fractal and fractional

MDPI

Article

Nonlocal Cauchy Problem via a Fractional Operator Involving Power Kernel in Banach Spaces

Ayşegül Keten [1], Mehmet Yavuz [1,*] and Dumitru Baleanu [2,3]

[1] Department of Mathematics-Computer, Faculty of Science, Necmettin Erbakan University, Konya 42090, Turkey; aketen@erbakan.edu.tr
[2] Department of Mathematics, Faculty of Arts and Sciences, Çankaya University, Ankara 06530, Turkey; dumitru@cankaya.edu.tr
[3] Institute of Space Sciences, Magurele, P.O.Box MG-23, Ro 077125 Bucharest, Romania
* Correspondence: mehmetyavuz@erbakan.edu.tr; Tel.: +90-332-323-8220 (ext. 5548)

Received: 25 April 2019; Accepted: 13 May 2019; Published: 16 May 2019

Abstract: We investigated existence and uniqueness conditions of solutions of a nonlinear differential equation containing the Caputo–Fabrizio operator in Banach spaces. The mentioned derivative has been proposed by using the exponential decay law and hence it removed the computational complexities arising from the singular kernel functions inherit in the conventional fractional derivatives. The method used in this study is based on the Banach contraction mapping principle. Moreover, we gave a numerical example which shows the applicability of the obtained results.

Keywords: existence-uniqueness conditions; nonlocal Cauchy problem; Caputo–Fabrizio fractional derivative; Banach space

---

1. Introduction and Some Preliminaries

Modeling real-life problems with fractional differential equations (FDE) has a significant role in recent years. Some significant definitions that deal with fractional derivatives have been developed by Coimbra, Davison–Essex, Riesz, Riemann–Liouville, Hadamard, Grunwald–Letnikov, and Caputo [1,2]. Novel solution methods of such problems have been investigated by using these fractional derivative operators [3–9]. Moreover, in the last decades, new fractional derivative operators have been defined by using an exponential kernel called Caputo–Fabrizio (CF) [10] and the Mittag–Leffler kernel called Atangana–Baleanu (AB) [11]. These operators are very efficient for modeling complex nonlinear fractional dynamical systems and solving them. Caputo and Fabrizio have given a different perspective to fractional operators by introducing a new fractional operator without a singular kernel. Actually, if the CF operator was compared with the classical Caputo derivative, it can be seen that the new derivative with an exponential kernel has rapid stabilization in accordance with the memory effect. This definition comes naturally from the constitutive equation relating to the flux and gradient by exponential damping functions. In addition to being a very useful mathematical definition, it is an operator that is highly preferred in terms of physical meaning [12,13].

Some illustrative applications of the CF operator in various fields where the nonlocality appears in real world phenomena and more information about CF operator can be found in [14–18]. On the other hand, the AB fractional operator is defined with Mittag–Leffler function (MLF) and since the MLF is considered a nonlocal function, the kernel of AB derivative is nonlocal. Some researchers have studied this operator by applying it to a physical problem [19], a model of groundwater [20], initial and boundary value problems [21], and comparing it with the Liouville–Caputo fractional operator in terms of the solutions of nonlinear fractional equations [22] and comparing it with the CF derivative operator [23] as well as many others [24–34].

It is well known that the Cauchy problem (CP) consists of a differential equation with initial conditions. Starting with the work of Peano in the 1890s, analysis have had a continuing interest in the Cauchy problem. One of the crucial problems in the theory of Cauchy problems was finding the conditions that guarantee the existence of solutions of CPs.

We considered the following Cauchy problem:

