MODIFIED VARIATIONAL ITERATION METHOD FOR STRAIGHT FINS WITH TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

by

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Original scientific paper
https://doi.org/10.2298/TSCI171017290I

The modified variational iteration method (MVIM) has been used to calculate the efficiency of straight fins with temperature dependent thermal conductivity. The obtained results are compared with homotopy analysis method (HAM), homotopy perturbation method (HPM), and Adomian decomposition method (ADM). It is used $w \neq 0$ auxiliary parameter to keep under control convergence region of solution series in MVIM. As a result, although MVIM and HAM give results close to each other; HPM and ADM give divergent results from analytical solution.

Key words: *modified variational iteration method, thermal conductivity, series solution*

Introduction

The mathematical modeling of events in nature is explained by using differential equations. The solutions of non-linear equations have a very important place in applied mathematics and physics, because, solutions of non-linear partial differential equations (NPDE) provide a very significant contribution to science about the character of physical phenomenon. The exact and numerical solutions, in particular, traveling wave solutions holds significant place in soliton theory. It is both difficult and takes a lot of time to find solutions of NPDE. Therefore, we do not always obtain analytical solutions of NPDE. In this case, we use some semi analytical methods which give series solution. In all of these methods, the solutions are written in the form of series. It has recently become more attractive to find numerical solutions of NPDE by using symbolical computer programs such as MAPLE, MATLAB, and MATHEMATICA.

There are many studies in literature for thermal parameters of the fins. Yu and Chen [1] solved non-linear fin problem by using differential transformation method. The thermal analysis of convective straight fins with variable heat transfer parameter was investigated by

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authors [2-5]. Arslanturk [3] generated the correlation equations by using ADM for temperature distribution in these types of fins. Moreover, Arslanturk [4] designed circular fins with variable heat transfer parameter and expressed their results with the correlation equations. Coskun and Atay [5] investigated efficiency of straight fins by using variational iteration method (VIM) and finite elements method. The HAM was used to obtain efficiencies of convective straight fins by scientists [2, 6]. Joneidi *et.al.* [7] solved the fin problem by using differential transformation method. The 2-D thermal analyses of the fins which loose heat by both convection and radiation was investigated by using quadrature method [8]. Yang *et al.* solved fin problem with hyperbolic profile by using decomposition method and showed efficiency of fin with graph [9].

Variational iteration method

The VIM was first presented by He [9]. This method is used to obtain approximate solutions which converge to analytical solution in linear and non-linear equations. The simplicity of this method is easily performed. The method was applied to homogeneous, non-homogeneous, linear and non-linear differential equations [10-13] by many scientists. The solutions of ODE and PDE are found with the help of integral multiplier which is called Lagrange multiplier. It is obtained a closer solution to analytical solution with appropriate section of integral multiplier. In recent years, this method was applied to find approximate solutions of non-linear and integral equations by scientist [14]. When this method is compared with other methods, there are two advantages of this method:

– it provides a physical perception for status of the mathematical model, and
– the found solutions are the best trial functions.

Analysis of VIM

To clarify the VIM, let as consider the general non-linear differential equation:

$$L\left[u(t)\right]+N\left[u(t)\right]=g(t) \tag{1}$$

where L and N are linear and non-linear operators, respectively, and $g(t)$ is a permanent function. The main feature of this method is to constitute a verification functional for eq. (1). The approximate solution is written by this method:

$$u_{n+1}\left(t\right)=u_n\left(t\right)+\int_0^t\lambda\left\{Lu_n\left(s\right)+N\widetilde{u_n}\left(s\right)-g\left(s\right)\right\}\mathrm{d}s,\quad n\geq 0 \tag{2}$$

where λ is a general Lagrange multiplier which can be determined optimally via the variational theory and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n=0$. If Lagrange multiplier is exactly defined, the analytical solutions can be found with an iteration step for linear problems. The δ to be a variational operator, if δ operator is applied to the eq. (2), we have:

$$\delta u_{n+1}\left(t\right)=\delta u_n\left(t\right)+\delta\int_0^t\lambda\left\{Lu_n\left(s\right)+N\widetilde{u_n}\left(s\right)-g\left(s\right)\right\}\mathrm{d}s\,,\quad n\geq 0 \tag{3}$$

Consequently, the approximation solution is given:

$$u(t)=\lim_{n\to\infty}u_n(t) \tag{4}$$

Some authors showed the convergence analysis of method by using the fixed point theorem [15].

