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On a three step crisis integro-differential equation

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Abstract

One of the interesting fractional integro-differential equations is the three step crisis equation which has been reviewed recently. In this paper, we investigate the existence of solutions for a three step crisis fractional integro-differential equation under some boundary conditions.

MSC: Primary 34A08; secondary 34A60

Keywords: Caputo derivation; Pointwise defined equation; Three steps crisis equation; Singularity

1 Preliminaries

It is well known that we can make better exact models for most natural phenomena by using fractional differential equations. Most researchers are working on fractional integro-differential equations (see, for example, [1, 2, 5–8, 10–21]).

In 2010, Agarwal et al. reviewed the existence of solutions $D^\alpha u(t) + f(t, u(t)) = 0$ with boundary conditions $u'(0) = \cdots = u^{(n-1)} = 0$ and $u(1) = \int_0^1 u(s)\,d\mu(s)$, where $n \geq 2$, $\alpha \in (n-1, n)$, $\mu(s)$ is a functional of bounded variation, f may have singularity at $t = 0$ and $\int_0^1 d\mu(s) < 1$ [3]. In 2012, Agarwal et al. studied positive solutions for the integral value problem $D^\alpha u_i(t) + f_i(t, u_1(t), u_2(t)) = 0$ with boundary conditions $u_i(0) = u_i'(0) = 0$ and $u_i(1) = \int_0^1 u_i(t)\,d\eta(t)$ for $i = 1, 2$, where $t \in (0,1)$, $\alpha \in (2,3]$, D^α is the Riemann–Liouville fractional derivative of order α, f_i is a real valued continuous map on $[0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$ and $\int_0^1 u_i(t)\,d\eta(t)$ denotes the Riemann–Stieltjes integral [4]. In 2013, the singular fractional problem $D^\alpha u + f(t, u, D^\gamma u, D^\mu u) + g(t, u, D^\gamma u, D^\mu u) = 0$ with boundary conditions $u(0) = u'(0) = u''(0) = u'''(0) = 0$ was reviewed, where $3 < \alpha < 4$, $0 < \gamma < 1$, $1 < \mu < 2$, D^α is the Caputo fractional derivative and f is a Caratheodory function on $[0,1] \times (0,\infty)^3$ [9].

Recently, the authors introduced a new model for investigating the fractional differential equations called three step crisis integro-differential equations [11]. By using the idea, we investigate the existence of solutions for the three step crisis integro-differential equation

$$D^\alpha x(t) + f\left(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi) x(\xi)\,d\xi\right) = 0 \tag{1}$$

with boundary conditions $x(0) = x'(T_0)$, $x(1) = x'(T_1)$ and $x''(0) = x^{(n)}(0) = 0$, where $\alpha > 1$ with $n = [\alpha] - 1$, $T_0, T_1, \beta, \lambda, \mu \in (0,1)$, $h \in L^1[0,1]$, D^α is the Caputo fractional derivative of order α, $f(t, x_1(t), \dots, x_5(t)) = f_1(t, x_1(t), \dots, x_4(t))$ on $[0, \lambda)$, $f(t, x_1(t), \dots, x_5(t)) =$

$f_2(t,x_1(t),\dots,x_4(t))$ on $[\lambda,\mu]$ and $f(t,x_1(t),\dots,x_5(t)) = f(t,x_1(t),\dots,x_4(t))$ on $(\mu,1]$ in which $f_1(t,\cdot,\cdot,\cdot,\cdot)$ and $f_3(t,\cdot,\cdot,\cdot,\cdot)$ are continuous on $[0,\lambda)$ and $(\mu,1]$, respectively, and $f_2(t,\cdot,\cdot,\cdot,\cdot)$ is singular at some points $t \in [\lambda,\mu]$. In this case, we use the symbol $f = [f_1, f_2, f_3, \lambda, \mu]$ [11].

As is well known, the Caputo fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}\,ds$, where $n = [\alpha] + 1$ (see, for example, [13]). Let Ψ be the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < \infty$ for all $t > 0$ [22]. One can check that $\psi(t) < t$ for all $t > 0$ [22]. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two maps. Then T is called an α-admissible map whenever $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ [22]. Let (X,d) be a complete metric space, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ a map. A self-map $T : X \to X$ is called an α-ψ-contraction whenever $\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$ for all $x,y \in X$ [22]. We need the following results.