$$x'(\xi) = f(\xi, x), \quad x(\xi_0) = x_0, \tag{1}$$

where $f : [\xi_0, \xi_0 + a] \times \mathscr{E} \to \mathscr{E}$, $a > 0$ and $\mathscr{E}$ is a Banach space. A survey of the research history of this problem shows that a solution of Equation (1) (a continuous function $x(\xi) : [\xi_0, \xi_0 + L] \to \mathscr{E}$ such that it satisfies Equation (1)) always exists if $dim\mathscr{E} < \infty$, L is small enough, and $f(\xi, x)$ is a continuous function (this result is also known as Peano's theorem), where $dim\mathscr{E}$ is the dimension of the Banach space $\mathscr{E}$. On the other hand, Cauchy problems in infinite dimensional spaces may have no solutions. That is, there is no guarantee of the validity of Peano's theorem in infinite-dimensional Banach spaces. Dieudonne [35] provided the first example of a continuous map from an infinitely dimensional Banach space c_0 for which there is no solution to the related Cauchy problem in Equation (1). Afterwards, Godunov [36] proved that Peano's theorem is false in every infinite-dimensional Banach space. More precisely, for every infinite-dimensional Banach space $\mathscr{E}$, $\xi_0 \in R$, $u_0 \in \mathscr{E}$, there exists a continuous mapping $f : R \times \mathscr{E} \to \mathscr{E}$, such that there exists no solution of Equation (1). Therefore, determining existence and uniqueness conditions (EUC) of solutions of a DE in Banach spaces is important. It is possible to find a few different approaches to EUC of real-life problems defined with non-integer order derivative in the literature. Among them, Balachandran and Trujillo [37] studied the existence of nonlinear FDEs solutions in the Caputo sense in Banach spaces. Lakshmikantham and Devi [38] discussed the general theory of FDEs in Banach spaces. Benchohra and Seba [39] studied the existence of solutions in Banach spaces for a class of initial value problems. In these mentioned studies, the Caputo derivative is considered. Wang et al. [40] developed two sufficient conditions for nonlocal controllability for fractional evolution systems. Lv et al. [41] employed about a new existence and uniqueness theorem for solutions of a special equation by using a Caputo fractional derivative in a Banach space.

In this study, after giving preliminary material, we obtained EUC of solutions of the following Cauchy problem with nonlocal initial conditions (nonlocal Cauchy problem) which includes the Caputo–Fabrizio operator in a Banach space $\mathscr{E}$. Let us consider:

$$^{CF}D^{\alpha}_{\mu}\omega(\mu) = T\omega(\mu) + h(\mu, \omega(\mu)), \quad 0 \le \mu \le 1, \tag{2}$$

$$\omega(0) = \int_0^1 g(\xi)\omega(\xi)d\xi, \tag{3}$$

where $^{CF}D^{\alpha}_{\mu}$ is the Caputo–Fabrizio derivative of order $\alpha \in (0,1)$, $g : [0,1] \to [0,1)$ is a continuous function, and $h : [0,1] \times \mathscr{E} \to \mathscr{E}$ is a given function, $T : \mathscr{E} \to \mathscr{E}$ is a given operator satisfying some assumptions that will be specified in Section 3.

2. Preliminaries

We begin by introducing some notations and basic terminology. $\mathscr{E}$ will always represent real Banach spaces. We will denote with the notation $L(\mathscr{E}, \mathscr{E})$ the Banach space of all linear bounded operators from $\mathscr{E}$ to $\mathscr{E}$. We also show with $C(J, \mathscr{E})$, the complete space of all continuous functions from $J = [0,1]$ to $\mathscr{E}$, with the norm $\|\omega\| = \max_{\xi \in J} \|\omega(\xi)\|$. Let c_0 be the Banach space of all null sequences with norm $\|\omega\| = \sup_{n \in N} |\omega_n|$. Let $L^1([0,1], \mathscr{E})$ be the Banach space of measurable functions $x : [0,1] \to \mathscr{E}$ which are Lebesgue integrable, equipped with norm $\|x\| = \int_0^1 \|x(s)\| ds$. Let $\beta = \int_0^1 g(\xi)d\xi$ and $R_+ = [0, +\infty)$. A function $w \in C(J, \mathscr{E})$ which satisfies Equations (2) and (3) is called as a solution of these equations .

Definition 1. *Let $\alpha \in (0,1)$, $b > 0$, and $\omega \in C^1(0,b)$. The Caputo–Fabrizio fractional derivative ${}^{CF}D^{\alpha}_{\mu}\omega(\mu)$ of order α is defined by:*

$$ {}^{CF}D^{\alpha}_{\mu}\omega(\mu) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^{\mu} e^{\left(-\frac{\alpha}{1-\alpha}(\mu-\xi)\right)}\omega'(\xi)d\xi, \quad \mu \geq 0, \tag{4} $$

where $M(\alpha)$ is a normalization constant depending on α [10,14].