Analysis of MVIM

The MVIM are obtained by varying of correction functional of VIM. In this method, we define an auxiliary parameter ($w \neq 0$) to control convergence of the solution series. This parameter firstly was used by Liao [16] for a general analytical method which is called the HAM [17, 18] by using homotopy which is one of the basic concepts of topology. The method is used to unravel non-linear differential equations, integral and integro-differential equations. We firstly use auxiliary parameter ($w \neq 0$) for VIM. Substituting w into eq. (2), we have:

$$u_{n+1}(t) = u_n(t) + w\int_0^t \lambda\left[Lu_n(s) + N\widetilde{u_n}(s) - g(s)\right]\mathrm{d}s\,, \quad n \geq 0 \tag{5}$$

Problem description

We consider a straight fin with a temperature-dependent thermal conductance, incidental constant cross-sectional area, A_c, parameter, P, and length, b, see fig. 1. The fin is appended to a base surface of temperature, T_b, extends into a fluid of temperature, T_a, and its tip is isolated. Thus, we have:

$$A_c \frac{\mathrm{d}}{\mathrm{d}x}\left[k(T)\frac{\mathrm{d}T}{\mathrm{d}x}\right] - Ph(T_b - T_a) = 0 \tag{6}$$

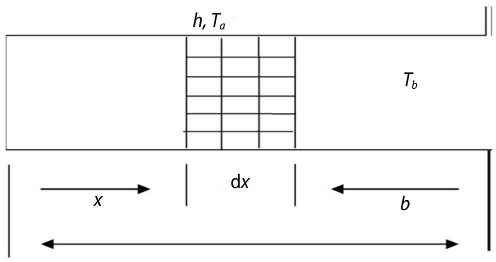

Figure 1. Figure of a straight fin

The thermal conductance of the fin material is supposed to be a linear function of temperature according to:

$$k(T) = k_a\left[1 + \lambda(T - T_a)\right] \tag{7}$$

where k_a is the thermal conductance at the ambient fluid temperature of the fin and λ is the parameter describing thermal conductivity variation.

We use dimensionless variables:

$$u = \frac{T - T_a}{T_b - T_a}, \quad \tau = \frac{x}{b}, \quad \varepsilon = \lambda(T_b - T_a), \quad N = \sqrt{\frac{hPb^2}{k_a A_c}} \tag{8}$$

thus, we acquire:

$$(1 + \varepsilon u)\frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + \varepsilon\left(\frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2}\right)^2 - N^2 u = 0, \quad u(1) = 1, \quad u'(0) = 0 \tag{9}$$

An application of MVIM

Let as consider correction functional for eq. (9):

$$u_{n+1}(\tau) = u_n(\tau) + \int_0^\tau \lambda(s)\left[(1 + \varepsilon u_n)(u_n)_{ss} + \varepsilon (u_n)_{ss}{}^2 - N^2(u_n)\right]\mathrm{d}s \tag{10}$$

if δ operator is applied to the eq. (10):

$$\delta u_{n+1}(\tau)=\delta u_n(\tau)+\int_0^\tau \lambda(s)\delta(u_n)_{ss}\,\mathrm{d}s+\epsilon\int_0^\tau \lambda(s)\delta(\widetilde{u_n})(\widetilde{u_n})_{ss}\,\mathrm{d}s+$$
$$+\varepsilon\int_0^\tau\int_0^\tau \lambda(s)\delta(\widetilde{u_n})_{ss}^{\,2}\,\mathrm{d}s-N^2\int_0^\tau \lambda(s)\delta u_n \mathrm{d}s \tag{11}$$

If necessary arrangements are made:

$$\delta u_{n+1}(\tau)=\delta u_n(\tau)+\lambda(s)\delta u_n^{'}(s)\big|_{\tau=s}-\lambda'(s)\delta u_n(s)\big|_{\tau=s}+$$
$$+\int_0^\tau \lambda''(s)\delta u_n(s)\mathrm{d}s-N^2\int_0^\tau \lambda(s)\delta u_n(s)\mathrm{d}s=0$$
$$\delta u_{n+1}(\tau)=\left[1-\lambda'(s)\right]\delta u_n(\tau)+\lambda(s)\,\delta u_n'(s)\big|_{\tau=s}+$$
$$+\int_0^\tau\left[\lambda''-N^2\lambda(s)\delta u_n(s)\right]\mathrm{d}s=0 \tag{12}$$