Lemma 1 ([23]) *Let $0 < n-1 \leq \alpha < n$. Then $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some constants $c_0,\dots,c_{n-1}$.*

Lemma 2 ([24]) *If E is a closed, bounded and convex subset of a Banach space X and $T : E \to E$ is completely continuous, then T has a fixed point in E.*

Lemma 3 ([22]) *Let (X,d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0,\infty)$ a map and $T : X \to X$ an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$, then T has a fixed point.*

2 Main results

Now, we are ready to state and prove our main results.

Lemma 4 *Let $\alpha > 1$, $n = [\alpha] + 1$, $T_0, T_1 \in (0,1)$ and $f \in L^1[0,1]$. Then $x(t) = \int_0^1 G(t,s)f(s)\,ds$ is the solution of the pointwise defined equation $D^\alpha x(t) + f(t) = 0$ with boundary conditions $x(0) = x'(T_0)$, $x(1) = x'(T_1)$ and $x''(0) = \cdots = x^{(n-1)}(0) = 0$, where $G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1+t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(T_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{t(T_0-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ whenever $0 \leq s \leq t$, $s \leq T_0$, $G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1+t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(T_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ whenever $0 \leq T_0 \leq s \leq t$, $s \leq T_1$, $G(t,s) = \frac{(1+t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(T_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ whenever $0 \leq t \leq s \leq T_1$, $s \geq T_0$, $G(t,s) = \frac{(1+t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(T_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{t(T_0-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ whenever $0 \leq t \leq s \leq T_0 \leq T_1$, $G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1+t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ whenever $0 \leq T_0 \leq T_1 \leq s \leq t$ and $G(t,s) = \frac{(1+t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ whenever $0 \leq t \leq s$, $s \geq T_1$.*

Proof Suppose that the equation $D^\alpha x(t) + f(t) = 0$ holds for all $t \in E \subset [0,1]$, where $m(E^c) = 0$ and m is the Lebesgue measure on $\mathbb{R}$. Let f_0 be a function such that $f_0 = f$ on E. It is easy to check that $I^\alpha(f(t)) = I^\alpha(f_0(t))$ for all $t \in [0,1]$. This implies that $I^\alpha(D^\alpha x(t)) = I^\alpha(-f_0(t))$ and by using Lemma 1 we get $x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds + c_0 + c_1 t$ for some constants c_0 and c_1. By using the boundary conditions, we obtain $x(0) = c_0$ and

$$x'(T_0) = -\frac{1}{\Gamma(\alpha-1)}\int_0^{T_0}(T_0-s)^{\alpha-1}f(s)\,ds + c_1.$$

Thus, $c_1 - c_0 = -\frac{1}{\Gamma(\alpha-1)}\int_0^{T_0}(T_0-s)^{\alpha-1}f(s)\,ds$. Since $x(1) = x'(T_1)$, we get

$$c_0 = \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-1}f(s)\,ds - \frac{1}{\Gamma(\alpha-1)}\int_0^{T_1}(T_1-s)^{\alpha-1}f(s)\,ds$$

and so

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds - \frac{1}{\Gamma(\alpha-1)} \int_0^{T_1} (T_1-s)^{\alpha-2} f(s)\,ds - \frac{1}{\Gamma(\alpha-1)} \int_0^{T_0} (T_0-s)^{\alpha-2} f(s)\,ds.$$

Hence,

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds \\ &\quad - \frac{1}{\Gamma(\alpha-1)} \int_0^{T_1} (T_1-s)^{\alpha-2} f(s)\,ds + \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds \\ &\quad - \frac{t}{\Gamma(\alpha-1)} \int_0^{T_1} (T_1-s)^{\alpha-2} f(s)\,ds - \frac{t}{\Gamma(\alpha-1)} \int_0^{T_0} (T_0-s)^{\alpha-2} f(s)\,ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds - \frac{1+t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s)\,ds \\ &\quad - \frac{1+t}{\Gamma(\alpha-1)} \int_0^{T_1} (T_1-s)^{\alpha-2} f(s)\,ds - \frac{t}{\Gamma(\alpha-1)} \int_0^{T_0} (T_0-s)^{\alpha-2} f(s)\,ds. \end{aligned}$$

Now it is easy to check that $x(t) = \int_0^1 G(t,s) f(s)\,ds$, where G is the given Green function. □