Definition 2. *Let $\alpha \in (0,1)$. The The Caputo–Fabrizio fractional integral ${}^{CF}I^{\alpha}_{\mu}\omega(\mu)$ of order α, is given by [14]:*

$$ {}^{CF}I^{\alpha}_{\mu}\omega(\mu) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\omega(\mu) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^{\mu} \omega(\xi)d\xi, \quad \mu \geq 0. \tag{5} $$

Remark 1. *Imposing [14]:*

$$ \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1, \tag{6} $$

it is obtained an explicit formula for $M(\alpha)$,

$$ M(\alpha) = \frac{2}{2-\alpha}. \tag{7} $$

Remark 2. *Let $\alpha \in (0,1)$. Then we have [14,42]:*

$$ {}^{CF}D^{\alpha}_{\mu}({}^{CF}I^{\alpha}_{\mu}\omega(\mu)) = \omega(\mu) \text{ and } {}^{CF}I^{\alpha}_{\mu}({}^{CF}D^{\alpha}_{\mu}\omega(\mu)) = \omega(\mu) - \omega(0). \tag{8} $$

3. Main Results

Now, we are ready to prove the uniqueness and existence of the solutions for Equations (2) and (3) under the following hypotheses:

(I) $T \in L(\mathscr{E}, \mathscr{E})$.
(II) $h \in C(J \times \mathscr{E}, \mathscr{E})$ and there exist a $p_h \in L^1([0,1], R^+)$ such that $\|h(\mu, v)\| \leq p_h(\mu)\|v\|$ for $\mu \in J$ and each $v \in \mathscr{E}$.
(III) $H : J \to \mathscr{E}$, $H(.) = h(., \omega(.))$ is a differentiable function, for any $\omega \in C^1(J, \mathscr{E})$.
(IV) There exists a constant L such that $\|T\| + L < 1 - \beta$ and:

$$ \|h(\xi, v) - h(\xi, \bar{v})\| \leq L\|v - \bar{v}\| \text{ for every } v, \bar{v} \in \mathscr{E}. $$

Lemma 1. *If the conditions (I), (II), and (III) are satisfied then Equations (2) and (3) are equivalent to the following equation:*

$$ \begin{aligned} \omega(\mu) &= a_{\alpha}[T\omega(\mu) + h(\mu, \omega(\mu))] + b_{\alpha} \int_0^{\mu} [T\omega(\varrho) + h(\varrho, \omega(\varrho))]d\varrho \\ &+ \frac{a_{\alpha}}{1-\beta} \int_0^1 g(\xi)\left[T\omega(\xi) + h(\xi, \omega(\xi))\right] d\xi + \frac{b_{\alpha}}{1-\beta} \int_0^1 \left[T\omega(\varrho) + h(\varrho, \omega(\varrho))\right] G(\varrho)d\varrho, \end{aligned} \tag{9} $$

where $G(\varrho) = \int_{\varrho}^1 g(\xi)d\xi$, $a_{\alpha} = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}$ and $b_{\alpha} = \frac{2\alpha}{(2-\alpha)M(\alpha)}$.

Proof. By considering Remark 1 and Equations (2) and (3), we get:

$$ \omega(\mu) = a_{\alpha}[T\omega(\mu) + h(\mu, \omega(\mu))] + b_{\alpha} \int_0^{\mu} [T\omega(\varrho) + h(\varrho, \omega(\varrho))]d\varrho + \omega(0). \tag{10} $$

So, by using the initial condition in Equation (3) we get:

