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A fixed point theorem on multiplicative metric space with integral-type inequality

Aziz Khan[a], Hasib Khan[b,c,*], Dumitru Baleanu[b,d,e,*], Hossein Jafari[f,g], Tahir Saeed Khan[a], Maysaa Alqurashi[h]

[a]*Department of Mathematics, University of Peshawar, P. O. Box 25000, Khybar Pakhtunkhwa, Pakistan.*

[b]*College of Engineering, Mechanics and Materials, Hohai University, 211100, Nanjing, P. R. China.*

[c]*Department of Mathematics, Shaheed Benazir Bhutto University, Sheringal, Dir Upper, 18000, Khybar Pakhtunkhwa, Pakistan.*

[d]*Department of Mathematics, Cankaya University, 06530 Ankara, Turkey.*

[e]*Institute of Space Sciences, P. O. BOX, MG-23, 76900 Magrrele-Bucharest, Romania.*

[f]*Department of Mathematics, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran.*

[g]*Department of Mathematical Sciences, University of South Africa, P. O. Box 392, UNISA 0003, South Africa.*

[h]*College of Science, Department of Mathematics, King Saud University, Riyad, Saudi Arabia.*

Abstract

In this paper, we prove fixed point theorems (FPTs) on multiplicative metric space (MMS) $(\mathcal{X}, \blacktriangle)$ by the help of integral-type contractions of self-quadruple mappings (SQMs), i.e., for $\wp_1, \wp_2, \wp_3, \wp_4 : \mathcal{X} \to \mathbb{R}$. For this, we assume that the SQMs are weakly compatible mappings and the pairs $(\wp_1, \wp_3)$ and $(\wp_2, \wp_4)$ satisfy the property $(\mathrm{CLR}_{\wp_3\wp_4})$. Further, two corollaries are produced from our main theorem as special cases. The novelty of these results is that for the unique common fixed point (CFP) of the SQMs $\wp_1, \wp_2, \wp_3, \wp_4$, we do not need to the assumption of completeness of the MMS $(\mathcal{X}, \blacktriangle)$. These results generalize the work of Abdou, [A. A. N. Abdou, J. Nonlinear Sci. Appl., **9** (2016), 2244–2257], and many others in the available literature.

Keywords: Multiplicative metric space, fractional integral inequalities, fixed point theorems.

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1. Introduction and preliminaries

By the virtue of different mathematical techniques and approaches, we observe a large number of FPTs in different types of spaces with applications to integral, differential equations. In these applications, the EPS of FODEs are very much popular in among the scientists, see [7, 8, 17]. In fixed point theory, Banach's FPT is one of the basic concepts for the solution of the equation $fx = x$.

*Corresponding author
Email addresses: azizkhan927@yahoo.com (Aziz Khan), hasibkhan13@yahoo.com (Hasib Khan), dumitru@cankaya.edu.tr (Dumitru Baleanu), jafarh@unisa.ac.za (Hossein Jafari), tsk@uop.edu.pk (Tahir Saeed Khan), maysaa@ksu.edu.sa (Maysaa Alqurashi)

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Theorem 1.1 ([9]). *If $(\mathcal{X}, \blacktriangle)$ is a complete metric space and $f : \mathcal{X} \to \mathcal{X}$ satisfies that $\blacktriangle(f(\mu^*), fx) \leqslant \nu\blacktriangle(\mu^*, x)$, for all $\mu^*, x \in \mathcal{X}$ and $\nu \in (0, 1)$, then f has a fixed point in $\mathcal{X}$.*

The Banach's FPT has gained much popularity due to valuable application in various scientific problems and research areas like image processing, the study of existence and uniqueness of the integral and differential equations and many more, equilibrium problems, selection and matching problems. Banach's FPT has generalized in further more new FPT by different approaches which have a lot of applications for example [2–4, 6, 14, 15, 19, 20, 22–25]. These generalizations can be made through by contractions or spaces. In the area of FPT Bhaskar and Lakshmikantham [15] have some contributions like coupled FPT which was further generalized to triple as well to quadruple FPT. In partially ordered metric spaces Berinde and Borcut [10], worked for triple FPT. Aydi et al. [6] worked on quadruple FPTs and also given applications in POMS. Liu [24] has made contribution in the field mixed g-monotone by quadruple FPTs and applications in POMS. Bota et al. [11], worked on coupled system of integral equations through by finite interval $[0, b]$. Nadler [26] studied multivalued contraction mappings for FPT. Stojakovic et al. [30] further generalized the work due to Nadler [26] and made some contribution in integral type contractions. Sarwar et al. [28] provided an FPT by integral type contractions and dynamic programming also as applications. Mustafa et al. [25] have given the concept of (ϕ, ψ)-contractions and applications in POMS. Samet and Jleli [18] have given generalization in metric spaces and also worked on Banach's FPT. Shatanawi et al. [29] worked coupled system by the use of two altering distance functions in POMS. Branciari [12] provided some special type integral contraction. Ali et al. [5] proved FPTs of generalized contractive multivalued mappings of integral type by using the admissible mappings. Chauhan et al. [13] proved FPTs by the help of integral-type contractions and provided the applications of their results. Recently, Khan et al. [21] introduced a generalized class of common limit range property for $2n$ self-mappings and proved fixed point theorems by means of some integral inequalities.

In this paper, we prove FPTs on MMS $(\mathcal{X}, \blacktriangle)$ by the help of integral-type contractions of SQMs $\wp_1, \wp_2, \wp_3, \wp_4 : \mathcal{X} \to \mathbb{R}$. For this work, we assume that the SQMs are weakly compatible mappings and the pairs $(\wp_1, \wp_3)$ and $(\wp_2, \wp_4)$ satisfy the property $(CLR_{\wp_3\wp_4})$. Further, two corollaries are produced from our main theorem as special cases. The novelty of these results is that for the unique fixed point of the SQMs $\wp_1, \wp_2, \wp_3, \wp_4$, we do not need to the assumption of completeness of the MMS $(\mathcal{X}, \blacktriangle)$. These results generalize the work of [1], and many others in the available literature. In this portion of the paper, we present related and necessary work from the available literature. The second section contains the main theorem of the paper as a new FPT by the help of integral type contraction on MMS for SQMs. Section 3 of the paper presents an application and the paper is summarized in Section 4.