Thus, it is obtained ODE system from eq. (12):

$$\begin{cases}1-\lambda'(\tau)=0\\ \lambda(\tau)=0\\ \lambda''(s)\big|_{\tau=s}-N^2\lambda(s)\big|_{\tau=s}=0\end{cases} \tag{13}$$

by using eq. (13), we have a Lagrange multiplier:

$$\lambda(s)=\frac{1}{2N}\left[\mathrm{e}^{N(s-\tau)}+\mathrm{e}^{N(\tau-s)}\right] \tag{14}$$

Substituting eq. (14) into eq. (10) yields the following equal:

$$u_{n+1}=u_n+\frac{w}{2N}\int_0^\tau\left[e^{N(s-\tau)}+e^{N(\tau-s)}\right]\left[(1+\varepsilon u_n)(u_n)_{ss}+\varepsilon(u_n)_{ss}^{\,2}-N^2(u_n)\right]\mathrm{d}s \tag{15}$$

given by Inc [2]:

$$u_0(\tau)=C+(1-C)\tau^2 \tag{16}$$

where C is an integration constant. Substituting eq. (16) into eq. (15) yields:

$$u_1(\tau)=C+(1-C)\tau^2+w\left[\frac{1}{3}\tau^2(-1+\tau)(6\varepsilon)(-1+C)+N^2\right]\cdot$$
$$\cdot\frac{\left\{2-2C-\tau CN^2\left[2\varepsilon(-1+C)+N^2\right]+2(-1+C)\cosh(\tau N)\right\}}{N^2}$$
$$u_2(\tau)=C+(1-C)\tau^2+w\left[\frac{1}{3}\tau^2(-1+\tau)(6\varepsilon)(-1+C)+N^2\right]+$$
$$+\frac{\left\{2-2C-\tau CN^2\left[2\varepsilon(-1+C)+N^2\right]+2(-1+C)\cosh(\tau N)\right\}}{N^2}+$$

$$+\cosh(\tau N)-\left[12\varepsilon(-1+C)+(2+\tau)N^2\right]w\sinh(\tau N)-$$

$$-5N^6\left(4\tau^2+12\tau C-4\tau^3C+w\left\{\tau^4(-1+C)\left[6\varepsilon(-1+C)+N^2\right]\right\}\right)+$$

$$+\frac{\left[6\left(-\tau N\left\{-4+C\left[4+2\varepsilon\tau(-1+C)N^2+\tau N^4\right]\right\}+4(-1+C)\sinh(\tau N)\right)\right]}{N^3}$$

It is chosen $w=0.01$ for appropriate value in figs. 2 and 3. We obtain the following tables and figures for values [2] which obtain by using ADM, HPM, HAM, and MVIM. Tables 1 and 2 depict the comparison of HAM, ADM, HPM, and MVIM for ε = –1, C = 0.9, N = 5 and ε = –2, C = 0.9, N = 2.5, respectively. Also, figs. 4 and 5 show comparison of HAM, ADM, HPM, and MVIM for ε = –1, C = 0.9, N = 5 and ε = –2, C = 0.9, N = 2, respectively.

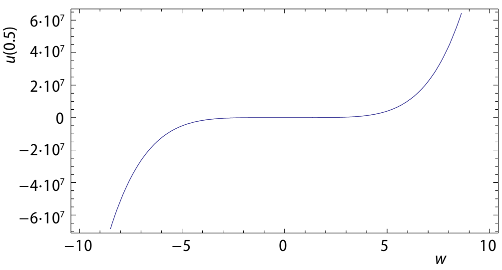

Figure 2. Graphic of $u(\tau)$ in $\tau = 0.5$ for $\varepsilon = 2$, $C = 0.9$, $N = 3$.