$$\begin{aligned}
\omega(0) &= \int_0^1 g(\xi)\omega(\xi)d\xi \\
&= \int_0^1 g(\xi)\left[a_\alpha[T\omega(\xi)+g(\xi,\omega(\xi))]+b_\alpha\int_0^\xi [T\omega(\varrho)+h(\varrho,\omega(\varrho))]d\varrho+\omega(0)\right]d\xi \\
&= \omega(0)\int_0^1 g(\xi)d\xi+\int_0^1 g(\xi)\left[a_\alpha[T\omega(\xi)+h(\xi,\omega(t))]+b_\alpha\int_0^\xi [T\omega(\varrho)+h(\varrho,\omega(\varrho))]d\varrho\right]d\xi \\
&= \frac{a_\alpha}{1-\beta}\int_0^1 g(\xi)[T\omega(\xi)+h(\xi,\omega(\xi))]d\xi+\frac{b_\alpha}{1-\beta}\int_0^1 g(\xi)\int_0^\xi [T\omega(\varrho)+h(\varrho,\omega(\varrho))]\,d\varrho d\xi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\omega(0) &= \frac{a_\alpha}{1-\beta}\int_0^1 g(\xi)[T\omega(\xi)+h(\xi,\omega(\xi))]d\xi+\frac{b_\alpha}{1-\beta}\int_0^1 [T\omega(\varrho)+h(\varrho,\omega(\varrho))]\left[\int_\varrho^1 g(\xi)d\xi\right]d\varrho \\
&= \frac{a_\alpha}{1-\beta}\int_0^1 g(\xi)[T\omega(\xi)+h(\xi,\omega(\xi))]d\xi+\frac{b_\alpha}{1-\beta}\int_0^1 [T\omega(\varrho)+h(\varrho,\omega(\varrho))]\,G(\varrho)d\varrho.
\end{aligned}$$

So, substituting $\omega(0)$ in Equation (10) we obtain:

$$\begin{aligned}
\omega(\mu) &= a_\alpha[T\omega(\mu)+h(\mu,\omega(\mu))]+b_\alpha\int_0^\mu [T\omega(\tau)+h(\tau,\omega(\tau))]d\tau \\
&+ \frac{a_\alpha}{1-\beta}\int_0^1 g(\xi)[T\omega(\xi)+h(\xi,\omega(\xi))]d\xi+\frac{b_\alpha}{1-\beta}\int_0^1 [T\omega(\varrho)+h(\varrho,\omega(\varrho))]\,G(\varrho)d\varrho.
\end{aligned}$$

Conversely if ω is a solution of Equation (10), then for every $\mu \in J$, according to Remark 2, we have:

$$\begin{aligned}
{}^{CF}D_\mu^\alpha\omega(\mu) &= {}^{CF}D_\mu^\alpha(a_\alpha[T\omega(\mu)+h(\mu,\omega(\mu))]+b_\alpha\int_0^\mu [T\omega(\varrho)+h(\varrho,\omega(\varrho))]d\varrho \\
&+ \frac{a_\alpha}{1-\beta}\int_0^1 g(\xi)[T\omega(\xi)+h(\xi,\omega(\xi))]d\xi+\frac{b_\alpha}{1-\beta}\int_0^1 [T\omega(\varrho)+h(\varrho,\omega(\varrho))]\,G(\varrho)d\varrho) \\
&= {}^{CF}D_\mu^\alpha(a_\alpha[T\omega(\mu)+h(\mu,\omega(\mu))]+b_\alpha\int_0^\mu [T\omega(\varrho)+h(\varrho,\omega(\varrho))]d\varrho) \\
&+ {}^{CF}D_\mu^\alpha(\frac{a_\alpha}{1-\beta}\int_0^1 g(\xi)[T\omega(\xi)+h(\xi,\omega(\xi))]d\xi+\frac{b_\alpha}{1-\beta}\int_0^1 [T\omega(\varrho)+h(\varrho,\omega(\varrho))]\,G(\varrho)d\varrho). \\
&= {}^{CF}D_\mu^\alpha(a_\alpha[T\omega(\mu)+h(\mu,\omega(\mu))]+b_\alpha\int_0^\mu [T\omega(\varrho)+h(\varrho,\omega(\varrho))]d\varrho)+\theta \\
&= {}^{CF}D_\mu^{\alpha\,CF}I_\mu^\alpha(T\omega(\mu)+h(\mu,\omega(\mu)))+\theta \\
&= T\omega(\mu)+h(\mu,\omega(\mu)).
\end{aligned}$$

It is obvious that $\omega(0)=\int_0^1 g(\xi)\omega(\xi)d\xi$. Therefore, the proof is completed. □

Theorem 1. *If the conditions (I), (II), (III), and (IV) are satisfied then the nonlocal Cauchy problem in Equations (2) and (3) has a unique solution on $C(J,\mathscr{E})$.*