Definition 1.2. Let $\mathcal{X}$ be a nonempty set. A multiplicative metric is mapping $\blacktriangle : \mathcal{X} \times \mathcal{X} \to R$ satisfying the following conditions:

(1) $\blacktriangle(z, y) \geqslant 1$ for all $z, y \in \mathcal{X}$ and $\blacktriangle(z, y) = 1$ if and only if $z = y$;

(2) $\blacktriangle(z, y) = \blacktriangle(y, z)$ for all $x, y \in \mathcal{X}$;

(3) $\blacktriangle(z, y) \leqslant \blacktriangle(z, t).\blacktriangle(t, y)$ for all $x, y, t \in \mathcal{X}$ (multiplicative triangle inequality).

Example 1.3. Let $\mathcal{X}$ be the collection of all n-tuples of positive real numbers. Let $\blacktriangle : \mathcal{R}^n_+ \times \mathcal{R}^n_+ \to \mathcal{R}$ be defined as follows:

$$d(z, y) = |\frac{z_1}{y_1}|.|\frac{z_2}{y_2}| \dots |\frac{z_n}{y_n}|,$$

where $z = (z_1, \dots, z_n)$, $y = (y_1, \cdots, y_n) \in \mathcal{R}^n$ and $|.| : \mathcal{R} \to \mathcal{R}$ is defined as follows:

$$|k| = \begin{cases} k, & \text{if } k \geqslant 1, \\ \frac{1}{k}, & \text{if } k < 1, \end{cases}$$

then, $(\mathcal{X}, \blacktriangle)$ is an MMS.

Definition 1.4. Let $(\mathcal{X}, \blacktriangle)$ be MMS. A mapping $\wp : \mathcal{X} \to \mathcal{X}$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $\blacktriangle(\wp z_1, \wp z_2) \leqslant \blacktriangle(z_1, z_2)^{\lambda}$, for all $z, y \in \mathcal{X}$.

Theorem 1.5 ([1])**.** *Let $\wp_1$, $\wp_2$, $\wp_3$, and $\wp_4$ be four self-mappings of a complete MMS $\mathcal{X}$ satisfying the following conditions:*

(1) $\wp_1(\mathcal{X}) \subset \wp_4(\mathcal{X})$ *and* $\wp_2(\mathcal{X}) \subset \wp_3(\mathcal{X})$*;*

(2) $(\wp_3, \wp_1)$ *and* $(\wp_4, \wp_2)$ *are compatible;*

(3) *one of* $\wp_1$, $\wp_2$, $\wp_3$ *and* $\wp_4$ *is continuous;*

(4) *for all* $z, y \in \mathcal{X}$

$$\blacktriangle(\wp_1 z, \wp_2 y) \leqslant \left[\varphi\Big(\max\Big\{\blacktriangle(\wp_3 z, \wp_4 y), \frac{\blacktriangle(\wp_3 z, \wp_1 x) d(\wp_4 y, \wp_2 y)}{1 + \blacktriangle(\wp_3 z, \wp_4 x)}, \frac{\blacktriangle(\wp_3 z, \wp_2 y)\blacktriangle(\wp_4 y, \wp_3 z)}{1 + \blacktriangle(\wp_3 z, \wp_4 y)}\Big\}\Big)\right]^{\lambda},$$

where $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous and monotone increasing function such that* $\varphi(0) < t$ *for all* $t > 0$.

Then $\wp_1$, $\wp_2$, $\wp_3$, and $\wp_4$ have a unique common fixed point in $\mathcal{X}$.

From [16, 27, 28] and the references therein, we take the following definitions.

Definition 1.6 ([28])**.** Let $(\mathcal{X}, \blacktriangle)$ be a metric space and $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4 : \mathcal{X} \to \mathcal{X}$ be self-quadruple mappings. The pairs $(\wp_1, \wp_2)$ and $(\wp_3, \wp_4)$ satisfy the CLR property with respect to mappings $\wp_3$ and $\wp_4$, denoted by $(CLR_{\wp_3\wp_4})$ if there exist two sequences $\hbar_n$ and μ_n in $\mathcal{X}$ such that

$$\lim_{n\to\infty} \wp_1(\hbar_n) = \lim_{n\to\infty} \wp_3(\hbar_n) = \lim_{n\to\infty} \wp_2(\mu_n) = \lim_{n\to\infty} \wp_4(\mu_n) = \nu \in \wp_3(\mathcal{X}) \cap \wp_4(\mathcal{X})$$

for $\hbar_n$, $\mu_n \in \mathcal{X}$ for all $n \in \mathbb{N} \cup \{0\}$.

Definition 1.7 ([28])**.** A coincidence point of a pair of self-mappings $\wp_1$, $\wp_2$,: $\mathcal{X} \to \mathcal{X}$ is a point $z \in \mathcal{X}$ for some $\wp_1 z = \wp_2 z$.

A common fixed point of pair of self-mappings $\wp_1$, $\wp_2$,: $\mathcal{X} \to \mathcal{X}$ is a point $\wp_1$, $\wp_2$,: $\mathcal{X} \to \mathcal{X}$ for which $\wp_1 z = \wp_2 z = z$.

Definition 1.8 ([28])**.** A pair of self-mappings $\wp_1$, $\wp_2$,: $\mathcal{X} \to \mathcal{X}$ is weakly compatible if they commute, at their coincidence point that is if there exists a point $z \in \mathcal{X}$ such that $\wp_1\wp_2 z = \wp_2\wp_1 z$.