By using some usual calculations, we find that $|G(t,s)| \le \frac{2+\alpha+T_0}{\Gamma(\alpha)}(1-s)^{\alpha-2}$ for all $t,s \in [0,1]$ and $|\frac{\partial G}{\partial t}(t,s)| \le \frac{3\alpha}{\Gamma(\alpha)}(1-s)^{\alpha-2}$ for all $t,s \in [0,1]$. Also, it is easy to see that $D^\mu x \in C[0,1]$ and $|D^\mu x| \le \frac{\|x'\|}{\Gamma(2-\mu)}$ whenever $x \in C^1[0,1]$. Here, $0 \le \mu \le 1$. Now, consider the Banach space $X = C^1[0,1]$ with the norm $\|x\|_* = \max\{\|x\|, \|x'\|\}$, $\|\cdot\|$ is the sup norm on $C[0,1]$. Assume that $f = [f_1, f_2, f_3, \lambda, \mu]$. Define $T : X \to X$ by

$$\begin{aligned} T_x(t) &= \int_0^1 G(t,s) f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\big(x(s)\big)\right) ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\Big(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\big(x(s)\big)\Big)\,ds \\ &\quad - \frac{1+t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\Big(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\big(x(s)\big)\Big)\,ds \\ &\quad - \frac{1+t}{\Gamma(\alpha-1)} \int_0^{T_1} (T_1-s)^{\alpha-2} f\Big(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\big(x(s)\big)\Big)\,ds \\ &\quad - \frac{t}{\Gamma(\alpha-1)} \int_0^{T_0} (T_0-s)^{\alpha-2} f\Big(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi, \phi\big(x(s)\big)\Big)\,ds \end{aligned}$$

for all $x \in X$ and $t \in [0,1]$. Note that the singular pointwise defined problem (1) has a solution if and only if T has a fixed point in X. We are going to investigate the singular pointwise defined problem (1) under two different conditions. Here. we present first one. In our second result we denote the map T by F.

Theorem 5 *Let* $f = [f_1, f_2, f_3, \lambda, \mu]$, $f_1(t,0,0,0,0,0) = 0$, $f_2(s,0,0,0,0,0) = 0$ *and* $f_3(u,0,0,0,0,0) = 0$ *for all* $t \in [0,\lambda]$, $s \in [0,\lambda]$ *and* $u \in [\mu,1]$. *Assume that there are nondecreasing maps*

$\Lambda, \Lambda' : X \to [0,\infty)$ and mappings $a_1, a_2, a_3, a_4 : (\lambda, \mu) \to [0,\infty)$ such that $\lim_{z\to 0^+} \frac{\Lambda(z)}{z} = q < \infty$, $\lim_{z\to 0^+} \frac{\Lambda'(z)}{z} = q < \infty$ and $\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4 \in L^1[\lambda,\mu]$, where $\hat{a}_i = (1-s)^{\alpha-2}$ for $i = 1,2,3,4$. Suppose that $|f_1(t,x_1,\dots,x_5) - f_1(t,y_1,\dots,y_5)| \le \sum_{i=1}^4 \Lambda(|x_i - y_i|)$,

$$\bigl|f_2(t,x_1,\dots,x_5) - f_2(t,y_1,\dots,y_5)\bigr| \le \sum_{i=1}^4 a_i(t)|x_i - y_i|$$

and $|f_3(t,x_1,\dots,x_5) - f_3(t,y_1,\dots,y_5)| \le \sum_{i=1}^4 \Lambda'(|x_i - y_i|)$ for almost all $t \in [0,1]$ and every $x_1, x_2, \dots, x_5, y_1, y_2, \dots, y_5 \in X$. If

$$\frac{4q}{\alpha-1}\bigl(1-(1-\lambda)^{\alpha-1}\bigr) + \sum_{i=1}^4 \|\hat{a}_i\|_{[\lambda,\mu]} + \frac{4q'}{\alpha-1}(1-\mu)^{\alpha-1} < \frac{\Gamma(\alpha)}{l\theta_0},$$

then the pointwise defined equation (1) with boundary conditions has a solution, where $\|h\|_1 = m_0$, $l = \max\{1, \frac{1}{\Gamma(2-\beta)}, m_0\}$ and $\theta_0 = \max\{3\alpha, 2+\alpha+T_0\}$.