Proof. We now consider the operator:

$$\psi : C(J,\mathscr{E}) \to C(J,\mathscr{E})$$

defined by:

$$\begin{aligned}(\psi\omega)(\mu) &= a_\alpha[T\omega(\mu)+h(\mu,\omega(\mu))]+b_\alpha\int_0^\mu[T\omega(\varrho)+h(\varrho,\omega(\varrho))]d\varrho \\ &+ \frac{a_\alpha}{1-\beta}\int_0^1 g(\xi)[T\omega(\xi)+h(\xi,\omega(\xi))]d\xi+\frac{b_\alpha}{1-\beta}\int_0^1[T\omega(\varrho)+h(\varrho,\omega(\varrho))]\,G(\varrho)d\varrho.\end{aligned}$$

We will show that the operator ψ is well defined via assumptions. For this aim, we must prove that $\psi(\omega)\in C(J,\mathscr{E})$ for every $\omega\in C(J,\mathscr{E})$.

Let $\mu_1,\mu_2\in J$, $\mu_1<\mu_2$. We deduce that:

$$\begin{aligned}\|(\psi\omega)\mu_1-(\psi\omega)\mu_2\| &= \|a_\alpha[T(\omega(\mu_1)-\omega(\mu_2))+h(\mu_1,\omega(\mu_1))-h(\mu_2,\omega(\mu_2))] \\ &+ b_\alpha\int_{\mu_1}^{\mu_2}[T\omega(\varrho)+h(\varrho,\omega(\varrho))]d\varrho\| \\ &\le a_\alpha\|T\|\|\omega(\mu_1)-\omega(\mu_2)\|+a_\alpha\|h(\mu_1,\omega(\mu_1))-h(\mu_2,\omega(\mu_2))\| \\ &+ b_\alpha\int_{\mu_1}^{\mu_2}\|T\omega(\varrho)+h(\varrho,\omega(\varrho))-h(\varrho,0)+h(\varrho,0)\|d\varrho \\ &\le a_\alpha\|T\|\|\omega(\mu_1)-\omega(\mu_2)\|+a_\alpha\|h(\mu_1,\omega(\mu_1))-h(\mu_2,\omega(\mu_1))\| \\ &+ a_\alpha L\|\omega(\mu_1)-\omega(\mu_2)\|+b_\alpha[\|T\|\|\omega\|+L\|\omega\|+\max_{\varrho\in J}\|h(\varrho,0)\|](\mu_2-\mu_1) \\ &\le a_\alpha(1-\beta)\|\omega(\mu_1)-\omega(\mu_2)\|+a_\alpha\|h(\mu_1,\omega(\mu_1))-h(\mu_2,\omega(\mu_1))\| \\ &+ b_\alpha[(1-\beta)\|\omega\|+\max_{\varrho\in J}\|h(\varrho,0)\|](\mu_2-\mu_1).\end{aligned}$$

As $\mu_1\to\mu_2$, the right-hand side of the above inequality tends to zero. Thus, ψ is well defined.

We must show that ψ is a contracting mapping. For this, let $\omega_1,\omega_2\in C(J,\mathscr{E})$, and $\mu\in J$. Then, we have:

$$\begin{aligned}\|(\psi\omega_1)\mu-(\psi\omega_2)\mu\| &\leqslant a_\alpha\|T\|\|\omega_1(\mu)-\omega_2(\mu)\|+a_\alpha\|h(\mu,\omega_1(\mu))-h(\mu,\omega_2(\mu))\| \\ &+ b_\alpha\left(\|T\|\|\omega_1-\omega_2\|+\int_0^\mu\|h(\varrho,\omega_1(\varrho))-h(\varrho,\omega_2(\varrho))\|d\varrho\right) \\ &+ \frac{a_\alpha}{1-\beta}\left(\|T\|\|\omega_1-\omega_2\|\beta+\int_0^1 g(\xi)\|h(\xi,\omega_1(\xi))-h(\xi,\omega_2(\xi))\|d\xi\right) \\ &+ \frac{b_\alpha}{1-\beta}\left(\|T\|\|\omega_1-\omega_2\|\beta+\int_0^1 G(\varrho)\|h(\varrho,\omega_1(\varrho))-h(\varrho,\omega_2(\varrho))\|d\varrho\right) \\ &\le a_\alpha\left(\|T\|+L\right)\|\omega_1(\mu)-\omega_2(\mu)\|+b_\alpha(\|T\|+L)\|\omega_1-\omega_2\| \\ &+ \frac{a_\alpha}{1-\beta}\left(\|T\|+L\right)\|\omega_1-\omega_2\|+\frac{b_\alpha}{1-\beta}\left(\|T\|+L\right)\|\omega_1-\omega_2\| \\ &\le \frac{\|T\|+L}{1-\beta}\|\omega_1-\omega_2\|.\end{aligned}$$