Definition 1.9 ([16])**.** Generalized altering distance function is a mapping $\tau : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying that

(i) τ is non-decreasing;

(ii) $\tau(t) = 0$ if and only if $t = 0$.

$$F = \{\tau : \mathbb{R}^+ \to \mathbb{R}^+ : \tau \text{ satisfies (i) and (ii)}\},$$

$$\begin{aligned}\Phi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+ : \ & \phi \text{ is right upper semi-continuous, nondecreasing,} \\ & \text{and for all } x > 0, \text{ we have } \tau x > \phi_1 x \text{ for } \phi_1 \in \Phi, \\ & \text{and } \tau x \text{ satisfies (i), (ii),} \\ & \text{and for any } \phi_2 \in \Phi, \text{ there exists an } \mathcal{L} > 0 \text{ such that } \mathcal{L}\phi_2\Big(\int_0^1 \Gamma(t)dt\Big) \leqslant 1\},\end{aligned}$$

$$\Psi_1 = \{\psi_1 : \mathbb{R}^+ \to \mathbb{R}^+ : \ \psi_1 \text{ satisfies } (A_1)\text{–}(A_3)\},$$

where

(A_1) ψ_1 is continuous and nondecreasing in each coordinate;

(A_2) $\psi_1(z,z,z,z,z,z) \leqslant z$ for all $z \geqslant 0$;

(A_3) $\psi_1 = 1$ if and only if all the components of ψ_1 are one.

$$\Psi_2 = \Big\{ \psi_2 : \mathbb{R}^{+^4} \to \mathbb{R}^+ : \psi_2 \text{ is continuous and } \Psi_2(t_1,t_2,t_3,t_4) = \begin{cases} 1, & \text{if any of the } t_i\text{'s is one, for } i = 1,2,3,4, \\ \max\{t_i\}, & \text{otherwise,} \end{cases} \Big\}.$$

2. Main results

In this section, we prove our main results, i.e., FPTs on MMS $(\mathcal{X}, \blacktriangle)$ by the help of integral-type contractions of SQMs $\wp_1, \wp_2, \wp_3, \wp_4 : \mathcal{X} \to \mathbb{R}$. For this purpose, we assume that the SQMs are weakly compatible mappings and the pairs $(\wp_1, \wp_3)$ and $(\wp_2, \wp_4)$ satisfy the property $(CLR_{\wp_3\wp_4})$. Further, two corollaries are drawn from our main theorem as special cases. These results generalize the work presented in [1], and many others in the available literature.

Theorem 2.1. *Let $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$ be SQMs of an MMS $(\mathcal{X}, \blacktriangle)$, satisfying the following conditions:*

(a) *$(\wp_1, \wp_3)$ and $(\wp_2, \wp_4)$ satisfy $(CLR_{\wp_3\wp_4})$ property.*

(b) *$(\wp_1, \wp_3)$ and $(\wp_2, \wp_4)$ are weakly compatible.*

(c) *$\wp_1$, $\wp_2$, $\wp_3$, and $\wp_4$ satisfy the inequality*

$$\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^\lambda(\wp_1\mu^*, \wp_2 x)} \Gamma(t)dt\Big) \leqslant \mathcal{L}\phi_1\Big(\int_0^{\psi_1(M(\mu^*,x))} \Gamma(t)dt\Big)\phi_2\Big(\int_0^{\psi_2(N(\mu^*,x))} \Gamma(t)dt\Big) \tag{2.1}$$

for μ^*, $x \in \mathcal{X}$, $\lambda \in (0,1)$, where

$$\psi_1(M(\mu^*,x)) = \frac{1}{\lambda}\max\Big\{\blacktriangle(\wp_1\mu^*, \wp_4 x), \blacktriangle(\wp_2 x, \wp_4 x), \blacktriangle(\wp_4 x, \wp_3\mu^*), \frac{\blacktriangle(\wp_1\mu^*, \wp_4 x)\blacktriangle(\wp_1\mu^*, \wp_3\mu^*)}{2}, \frac{\blacktriangle(\wp_2 x, \wp_3\mu^*)\blacktriangle(\wp_1\mu^*, \wp_2 x)}{1+\blacktriangle(\wp_2 x, \wp_1\mu^*)}, \frac{\blacktriangle(\wp_1\mu^*, \wp_4 x)\blacktriangle(\wp_3\mu^*, \wp_4 x)}{1+\blacktriangle(\wp_2\mu^*, \wp_3\mu^*)}\Big\}^\lambda,$$

$$\psi_2(N(\mu^*,x)) = \psi_2\Big(\blacktriangle(\wp_1\mu^*, \wp_4 x), \blacktriangle(\wp_2 x, \wp_4 x), \blacktriangle(\wp_4 x, \wp_3\mu^*), \frac{\blacktriangle(\wp_1\mu^*, \wp_4 x)\blacktriangle(\wp_3\mu^*, \wp_4 x)}{1+\blacktriangle(\wp_2 x, \wp_3\mu^*)}\Big).$$

Then, the SQMs $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$ have a unique CFP $x \in \mathcal{X}$.

Proof. By the help of our supposition of the $(CLR_{\wp_3\wp_4})$ property of the pairs $(\wp_1, \wp_2)$, and $(\wp_3, \wp_4)$, we assume two sequences $\{\mu_n^*\}$ and $\{x_n\}$ in the metric space $(\mathcal{X}, \blacktriangle)$ such that

$$\lim_{n\to\infty} \wp_1(\mu_n^*) = \lim_{n\to\infty} \wp_3(\mu_n^*) = \lim_{n\to\infty} \wp_2 x_n = \lim_{n\to\infty} \wp_4 x_n = z, \quad \text{for } z \in \wp_3(\mathcal{X}) \cap \wp_4(\mathcal{X}). \tag{2.2}$$

For $z \in \wp_3(\mathcal{X}) \cap \wp_4(\mathcal{X})$, we may have $z = \wp_3 u$ for some $u \in \mathcal{X}$. Therefore,

$$\lim_{n\to\infty} \wp_1(\mu_n^*) = \lim_{n\to\infty} \wp_3(\mu_n^*) = \lim_{n\to\infty} \wp_2 x_n = \lim_{n\to\infty} \wp_4 x_n = z = \wp_3 u. \tag{2.3}$$