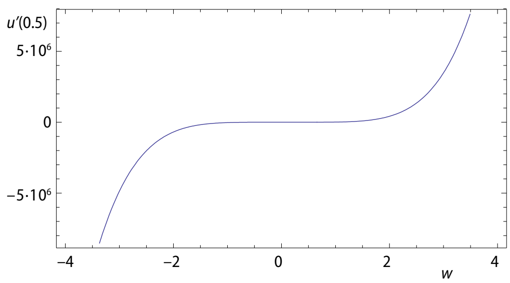

Figure 3. Graphic of $u'(\tau)$ in $\tau = 0.5$ for $\varepsilon = 2$, $C = 0.9$, $N = 3$

Table 1. Comparison of HAM, ADM, HPM, and MVIM for $\varepsilon = -1, C = 0.9, N = 5$

τ	HAM (h = 0.03)	ADM	HPM	MVIM (w = 0.01)
0.1	0.884668	1.43634	4.01327	0.855846
0.2	0.838876	4.06611	16.6673	0.81413
0.3	0.763232	12.7267	50.1048	0.774738
0.4	0.658736	37.0315	127.522	0.737539
0.5	0.526797	97.3252	291.169	0.702384
0.6	0.369198	232.718	613.184	0.669086
0.7	0.188102	513.057	1210.92	0.637403
0.8	−0.013967	1056.09	2268.02	0.606999
0.9	−0.234141	2051.34	2062.43	0.577379
1.0	−0.469221	3792.32	7003.12	0.547793

Figure 4. Comparison of HAM, ADM, HPM, and MVIM for $\varepsilon = -1$, $C = 0.9$, $N = 5$

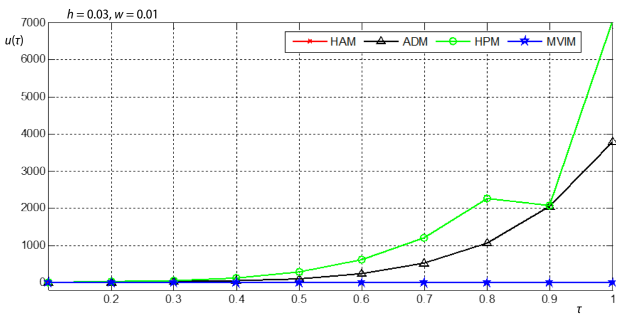

Table 2. Comparison of HAM, ADM, HPM, and MVIM for $\varepsilon = -2$, $C = 0.9$, $N = 2.5$

τ	HAM ($h = 0.05$)	ADM	HPM	MVIM ($w = 0.05$)
0.1	0.895067	1.55523	1.46417	0.840809
0.2	0.880286	3.84041	10.5889	0.784446
0.3	0.855714	8.76227	24.7295	0.730755
0.4	0.821442	34.9561	48.4612	0.679568
0.5	0.777599	34.9561	86.0491	0.630709
0.6	0.724353	63.5013	143.457	0.583991
0.7	0.661905	110.049	228.712	0.539208
0.8	0.590496	183.342	352.365	0.496141
0.9	0.510402	295.334	528.049	0.454541
1.0	0.421933	462.068	773.137	0.414413

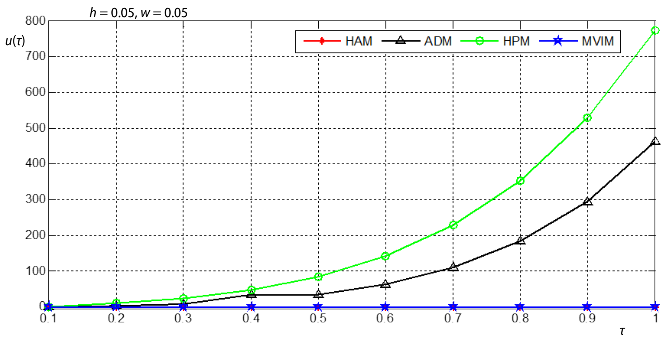

Figure 5. Comparison of HAM, ADM, HPM, and MVIM for $\varepsilon = -2$, $C = 0.9$, $N = 2$

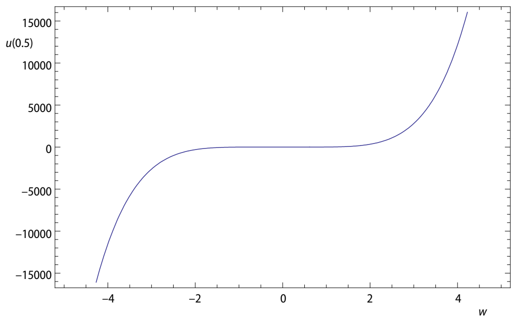

Figure 6. Graphic of $u(\tau)$ in $\tau = 0.5$ for $\varepsilon = 1$, $C = 0.5$, $N = 1$

It is investigated convergence interval of $u_3(\tau)$ solution series corresponding to the value of w which is a convergence control parameter in fig. 6. In both cases, the convergence interval of solution series does not change. It is reached closer results by HAM. This case shows that the method is reliable.