Since $0<\frac{\|T\|+L}{1-\beta}<1$, then ψ is a contraction mapping and therefore there exists a unique fixed point $\omega\in C(J,\mathscr{E})$ such that $\psi(\omega(\mu))=\omega(\mu)$. Any fixed point of ψ is the solution of Equations (2) and (3). □

Example 1. *Let us consider the infinite system of scalar fractional functional differential equations:*

$$\begin{aligned}{}^{CF}D_\mu^\alpha\omega_n(\mu) &= \frac{\omega_n(\mu)}{50.2^n}+\frac{\mu sin\omega_n(\mu)-\omega_n(\mu)}{e^\mu-1+50.2^n},\quad \mu\in J,\quad \alpha\in(0,1), \\ \omega_n(0) &= \int_0^1\frac{1}{4}\omega_n(\mu)d\mu,\quad n=1,2,3,\ldots\end{aligned}\tag{11}$$

Let $\mathscr{E}$ be the Banach space c_0. Then the infinite system Equation (11) can be regarded as a problem of for Equations (2) and (3) in $\mathscr{E}$. In this case:

$$\omega = (\omega_1, \omega_2, \omega_3, ...), \quad T(\omega) := (\frac{\omega_n}{50.2^n})_{n=1}^{\infty},$$

$$h(\mu,\omega) := (h_1(\mu,\omega), h_2(\mu,\omega), h_3(\mu,\omega), ...) \quad \textit{in which} \quad h_n(\mu,\omega) = \frac{\mu sin\omega_n - \omega_n}{e^{\mu} - 1 + 50.2^n}$$

$$\textit{and} \quad g(\mu) = \frac{1}{4}.$$

Therefore, it is obvious that $T \in L(\mathscr{E},\mathscr{E})$ and $h \in C(J \times \mathscr{E}, \mathscr{E})$. On the other hand, for every $\mu \in J$ and $\omega \in \mathscr{E}$, we have:

$$\begin{aligned} \|h(\mu,\omega)\| = \sup_{n\in N} \|h_n(\mu,\omega)\| &= \sup_{n\in N} |\frac{\mu sin\omega_n - \omega_n}{e^{\mu} - 1 + 50.2^n}| \\ &\leq \sup_{n\in N} \frac{\mu|sin\omega_n| + |\omega_n|}{e^{\mu} - 1 + 50.2^n} \\ &= \frac{1}{e^{\mu}} 2\|\omega\|, \end{aligned}$$

$p_h(\mu) = 2e^{-\mu} \in L^1([0,1], R^+)$. *Moreover, for every $\mu \in J$, $H(\mu) = h(\mu,\omega(\mu))$ is a differentiable function and:*

$$\begin{aligned} \|h(\mu,\omega) - h(\mu,\varpi)\|| &= \sup_{n\in N} |\frac{\mu sin\omega_n - \omega_n}{e^{\mu} - 1 + 50.2^n} - \frac{\mu sin\varpi_n - \varpi_n}{e^{\mu} - 1 + 50.2^n}| \\ &\leq \frac{1}{100} \sup_{n\in N} |\mu sin\omega_n - \omega_n - \mu sin\varpi_n + \varpi_n| \\ &\leq \frac{2}{100} \|\omega - \varpi\| \ \textit{for all} \ \ \omega, \varpi \in \mathscr{E}. \end{aligned}$$

Since $L = \frac{2}{100}$, $\beta = \frac{1}{4}$ then we obtain that:

$$\|T\| + L \leq \frac{3}{100} < 1 - \beta = \frac{3}{4}.$$

Thus, by Theorem 1, we can show that the infinite system in Equation (11) has a unique solution.

4. Conclusions

In the present study, the existence and uniqueness conditions of the special type nonlinear fractional differential equations were obtained in the Caputo–Fabrizio fractional derivative sense. These conditions were constructed in Banach spaces via the Banach contraction principle mapping. Moreover, the applicability and the effectiveness of the results were confirmed with an illustrative numerical example.