We show that $\wp_1 u = \wp_3 u$. For this, we follow by the contradiction, that is, $\wp_1 u \neq \wp_3 u$ and define the following sequences:

$$x_{2n} = \wp_1(\mu^*_{2n}) = \wp_4\mu^*_{2n+1} \quad \text{and} \quad x_{2n+1} = \wp_2(\mu^*_{2n+1}) = \wp_3(\mu^*_{2n+2}).$$

By the help of (2.1), we have

$$\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^\lambda(\wp_1 u, \wp_2 x_n)} \Gamma(t)dt\Big) \leqslant \mathcal{L}\phi_1\Big(\int_0^{\psi_1(M(u,x_n))} \Gamma(t)dt\Big)\phi_2\Big(\int_0^{\psi_2(N(u,x_n))} \Gamma(t)dt\Big) \tag{2.4}$$

for $\mu^* = u$ and $x = x_n$ in the inequality (2.4), where

$$\begin{aligned}\psi_1(M(u,x_n)) = \frac{1}{\lambda}\max\Big\{&\blacktriangle(\wp_1 u, \wp_4 x_n),\ \blacktriangle(\wp_2 x_n, \wp_4 x_n),\ \blacktriangle(\wp_4 x_n, \wp_3 u),\\ &\frac{\blacktriangle(\wp_1 u, \wp_4 x_n)\blacktriangle(\wp_1 u, \wp_3 u)}{2},\ \frac{\blacktriangle(\wp_2 x_n, \wp_3 u)\blacktriangle(\wp_1 u, \wp_2 x_n)}{1+\blacktriangle(\wp_2 x_n, \wp_1 u)},\\ &\frac{\blacktriangle(\wp_1 u, \wp_4 x_n)\blacktriangle(\wp_3 u, \wp_4 x_n)}{1+\blacktriangle(\wp_2 u, \wp_3 u)}\Big\}^\lambda,\end{aligned} \tag{2.5}$$

$$\psi_2(N(u,x_n)) = \psi_2\Big(\blacktriangle(\wp_1 u, \wp_4 x_n),\ \blacktriangle(\wp_2 x_n, \wp_4 x_n),\ \blacktriangle(\wp_4 x_n, \wp_3 u),\ \frac{\blacktriangle(\wp_1 u, \wp_4 x_n)\blacktriangle(\wp_3 u, \wp_4 x_n)}{1+\blacktriangle(\wp_2 x_n, \wp_3 u)}\Big). \tag{2.6}$$

Taking the $\lim_{n\to\infty}$ in (2.5), (2.6) and (2.4), respectively, we get

$$\begin{aligned}\lim_{n\to\infty}\psi_1(M(u,x_n)) &= \frac{1}{\lambda}\lim_{n\to\infty}\max\Big\{\blacktriangle(\wp_1 u, \wp_4 x_n),\ \blacktriangle(\wp_2 x_n, \wp_4 x_n),\ \blacktriangle(\wp_4 x_n, \wp_3 u),\\ &\qquad \frac{\blacktriangle(\wp_1 u, \wp_3 u)\blacktriangle(\wp_1 u, \wp_4 x_n)}{2},\ \frac{\blacktriangle(\wp_2 x_n, \wp_3 u)\blacktriangle(\wp_1 u, \wp_2 x_n)}{1+\blacktriangle(\wp_2 x_n, \wp_1 u)},\\ &\qquad \frac{\blacktriangle(\wp_1 u, \wp_4 x_n)\blacktriangle(\wp_3 u, \wp_4 x_n)}{1+\blacktriangle(\wp_2 x_n, \wp_3 u)}\Big\}^\lambda\\ &= \frac{1}{\lambda}\max\Big\{\blacktriangle(\wp_1 u, z),\ \blacktriangle(z,z),\ \blacktriangle(z,z),\ \frac{\blacktriangle(\wp_1 u, z)\blacktriangle(\wp_1 u, x)}{2},\\ &\qquad \frac{\blacktriangle(z,z)\blacktriangle(\wp_1 u, z)}{1+\blacktriangle(z,\wp_1 u)},\ \frac{\blacktriangle(\wp_1 u, z)\blacktriangle(z,z)}{1+\blacktriangle(z,\wp_1 u)}\Big\}^\lambda\\ &= \frac{1}{\lambda}\max\{\blacktriangle(\wp_1 u, z), 1, 1, \blacktriangle(\wp_1 u, z), 1, 1\}^\lambda = \frac{1}{\lambda}\blacktriangle^\lambda(\wp_1 u, z),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\lim_{n\to\infty}\psi_2(N(u,x_n)) &= \lim_{n\to\infty}\psi_2\Big(\blacktriangle(\wp_1 u, \wp_4 x_n),\ \blacktriangle(\wp_2 x_n, \wp_4 x_n),\ \blacktriangle(\wp_4 x_n, \wp_3 u),\\ &\qquad \frac{\blacktriangle(\wp_1 u, \wp_4 x_n)\blacktriangle(\wp_3 x_n, \wp_4 x_n)}{1+\blacktriangle(\wp_2 u, \wp_3 x_n)}\Big)\\ &= \psi_2\Big(\blacktriangle(\wp_1 u, z),\ \blacktriangle(z,z),\ \blacktriangle(z,\ z),\ \frac{\blacktriangle(\wp_1 u,\ z)\blacktriangle(z,\ z)}{1+\blacktriangle(\wp_2 u,\ z)}\Big)\\ &= \psi_2\Big(\blacktriangle(\wp_2 u, z),\ 1,\ 1, 1\Big) = 1,\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\lim_{n\to\infty}\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^\lambda(\wp_1 u, \wp_2 x_n)} \Gamma(t)dt\Big) &\leqslant \mathcal{L}\lim_{n\to\infty}\phi_1\Big(\int_0^{\psi_1(M(u,x_n))} \Gamma(t)dt\Big)\lim_{n\to\infty}\phi_2\Big(\int_0^{\psi_2(N(u,x_n))} \Gamma(t)dt\Big)\\ &= \mathcal{L}\lim_{n\to\infty}\phi_1\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^\lambda(\wp_1 u, z)} \Gamma(t)dt\Big)\lim_{n\to\infty}\phi_2\Big(\int_0^{1} \Gamma(t)dt\Big).\end{aligned} \tag{2.9}$$