Proof Let $x_1, x_2 \in X$ and $t \in [0,1]$. Then we have

$$\begin{aligned}
\bigl|T_{x_1}(t) - T_{x_2}(t)\bigr| &\le \int_0^1 \bigl|G(t,s)\bigr| \biggl| f\biggl(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\,d\xi\biggr) \\
&\quad - f\biggl(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\,d\xi\biggr)\biggr|\,ds \\
&\le \int_0^\lambda \bigl|G(t,s)\bigr| \biggl| f_1\biggl(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\,d\xi\biggr) \\
&\quad - f_1\biggl(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\,d\xi\biggr)\biggr|\,ds \\
&\quad + \int_\lambda^\mu \bigl|G(t,s)\bigr| \biggl| f_2\biggl(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\,d\xi\biggr) \\
&\quad - f_2\biggl(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\,d\xi\biggr)\biggr|\,ds \\
&\quad + \int_\mu^1 \bigl|G(t,s)\bigr| \biggl| f_3\biggl(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\,d\xi\biggr) \\
&\quad \times f_3\biggl(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\,d\xi\biggr)\biggr|\,ds \\
&\le \int_0^\lambda \bigl|G(t,s)\bigr| \Bigl[\Lambda\bigl(\bigl|x_1(s) - x_2(s)\bigr|\bigr) + \Lambda\bigl(\bigl|x_1'(s) - x_2'(s)\bigr|\bigr) \\
&\quad + \Lambda\bigl(\bigl|D^\beta(x_1 - x_2)(s)\bigr|\bigr) + \Lambda\biggl(\biggl|\int_0^s h(\xi)\bigl(x_1(\xi) - x_2(\xi)\bigr)\,d\xi\biggr|\biggr)\biggr]\,ds \\
&\quad + \int_\lambda^\mu \bigl|G(t,s)\bigr| \Bigl[a_1(s)\bigl|x_1(s) - x_2(s)\bigr| + a_2(s)\bigl|x_1'(s) - x_2'(s)\bigr| \\
&\quad + a_3(s)\bigl|D^\beta(x_1 - x_2)(s)\bigr| + a_4(s)\biggl|\int_0^s h(\xi)\bigl(x_1(\xi) - x_2(\xi)\bigr)\,d\xi\biggr|\biggr]\,ds \\
&\quad + \int_\mu^\lambda \bigl|G(t,s)\bigr| \Bigl[\Lambda'\bigl(\bigl|x_1(s) - x_2(s)\bigr|\bigr) + \Lambda'\bigl(\bigl|x_1'(s) - x_2'(s)\bigr|\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ \Lambda'\big(\big|D^\beta(x_1-x_2)(s)\big|\big) + \Lambda'\left(\left|\int_0^s h(\xi)\big(x_1(\xi)-x_2(\xi)\big)\,d\xi\right|\right)\Bigg]\,ds \\
&\le \int_0^\lambda \big|G(t,s)\big|\Bigg[\Lambda\big(\|x_1-x_2\|\big) + \Lambda\big(\big\|x_1'-x_2'\big\|\big) \\
&+ \Lambda\big(\big\|D^\beta(x_1-x_2)\big\|\big) + \Lambda\left(\int_0^s \big|h(\xi)\big|\|x_1-x_2\|\,d\xi\big|\right)\Bigg]\,ds \\
&+ \int_\lambda^\mu \big|G(t,s)\big|\Bigg[a_1(s)\|x_1-x_2\| + a_2(s)\big\|x_1'-x_2'\big\| \\
&+ a_3(s)\big\|D^\beta(x_1-x_2)\big\| + a_4(s)\int_0^s \big|h(\xi)\big|\|x_1-x_2\|\,d\xi\big|\Bigg]\,ds \\
&+ \int_\mu^\lambda \big|G(t,s)\big|\Bigg[\Lambda'\big(\|x_1-x_2\|\big) + \Lambda'\big(\big\|x_1'-x_2'\big\|\big) \\
&+ \Lambda'\big(|D^\beta\|x_1-x_2\|\big) + \Lambda'\left(\int_0^s \big|h(\xi)\big|\|x_1-x_2\|\,d\xi\big|\right)\Bigg]\,ds \\
&\le \int_0^\lambda \big|G(t,s)\big|\Bigg[\Lambda\big(\|x_1-x_2\|\big) + \Lambda\big(\big\|x_1'-x_2'\big\|\big) \\
&+ \Lambda\left(\frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)}\right) + \Lambda\big(m_0\|x_1-x_2\|\big)\Bigg]\,ds \\
&+ \int_\lambda^\mu \big|G(t,s)\big|\Bigg[a_1(s)\|x_1-x_2\| + a_2(s)\big\|x_1'-x_2'\big\| \\
&+ a_3(s)\frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)} + a_4(s)m_0\|x_1-x_2\|\Bigg]\,ds \\
&+ \int_\mu^\lambda \big|G(t,s)\big|\Bigg[\Lambda'\big(\|x_1-x_2\|\big) + \Lambda'\big(\big\|x_1'-x_2'\big\|\big) \\
&+ \Lambda'\left(\frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)}\right) + \Lambda'\big(m_0\|x_1-x_2\|\big)\Bigg]\,ds \\
&\le \int_0^\lambda \big|G(t,s)\big|\Bigg[\Lambda\big(l\|x_1-x_2\|_*\big) + \Lambda\big(l\|x_1-x_2\|_*\big) \\
&+ \Lambda\left(\frac{l\|x_1-x_2\|_*}{\Gamma(2-\beta)}\right) + \Lambda\big(m_0 l\|x_1-x_2\|_*\big)\Bigg]\,ds \\
&+ \int_\lambda^\mu \big|G(t,s)\big|\Bigg[a_1(s)l\|x_1-x_2\|_* + a_2(s)l\|x_1-x_2\|_* \\
&+ a_3(s)\frac{l\|x_1-x_2\|_*}{\Gamma(2-\beta)} + a_4(s)m_0 l\|x_1-x_2\|_*\Bigg]\,ds \\
&+ \int_\mu^\lambda \big|G(t,s)\big|\Bigg[\Lambda'\big(l\|x_1-x_2\|_*\big) + \Lambda'\big(l\|x_1-x_2\|_*\big) \\
&+ \Lambda'\left(\frac{l\|x_1-x_2\|_*}{\Gamma(2-\beta)}\right) + \Lambda'\big(m_0 l\|x_1-x_2\|_*\big)\Bigg]\,ds, \qquad (*)
\end{aligned}
$$