Author Contributions: A.K. and M.Y. conceived the manuscripts, obtained the solution and wrote the paper. D.B. analyzed the paper; all authors read and approved the final manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Podlubny, I. *Fractional Differential Equation: An Introduction to Fractional Derivatives, Fractional Differential Equations, to methods of Their Solution and Some of Their Applications*; Academic Press: Cambridge, MA, USA, 1999; p. 198.
2. Baleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J.J. *Fractional Calculus: Models and Numerical Methods*; World Scientific Publishing: Singapore, 2012.
3. Yavuz, M. Novel Solution Methods for Initial Boundary Value Problems of Fractional Order with Conformable Differentiation. *Int. J. Optim. Control Theor. Appl. (IJOCTA)* **2018**, *8*, 1–7. [CrossRef]
4. Yavuz, M.; Özdemir, N. Numerical inverse Laplace homotopy technique for fractional heat equations. *Therm. Sci.* **2018**, *22*, 185–194. [CrossRef]
5. Avci, D.; Eroglu, B.B.I.; Özdemir, N. Conformable heat equation on a radial symmetric plate. *Therm. Sci.* **2017**, *21*, 819–826. [CrossRef]
6. Özdemir, N.; Yavuz, M. Numerical solution of fractional Black-Scholes equation by using the multivariate Padé approximation. *Acta Phys. Pol. A* **2017**, *132*, 1050–1053. [CrossRef]
7. Baskonus, H.M.; Mekkaoui, T.; Hammouch, Z.; Bulut, H. Active control of a chaotic fractional order economic system. *Entropy* **2015**, *17*, 5771–5783. [CrossRef]
8. Yavuz, M.; Yaşkıran, B. Approximate-analytical solutions of cable equation using conformable fractional operator. *New Trends Math. Sci.* **2017**, *5*, 209–219. [CrossRef]
9. Evirgen, F.; Özdemir, N. A fractional order dynamical trajectory approach for optimization roblem with HPM. In *Fractional Dynamics and Control*; Baleanu, D., Machado, J.A.T., Luo, A.C.J., Eds.; Springer: New York, NY, USA, 2012; pp. 145–155.
10. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 1–13.
11. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
12. Hristov, J. Derivatives with non-singular kernels from the Caputo–Fabrizio definition and beyond: Appraising analysis with emphasis on diffusion models. *Front. Fract. Calc.* **2018**, *1*, 270–342.
13. Hristov, J. Space-fractional diffusion with a potential power-law coefficient: Transient approximate solution. *Prog. Fract. Differ. Appl.* **2017**, *3*, 19–39. [CrossRef]
14. Losada, J.; Nieto, J.J. Properties of a new fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 87–92.
15. Yavuz, M.; Özdemir, N. A different approach to the European option pricing model with new fractional operator. *Math. Model. Nat. Phenom.* **2018**, *13*, 12. [CrossRef]
16. Hristov, J. Steady-state heat conduction in a medium with spatial non-singular fading memory: Derivation of Caputo–Fabrizio space-fractional derivative with Jeffrey's kernel and analytical solutions. *Therm. Sci.* **2017**, *21*, 827–839. [CrossRef]
17. Atangana, A.; Alkahtani, B.S.T. New Model of Groundwater Flowing within a Confine Aquifer: Application of Caputo–Fabrizio Derivative. *Arab. J. Geosci.* **2016**, *9*, 8. [CrossRef]
18. Abdeljawad, T.; Baleanu, D. On fractional derivatives with exponential kernel and their discrete versions. *Rep. Math. Phys.* **2017**, *80*, 11–27. [CrossRef]
19. Gómez-Aguilar, J.F.; Escobar-Jiménez, R.F.; López-López, M.G.; Alvarado-Martínez, V.M. Atangana-Baleanu fractional derivative applied to electromagnetic waves in dielectric media. *J. Electromagn. Waves Appl.* **2016**, *30*, 1937–1952. [CrossRef]
20. Alqahtani, R.T. Atangana-Baleanu derivative with fractional order applied to the model of groundwater within an unconfined aquifer. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3647–3654. [CrossRef]
21. Al-Salti, F.A.-M.N.; Karimov, E. Initial and boundary value problems for fractional differential equations involving Atangana-Baleanu derivative. *arXiv* **2017**, arXiv:1706.00740.
22. Yavuz, M.; Ozdemir, N.; Baskonus, H.M. Solutions of partial differential equations using fractional operator involving Mittag-Leffler kernel. *Eur. Phys. J. Plus* **2018**, *133*, 215. [CrossRef]
23. Sheikh, N.A.; Ali, F.; Saqib, M.; Khan, I.; Jan, S.A.A.; Alshomrani, A.S.; Alghamdi, M.S. Comparison and analysis of the Atangana-Baleanu and Caputo–Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction. *Results Phys.* **2017**, *7*, 789–800. [CrossRef]