By the use of (2.3), (2.7) and (2.8) in (2.9), we have

$$\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^{\lambda}(\wp_1 u, z)}\Gamma(t)dt\Big) \leqslant \varphi_1\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^{\lambda}(\wp_1 u, z)}\Gamma(t)dt\Big),$$

which is a contradiction of $\tau(t) > \varphi_1(t)$. This contradiction is due to our supposition $\wp_1 u \neq \wp_3 u$, and hence $\wp_1 u = \wp_3 u$. Also from (2.2), we have $z \in \wp_4(\mathcal{X})$. This implies $z = \wp_4(\nu)$, for some $\nu \in X$. Now, we show that $\wp_4(\nu) = \wp_2(\nu)$, for this we assume the contrary path, that is, $\wp_4(\nu) \neq \wp_2(\nu)$. By putting $\mu^* = \mu_n^*$ and $z = \nu$ in (2.4), and following the same lines as above for the proof of $\wp_3 u = \wp_1 u = z$, we can get $\wp_4(\nu) = \wp_2(\nu) = x$. Consequently, we have

$$\wp_4(\nu) = \wp_2(\nu) = \wp_3 u = \wp_1 u = z.$$

Since, $(\wp_1, \wp_2)$ and $(\wp_3, \wp_4)$ are weakly compatible. Therefore, $\wp_3 u = \wp_1 u$ implies $\wp_1\wp_3 u = \wp_3\wp_1 u$ which implies $\wp_1 z = \wp_3 z$. Similarly, we have $\wp_4 z = \wp_2 z$. Next, we show that z is a CFP of $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$. For this, let assume that $\wp_1 z \neq z$, and putting $\mu^* = z$ and $x = \nu$ in (2.4), we have

$$\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^{\lambda}(\wp_1 z, \wp_2(\nu))}\Gamma(t)dt\Big) \leqslant \mathcal{L}\varphi_1\Big(\int_0^{\psi_1(M(z,\nu))}\Gamma(t)dt\Big)\varphi_2\Big(\int_0^{\psi_2(N(z,\nu))}\Gamma(t)dt\Big),$$

where,

$$\begin{aligned}
\psi_1(M(z,\nu)) &= \frac{1}{\lambda}\max\Big\{\blacktriangle(\wp_1 z, \wp_4(\nu)),\ \blacktriangle(\wp_2(\nu), \wp_4(\nu)),\ \blacktriangle(\wp_4(\nu), \wp_3 z),\ \frac{\blacktriangle(\wp_1 z, \wp_4(\nu))\blacktriangle(\wp_1 z, \wp_3 z)}{2},\\
&\qquad \frac{\blacktriangle(\wp_2(\nu), \wp_3 z)\blacktriangle(\wp_1 z, \wp_2(\nu))}{1+\blacktriangle(\wp_2(\nu), \wp_1 z)}, \frac{\blacktriangle(\wp_1 z, \wp_4(\nu))\blacktriangle(\wp_3 z, \wp_4(\nu))}{1+\blacktriangle(\wp_2(\nu), \wp_3 z)}\Big\}^{\lambda}\\
&= \frac{1}{\lambda}\max\Big\{\blacktriangle(\wp_1 z, z),\ \blacktriangle(z,z),\ \blacktriangle(z, \wp_1 z), \frac{\blacktriangle(\wp_1 z, z)\blacktriangle(\wp_1 z, \wp_1 z)}{2}, \frac{\blacktriangle(z, \wp_1 z)\blacktriangle(\wp_1 z, z)}{1+\blacktriangle(z, \wp_1 z)},\\
&\qquad \frac{\blacktriangle(\wp_1 z, z)\blacktriangle(\wp_1 z, \wp_4(\nu))}{1+\blacktriangle(z, \wp_3 z)}\Big\}^{\lambda} = \frac{1}{\lambda}\blacktriangle^{\lambda}(\wp_1 z, z),
\end{aligned}$$

and

$$\begin{aligned}
\psi_2(N(z,\nu)) &= \psi_2\Big(\blacktriangle(\wp_1 z, \wp_4(\nu)),\ \blacktriangle(\wp_2(\nu), \wp_4(\nu)),\ \blacktriangle(\wp_4(\nu), \wp_3 z),\ \frac{\blacktriangle(\wp_1 z, \wp_4(\nu))\blacktriangle(\wp_3 z, \wp_4(\nu))}{1+\blacktriangle(\wp_2(\nu), \wp_3 z)}\Big)\\
&= \psi_2\Big(\blacktriangle(\wp_1 z, z),\ \blacktriangle(z,z),\ \blacktriangle(z, \wp_1 z), \frac{\blacktriangle(\wp_1 z, z)\blacktriangle(\wp_1 z, z)}{1+\blacktriangle(z, \wp_1 z)}\Big)\\
&= \psi_2\Big(\blacktriangle(\wp_1 z, z),\ 1,\ \blacktriangle(z, \wp_1 z), \frac{\blacktriangle(\wp_1 z, z)\blacktriangle(\wp_1 z, z)}{1+\blacktriangle(z, \wp_1 z)}\Big) = 1.
\end{aligned}$$

By the use of (2.3), (2.7) and (2.8) in (2.9), we have

$$\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^{\lambda}(\wp_1 z, z)}\Gamma(t)dt\Big) \leqslant \varphi_1\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^{\lambda}(\wp_1 z, z)}\Gamma(t)dt\Big)$$

for $\lambda \in (0,1)$ which is a contradiction of $\tau(t) > \varphi_1(t)$. This contradiction is due to our supposition that $\wp_1 z \neq z$, and hence $\wp_1 z = \wp_3 z = z$. Similarly, we can show $\wp_2 z = \wp_4 z = z$. Ultimately, we have $\wp_1 z = \wp_3 z = \wp_2 z = \wp_4 z = z$.