Fin efficiency

The heat dissipation of a fin can be obtained by integrating the convection heat loss from the fin surface [19]:

$$Q = \int_0^b P(T - T_a)\mathrm{d}x = b(T_b - T_a)\int_0^1 Pu(\tau)\mathrm{d}\tau \tag{17}$$

The efficiency of the fins is defined as the ratio of the actual heat transfer rate to the heat transfer rate of the entire surface that depends on the temperature at the base of the fin T_b . So,

$$\eta = \frac{Q}{Q_{\text{ideal}}} = \frac{\int_0^b P(T - T_a)\mathrm{d}x}{Pb(T_b - T_a)} = \int_0^1 u(\tau)\mathrm{d}\tau \tag{18}$$

Thus the efficiency of straight fins is obtained as an analytical expression:

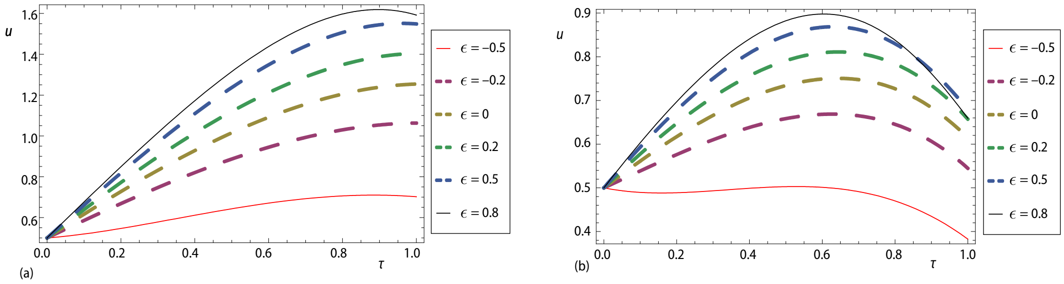

Figure 7. Temperature distribution in convective fins with variable thermal conductivity and fixed values (a) N = 0.5, C = 0.5, w = 0.5 for the 3rd-order of approximation and (b) N = 0.7, C = 0.5, w = 0.5 for the 3rd-order of approximation.

$$\eta=\frac{1}{3}+\frac{2C}{3}+\frac{2}{N^2}-\frac{2C}{N^2}-\frac{20N^6}{3}+5CN^6+N^2w+N^8w-CN^8w+$$
$$+\frac{w\varepsilon}{6}-\frac{Cw\varepsilon}{6}-6N^6w\varepsilon+12CN^6w\varepsilon-6C^2N^6w\varepsilon-60N^6\tau C+\frac{2C}{N^4\varepsilon\tau}-$$
$$-\frac{2}{N^4\varepsilon\tau}+\frac{24}{N^4\tau}-\frac{24C}{N^4\tau}+\frac{12C}{N^6\varepsilon\tau^2}-\frac{12C^2}{N^6\varepsilon\tau^2}-\frac{6C}{N^8\tau}+\cosh[\tau N]-\frac{2}{N^2}\cosh[\tau N]+$$
$$+\frac{2C}{N^2}\cosh[\tau N]-\frac{24}{N^3}\sinh[\tau N]+\frac{24C}{N^3}\sinh[\tau N]-\frac{5}{2}N^2w\sinh[\tau N]+$$
$$+12\,w\varepsilon\sinh[\tau N]-12C\,w\varepsilon\sinh[\tau N]+\ldots$$

Conclusion

In this study, we apply the basic idea of MVIM and then, we use to define heat distribution in surfaces which are independent from thermal conductivity. This problem is unraveled by using the MVIM and the problem is classed with numerical solutions which obtain by using HAM, ADM, and HPM by Inc [2], Arslanturk [3], Rajabi [20], respectively. The obtained solutions are given in figs. 5 and 6 and tabs. 1 and 2. It is observed that results of MVIM are closer to results of HAM for different cases of ε, C, N, and w values. As seen from the tabs. 1 and 2 and figs. 4-6, ADM and HPM give divergent results from analytical solution. Fin efficiency is shown in fig. 7.

Acknowledgment

This work was supported by the China Government Young Talent Program.

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Paper submitted: October 17, 2017
Paper revised: November 26, 2017
Paper accepted: December 7, 2017