24. Hristov, J. On the Atangana-Baleanu Derivative and Its Relation to the Fading Memory Concept: The Diffusion Equation Formulation. In *Fractional Derivatives with Mittag-Leffler Kernel*; Springer: Cham, Switzerland, 2019; pp. 175–193.
25. Hristov, J. Response functions in linear viscoelastic constitutive equations and related fractional operators. *Math. Model. Nat. Phenom.* **2019**, *14*, 305. [CrossRef]
26. Dos Santos, M. Non-Gaussian Distributions to Random Walk in the Context of Memory Kernels. *Fractal Fract.* **2018**, *2*, 20. [CrossRef]
27. Dos Santos, M. Fractional Prabhakar Derivative in Diffusion Equation with Non-Static Stochastic Resetting. *Physics* **2019**, *1*, 40–58. [CrossRef]
28. Yavuz, M.; Özdemir, N. European vanilla option pricing model of fractional order without singular kernel. *Fractal Fract.* **2018**, *2*, 3. [CrossRef]
29. Argub, O.A.; Al-Smadi, M. Atangana-Baleanu fractional approach to the solutions of Bagley-Torvik and Painlevé equations in Hilbert space. *Chaos Solitons Fractals* **2018**, *117*, 161–167.
30. Argub, O.A.; Maayah, B. Numerical solutions of integrodifferential equations of Fredholm operator type in the sense of the Atangana–Baleanu fractional operator. *Chaos Solitons Fractals* **2018**, *117*, 117–124. [CrossRef]
31. Argub, O.A. Solutions of time-fractional Tricomi and Keldysh equations of Dirichlet functions types in Hilbert space. *Numer. Methods Part. Differ. Equ.* **2018**, *34*, 1759–1780.
32. Yavuz, M. Characterizations of two different fractional operators without singular kernel. *Math. Model. Nat. Phenom.* **2019**, *14*, 302. [CrossRef]
33. Argub, O.A. Numerical solutions for the Robin time-fractional partial differential equations of heat and fluid flows based on the reproducing kernel algorithm. *Int. J. Numer. Methods Heat Fluid Flow* **2018**, *28*, 828–856.
34. Argub, O.A. Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions. *Comput. Math. Appl.* **2017**, *73*, 1243–1261.
35. Dieudonne, J. Deux examples singuliers equations differentielles. *Acta. Sci. Math. (Szeged)* **1950**, *12*, 38–40.
36. Godunov, A.N. Peano's theorem in Banach spaces. *Funct. Anal. Appl.* **1975**, *9*, 53–55. [CrossRef]
37. Balachandran, K.; Trujillo, J.J. The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 4587–4593. [CrossRef]
38. Lakshmikantham, V.; Devi, J.V. Theory of fractional differential equations in a Banach space. *Eur. J. Pure Appl. Math.* **2008**, *1*, 38–45.
39. Benchohra, M.; Seba, D. Impulsive fractional differential equations in Banach spaces. *Electron. J. Qual. Theory Differ. Equ.* **2009**, *8*, 1–14. [CrossRef]
40. Wang, J.; Fan, Z.; Zhou, Y. Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces. *J. Optim. Theory Appl.* **2012**, *154*, 292–302. [CrossRef]
41. Lv, Z.W.; Liang, J.; Xiao, T.J. Solutions to fractional differential equations with nonlocal initial condition in Banach spaces. *Adv. Differ. Equ.* **2010**, *2010*, 340349. [CrossRef]
42. Baleanu, D.; Agheli, B.; al Qurashi, M.M. Fractional advection differential equation within Caputo and Caputo–Fabrizio derivatives. *Adv. Mech. Eng.* **2016**, *8*, 1–8. [CrossRef]