Finally, we prove that the common fixed point of the SQMs $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$ is unique. For this, we again presume a contrary path, that is, let there exist two different fixed points such that

$$\wp_1 z_1 = \wp_3 z_1 = z_1, \quad \wp_2 z_2 = \wp_4 z_2 = z_2$$

for some $z_1, z_2 \in \mathcal{X}$ such that $z_1 \neq z_2$. By putting $\mu^* = z_1$ and $x = z_2$ in (2.4), we have

$$\begin{aligned}\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^\lambda(\wp_1 z_1, \wp_2 z_2)} \Gamma(t)dt\Big) &= \tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle(z_1,z_2)} \Gamma(t)dt\Big) \\ &\leqslant \mathcal{L}\phi_1\Big(\int_0^{\psi_1(M(z_1,z_2))} \Gamma(t)dt\Big)\phi_2\Big(\int_0^{\psi_2(N(z_1,z_2))} \Gamma(t)dt\Big),\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}\psi_1(M(z_1,z_2)) &= \frac{1}{\lambda}\max\Big\{\blacktriangle(\wp_1 z_1, \wp_3 z_1),\ \blacktriangle(\wp_2 z_2, \wp_4 z_2),\ \blacktriangle(\wp_4 z_2, \wp_3 z_1), \\ &\qquad \frac{\blacktriangle(\wp_1 z_1, \wp_4 z_2)\blacktriangle(\wp_1 z_1, \wp_3 z_1)}{2}, \frac{\blacktriangle(\wp_2 z_2, \wp_3 z_1)\blacktriangle(\wp_1 z_1, \wp_2 z_2)}{1+\blacktriangle(\wp_2 z_2, \wp_1 z_1)}, \\ &\qquad \frac{\blacktriangle(\wp_1 z_1, \wp_4 z_2)\blacktriangle(\wp_3 z_1, \wp_4 z_2)}{1+\blacktriangle(\wp_2 z_1, \wp_3 z_1)}\Big\}^\lambda \\ &= \frac{1}{\lambda}\max\Big\{\blacktriangle(z_1,z_1),\ \blacktriangle(z_2,z_2),\ \blacktriangle(z_2,z_1), \frac{\blacktriangle(z_1,z_2)\blacktriangle(z_1,z_1)}{2}, \frac{\blacktriangle(z_2,z_1)\blacktriangle(z_1,z_2)}{1+\blacktriangle(z_2,z_1)}, \\ &\qquad \frac{\blacktriangle(z_1,z_2)\blacktriangle(z_1,z_2)}{1+\blacktriangle(z_2,z_1)}\Big\}^\lambda = \frac{1}{\lambda}\blacktriangle^\lambda(z_1,z_2),\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\psi_2(N(z_1,z_2)) &= \psi_2\Big(\blacktriangle(\wp_1 z_1, \wp_4 z_2),\ \blacktriangle(\wp_2 z_2, \wp_4 z_2),\ \blacktriangle(\wp_4 z_2, \wp_3 z_1), \frac{\blacktriangle(\wp_1 z_1, \wp_4 z_2)\blacktriangle(\wp_3 z_1, \wp_4 z_2)}{1+\blacktriangle(\wp_2 z_2, \wp_3 z_1)}\Big) \\ &= \psi_2\Big(\blacktriangle(z_1,z_2),\ \blacktriangle(z_2,z_2),\ \blacktriangle(z_2,z_1), \frac{\blacktriangle(z_1,z_2)\blacktriangle(z_1,z_2)}{1+\blacktriangle(z_2,z_1)}\Big) \\ &= \psi_2\Big(\blacktriangle(z_1,z_2),\ 1,\ \blacktriangle(z_2,z_1), \frac{\blacktriangle(z_1,z_2)\blacktriangle(z_1,z_2)}{1+\blacktriangle(z_2,z_1)}\Big) = 1.\end{aligned} \tag{2.12}$$

By the use of (2.11), (2.12) in (2.10), we have

$$\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^\lambda(z_1,z_2)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^\lambda(z_1,z_2)} \Gamma(t)dt\Big),$$

which is a contradiction of the fact $\tau(t) > \phi_1(t)$. Thus, $z_1 = z_2$ and therefore, the CFP of the SQMs $\wp_1, \wp_2, \wp_3, \wp_4$ is unique. □

If we assume $\phi_2(t) = \frac{t}{\mathcal{L}}$, then we have the following corollary.

Corollary 2.2. *Let $\wp_1, \wp_2, \wp_3, \wp_4$ be SQMs of an MMS $(\mathcal{X}, \blacktriangle)$, satisfying the following conditions:*

(a) *$\{\wp_1, \wp_3\}$ and $\{\wp_2, \wp_4\}$ satisfy $(\mathrm{CLR}_{\wp_3\wp_4})$ property;*

(b) *$\{\wp_1, \wp_3\}$ and $\{\wp_2, \wp_4\}$ are weakly compatible;*

(c) *$\wp_1, \wp_2, \wp_3$, and $\wp_4$ satisfy the inequality*

$$\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^\lambda(\wp_1\mu^*, \wp_2 x)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\psi_1(M(\mu^*,x))} \Gamma(t)dt\Big)\int_0^{\psi_2(N(\mu^*,x))} \Gamma(t)dt$$

for $\mu^*, x \in \mathcal{X}$, $\lambda \in (0,1)$, where

$$\begin{aligned}\psi_1(M(\mu^*,x)) = \frac{1}{\lambda}\max\Big\{&\blacktriangle(\wp_1\mu^*, \wp_4 x),\ \blacktriangle(\wp_2 x, \wp_4 x),\ \blacktriangle(\wp_4 x, \wp_3\mu^*), \\ &\frac{\blacktriangle(\wp_1\mu^*, \wp_4 x)\blacktriangle(\wp_1\mu^*, \wp_3\mu^*)}{2}, \frac{\blacktriangle(\wp_2 x, \wp_3\mu^*)\blacktriangle(\wp_1\mu^*, \wp_2 x)}{1+\blacktriangle(\wp_2 x, \wp_1\mu^*)}, \\ &\frac{\blacktriangle(\wp_1\mu^*, \wp_4 x)\blacktriangle(\wp_3\mu^*, \wp_4 x)}{1+\blacktriangle(\wp_2\mu^*, \wp_3\mu^*)}\Big\}^\lambda,\end{aligned}$$

$$\psi_2(N(\mu^*,x)) = \psi_2\Big(\blacktriangle(\wp_1\mu^*,\wp_4 x), \blacktriangle(\wp_2 x,\wp_4 x), \blacktriangle(\wp_4 x,\wp_3\mu^*), \frac{\blacktriangle(\wp_1\mu^*,\wp_4 x)\blacktriangle(\wp_3\mu^*,\wp_4 x)}{1+\blacktriangle(\wp_2 x,\wp_3\mu^*)}\Big).$$

Then, the SQMs $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$ have a unique CFP $x \in \mathcal{X}$.

