum_{n=0}^{\infty} (n+1) a_{n+1} \xi^n + \gamma \sum_{n=0}^{\infty} (n+5)(n+4)(n+3)$$

$$\times (n+2)(n+1) a_{n+5} \xi^n = 0. \tag{45}$$

Comparing coefficients in Eq. (45) when $n = 0$, we obtain

$$a_5 = \frac{1}{120\gamma} \left(\frac{\Gamma(2 - \frac{12\alpha}{5})}{\Gamma(2 - \frac{7\alpha}{5})} a_0 + \beta a_0^2 a_1 \right), \tag{46}$$

when $n \geq 1$, we have

$$a_{n+5} = \frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)\gamma}$$

$$\times \left\{ \frac{\Gamma(2 - \frac{12\alpha}{5} + \frac{n\alpha}{5})}{\Gamma(2 - \frac{7\alpha}{5} + \frac{n\alpha}{5})} a_n \right.$$

$$+ \beta \sum_{k=0}^{n} \sum_{j=0}^{k} a_j (n - k + 1) a_{n-k+1} \Bigg\}.$$

Thus, the power series solution for Eq. (28) can be represented in the form:

$$f(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + a_4 \xi^4 + a_5 \xi^5$$

$$+ \sum_{n=1}^{\infty} a_{n+5} \xi^{n+5}$$

$$= a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + a_4 \xi^4 + \frac{1}{120\gamma} \left(\frac{\Gamma(2 - \frac{12\alpha}{5})}{\Gamma(2 - \frac{7\alpha}{5})} a_0 \right.$$

$$+ \beta a_0^2 a_1 \Bigg) \xi^5 + \sum_{n=1}^{\infty} \frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)\gamma}$$

$$\times \left\{ \frac{\Gamma(2 - \frac{12\alpha}{5} + \frac{n\alpha}{5})}{\Gamma(2 - \frac{7\alpha}{5} + \frac{n\alpha}{5})} a_n + \beta \sum_{k=0}^{n} \sum_{j=0}^{k} a_j \right.$$

$$\times (n - k + 1) a_{n-k+1} \Bigg\} \xi^{n+5}$$

Consequently, we acquire the exact power series solution for Eq. (28) as

$$u(x, t) = a_0 t^{-\frac{2\alpha}{5}} + a_1 x t^{-\frac{3\alpha}{5}} + a_2 x^2 t^{-\frac{4\alpha}{5}} + a_3 x^3 t^{-\alpha}$$

$$+ a_4 x^4 t^{-\frac{6\alpha}{5}} + \frac{1}{120\gamma} \left(\frac{\Gamma(2 - \frac{12\alpha}{5})}{\Gamma(2 - \frac{7\alpha}{5})} a_0 + \beta a_0^2 a_1 \right) x^5 t^{-\frac{7\alpha}{5}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(n+5)(n+4)(n+3)(n+2)(n+1)\gamma}$$

$$\times \left\{ \frac{\Gamma(2 - \frac{12\alpha}{5} + \frac{n\alpha}{5})}{\Gamma(2 - \frac{7\alpha}{5} + \frac{n\alpha}{5})} a_n + \beta \sum_{k=0}^{n} \sum_{j=0}^{k} a_j \right. \tag{47}$$

$$\times (n - k + 1) a_{n-k+1} \Bigg\} x^{n+5} t^{-\frac{(n+7\alpha)}{5}}$$

6 Physical interpretation of the power series solution for Eqs. (47)

In order to have clear and proper understanding of the physical properties of the power series solution, the 3-D, 2-D and contour plots for the solution Eqs. (47), are plotted in Figures 1-4 by using suitable parameter values.
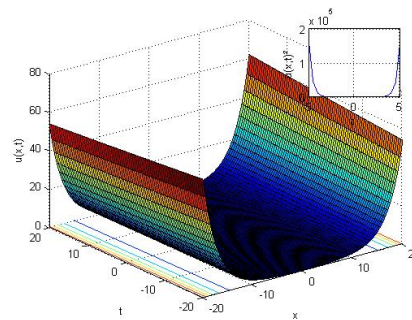

Figure 1: 3D plot of (47) $a_0 = a_1 = a_2 = 1, a_3 = 0.5, a_4 = 1.7, \beta = 2, \gamma = 1, \alpha = 0.5, \Gamma = 0.85$
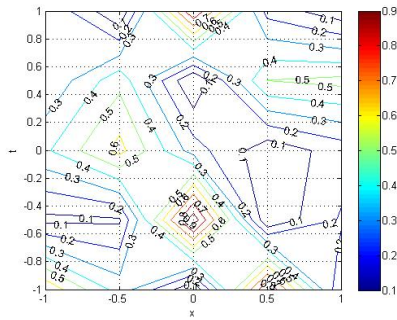
Figure 2: Contour plot of (47) $a_0 = a_1 = a_2 = 1$, $a_3 = 0.5$, $a_4 = 1.7$, $\beta = 2$, $\gamma = 1$, $\alpha = 0.5$, $\Gamma = 0.85$
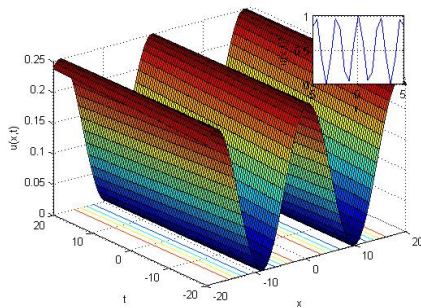

Figure 3: 3D plot of (47) $a_0 = a_1 = a_2 = 0.8$, $a_3 = a_4 = 1$, $\beta = 1.2$, $\gamma = 3$, $\alpha = 0.9$, $\Gamma = 0.1$
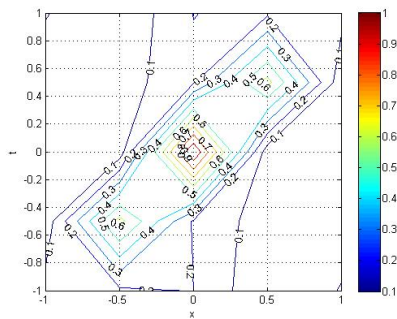

Figure 4: Contour plot of (47) $a_0 = a_1 = a_2 = 0.8$, $a_3 = a_4 = 1$, $\beta = 1.2$, $\gamma = 3$, $\alpha = 0.9$, $\Gamma = 0.1$

7 Concluding remarks

In this research, we analyzed time fractional SMK by means of Lie symmetry analysis using the RL derivative. We reduced the governing equation to a nonlinear ODE of fractional order. The obtained fractional ODE was solved using a power series technique. Ibragimov's nonlocal conservation theorem was applied to establish Cls for the governing equation. Some 3-D, 2-D, and contour plots were also presented.

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