By considering $\wp_1 = \wp_3$ and $\wp_2(t) = \wp_4(t) = t$ in Theorem 2.1, we have the following result.

Corollary 2.3. *Let $\wp_1$ be a self-mapping of an MMS $(\mathcal{X}, \blacktriangle)$, satisfying the following integral condition*

$$\tau\Big(\int_0^{\frac{1}{\lambda}\blacktriangle^{\lambda}(\wp_1\mu^*,\wp_1 x)}\Gamma(t)dt\Big) \leqslant \mathcal{L}\phi_1\Big(\int_0^{\psi_1(M(\mu^*,x))}\Gamma(t)dt\Big)\phi_2\Big(\int_0^{\psi_2(N(\mu^*,x))}\Gamma(t)dt\Big)$$

for μ^*, $x \in \mathcal{X}$, $\lambda \in (0,1)$, where

$$\psi_1(M(\mu^*,x)) = \frac{1}{\lambda}\max\Big\{\blacktriangle(\wp_1\mu^*,x), \blacktriangle(\wp_1 x,x), \blacktriangle(x,\mu^*), \frac{\blacktriangle(\wp_1\mu^*,x)\blacktriangle(\wp_1\mu^*,\mu^*)}{2}, \frac{\blacktriangle(\wp_1 x,\mu^*)\blacktriangle(\wp_1\mu^*,\wp_1 x)}{1+\blacktriangle(\wp_1 x,\wp_1\mu^*)}, \frac{\blacktriangle(\wp_1\mu^*,x)\blacktriangle(\mu^*,x)}{1+\blacktriangle(\wp_1\mu^*,\mu^*)}\Big\}^{\lambda},$$

$$\psi_2(N(\mu^*,x)) = \psi_2\Big(\blacktriangle(\wp_1\mu^*,x), \blacktriangle(\wp_1 x,x), \blacktriangle(x,\mu^*), \frac{\blacktriangle(\wp_1\mu^*,x)\blacktriangle(\mu^*,x)}{1+\blacktriangle(\wp_1 x,\mu^*)}\Big).$$

Then, $\wp_1$ has a unique CFP $x \in \mathcal{X}$.

3. Applications

In this section, we give an application of our new FPT 2.1 on MMS $(\mathcal{X}, \blacktriangle)$ by the help of integral-type contractions of the SQMs $\wp_1, \wp_2, \wp_3, \wp_4 : \mathcal{X} \to \mathbb{R}$. For this, we define SQMs $\wp_1, \wp_2, \wp_3, \wp_4 : \mathcal{X} \to \mathbb{R}$ and show that the SQMs are weakly compatible mappings and the pairs $(\wp_1, \wp_3)$ and $(\wp_2, \wp_4)$ satisfy the property $(CLR_{\wp_3\wp_4})$. Finally, we check the integral inequality (2.1), as follows:

Example 3.1. Let $(\mathcal{X} = [1,\infty), \blacktriangle)$ be a metric space with $\blacktriangle(x,y) = |\frac{z}{y}|$, for z, $y \in \mathcal{X}$. Define $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$ as under:

$$\wp_1 z = \begin{cases} 2z, & \text{if } z \in [1,2), \\ 2.5, & \text{if } z \in (2,\infty), \end{cases} \qquad \wp_3 z = \begin{cases} 2z, & \text{if } z \in [1,2), \\ 2.7, & \text{if } z \in (2,\infty), \end{cases} \tag{3.1}$$

$$\wp_2 z = \begin{cases} 2z, & \text{if } z \in [1,2), \\ 2.5, & \text{if } z \in (2,\infty), \end{cases} \qquad \wp_4 z = \begin{cases} 2z, & \text{if } z \in [1,2), \\ 2.5, & \text{if } z \in (2,\infty). \end{cases} \tag{3.2}$$

Let us consider the sequence

$$\{x_n\} = \{1 + \frac{0.9}{3n^2+1.5}\}. \tag{3.3}$$

By the help of (3.1), (3.2) and (3.3), we have:

$$\begin{aligned} \lim_{n\to\infty}\wp_1(x_n) &= \lim_{n\to\infty}\wp_1(1+\frac{0.9}{3n^2+1.5}) = 2\lim_{n\to\infty}(1+\frac{0.9}{3n^2+1.5}) = 2, \\ \lim_{n\to\infty}\wp_3(x_n) &= \lim_{n\to\infty}\wp_3(1+\frac{0.9}{3n^2+1.5}) = 2\lim_{n\to\infty}(1+\frac{0.9}{3n^2+1.5}) = 2, \\ \lim_{n\to\infty}\wp_2(x_n) &= \lim_{n\to\infty}\wp_2(1+\frac{0.9}{3n^2+1.5}) = 2\lim_{n\to\infty}(1+\frac{0.9}{3n^2+1.5}) = 2, \\ \lim_{n\to\infty}\wp_4(x_n) &= \lim_{n\to\infty}\wp_4(1+\frac{0.9}{3n^2+1.5}) = 2\lim_{n\to\infty}(1+\frac{0.9}{3n^2+1.5}) = 2. \end{aligned} \tag{3.4}$$

From (3.4), it is proved that the SQMs $\wp_1$, $\wp_3$, $\wp_2$, $\wp_4$ share property $(CLR_{\wp_3\wp_4})$ and are weakly compatible. Further, we need to show that $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$ satisfy the inequality (2.4). For this, we study two cases, i.e., when $x \in [0, 2)$, and $x \in (2, \infty)$.

We consider $\tau(t) = 0.9t$, $\lambda = 0.5$, $\phi_1(t) = 0.85t$, $\phi_2(t) = 0.8t$, $\Gamma(t) = 2t$.

Case I. For $z \in [1, 2)$, we have $\wp_1 z = \wp_2 z = \wp_3 z = \wp_4 z = 2z$, which implies $\blacktriangle(\wp_1, \wp_2) = 1$, $\psi_1(M) = 1$ and $\psi_2(N) = 1$. Therefore, the inequality is satisfied.

Case II. For $z = \mu^* = x \in [2, \infty)$, we have $\wp_1 z = 2.5$, $\wp_3 z = 2.7$, $\wp_2 z = 2.5$, $\wp_4 z = 2.5$, and

$$\begin{aligned}\psi_1(M(\mu^*, x)) &= \frac{1}{\lambda}\max\Big\{\blacktriangle(\wp_2 z, \wp_4 z),\ \blacktriangle(\wp_1(z), \wp_4 x),\ \blacktriangle(\wp_4 z, \wp_3(z)),\\ &\qquad \frac{\blacktriangle(\wp_1(z), \wp_4 z)\blacktriangle(\wp_1(z), \wp_3(z))}{2},\ \frac{\blacktriangle(\wp_2 z, \wp_3(z))\blacktriangle(\wp_1(z), \wp_2 z)}{1+\blacktriangle(\wp_2 z, \wp_1 z)},\\ &\qquad \frac{\blacktriangle(\wp_1 z, \wp_4 z)\blacktriangle(\wp_3 z, \wp_4 z)}{1+\blacktriangle(\wp_3 z, \wp_4 z)}\Big\}^{\lambda}\\ &= \frac{1}{0.5}\max\Big\{\blacktriangle(2.5, 2.5),\ \blacktriangle(2.5, 2.5),\ \blacktriangle(2.5, 2.7),\ \frac{\blacktriangle(2.5, 2.5)\blacktriangle(2.5, 2.7)}{2},\\ &\qquad \frac{\blacktriangle(2.5, 2.7)\blacktriangle(2.5, 2.7)}{1+\blacktriangle(2.5, 2.5)},\ \frac{\blacktriangle(2.5, 2.5)\blacktriangle(2.7, 2.5)}{1+\blacktriangle(2.7, 2.5)}\Big\}^{0.5} = 2.07,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\psi_2(N(\mu^*, x)) &= \psi_2\Big\{\blacktriangle(\wp_1 z, \wp_4 z),\ \blacktriangle(\wp_2 z, \wp_4 z),\ \blacktriangle(\wp_4 z, \wp_3 z),\\ &\qquad \frac{\blacktriangle(\wp_1 z, \wp_4 z)\blacktriangle(\wp_3 z, \wp_4 z)}{1+\blacktriangle(\wp_2 z, \wp_3 z)}\Big\}\\ &= \psi_2\Big\{\blacktriangle(2.5, 2.5),\ \blacktriangle(2.5, 2.5),\ \blacktriangle(2.5, 2.7),\ \frac{\blacktriangle(2.5, 2.5)\blacktriangle(2.7, 2.5)}{1+\blacktriangle(2.5, 2.7)}\Big\} = 1.\end{aligned} \tag{3.6}$$

By the help of (2.4), (3.5) and (3.6), $\tau(t) = 0.9t$, $\phi_1(t) = 0.85t$, $\phi_2(t) = 0.8t$, $\Gamma(t) = 2t$, we have

$$\begin{aligned}3.60 = \tau\Big(\int_0^{\frac{1}{0.5}\blacktriangle^{\lambda=0.5}(2.5,2.5)} 2t\,dt\Big) &\leqslant \mathcal{L}\phi_1\Big(\int_0^{\psi_1(M(\mu^*,x))}\Gamma(t)\,dt\Big)\phi_2\Big(\int_0^{\psi_2(N(\mu^*,x))}\Gamma(t)\,dt\Big)\\ &= \mathcal{L}\phi_1\Big(\int_0^{2.07} 2t\,dt\Big)\phi_2\Big(\int_0^{1} 2t\,dt\Big) \leqslant 3.642.\end{aligned}$$

Thus, the inequality (2.1) is also satisfied. Thus, all the conditions of Theorem 2.1 are fulfilled and therefore, the SQMs $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$ have a unique CFP in the MMS $(\mathcal{X}, \blacktriangle)$.

4. Conclusion

New results of FPTs with applications are widely observed in literature. One of the applications of FPTs is the EUS of FODEs, see [7, 8, 17], this area is now a hot research oriented area. In the EPS of FODEs, we see a good role of BFPT. This theorem has been generalized by a large number of scientists by the help of a large number of contractions or by the help of generalizing the spaces and very much useful and interesting new FPTs were obtained. For instance, Jleli and Samet [18] generalized BFPT considering generalized metric space. Mustafa et al. [25] proved FPTs for the generalized (ϕ, ψ)-contraction in POMs and provided applications of their results. Inspired from these FPTs, in this paper, we proved FPTs MMS $(\mathcal{X}, \blacktriangle)$ by the help of integral-type contractions of SQMs $\wp_1, \wp_2, \wp_3, \wp_4 : \mathcal{X} \to \mathbb{R}$. For this, we assumed that the SQMs are weakly compatible mappings and the pairs $(\wp_1, \wp_3)$ and $(\wp_2, \wp_4)$ satisfy the property $(CLR_{\wp_3\wp_4})$. Further, two corollaries were produced from our main theorem as special cases. The novelty of these results is that for the unique common fixed point of the SQMs $\wp_1, \wp_2, \wp_3, \wp_4$, we do not need to the assumption of completeness of the MMS $(\mathcal{X}, \blacktriangle)$. These results generalize the work presented in [1], and many others in the available literature.

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