where $m_0 = \int_0^1 |h(\xi)|\,d\xi$ and $l = \max\{1, \frac{1}{\Gamma(2-\beta)}, m_0, \theta_0 + \theta_1\}$. On the other hand, $\lim_{z\to 0^+} \frac{\Lambda(z)}{z} = q$ and so for each $\epsilon > 0$ there exists $0 < \delta_\Lambda = \delta(\epsilon, \Lambda)$ such that $|\frac{\Lambda(z)}{z} - q| < \epsilon$ for all $0 < z \le \delta_\Lambda$. Thus, $0 < z \le \delta_\Lambda$ implies $|\frac{\Lambda(z)}{z}| - q \le |\frac{\Lambda(z)}{z} - q| < \epsilon$. Hence, $|\Lambda(z)| < (\epsilon + q)|z|$.

By choosing $0 < z \le \delta_1 := \min\{\delta_\Lambda, \epsilon\}$, we have

$$\left|\Lambda(z)\right| < (\epsilon + q)|z| < (\epsilon + q)\epsilon. \tag{2}$$

For Λ' we have similar conclusion, that is,

$$\left|\Lambda'(z)\right| < \left(\epsilon + q'\right)\epsilon \tag{3}$$

for all $0 < z \le \delta_1 := \min\{\delta_{\Lambda'}, \epsilon\}$. Let $\epsilon > 0$ be given, $l\|x_1 - x_2\|_* < \min\{\delta_1, \delta_2\}$ and $x_1 \to x_2$. By using (2) and (3), we get $\Lambda(l\|x_1 - x_2\|_*) < (\epsilon + q)\epsilon$ and $\Lambda'(l\|x_1 - x_2\|_*) < (\epsilon + q')\epsilon$. Now by using (*), we obtain

$$\begin{aligned}
&\left|T_{x_1}(t) - T_{x_2}(t)\right| \\
&\quad\le 4(q+\epsilon)\epsilon \int_0^\lambda \left|G(t,s)\right| ds + \epsilon \int_\lambda^\mu \left[a_1(s) + \cdots + a_4(s)\right]\left|G(t,s)\right| ds \\
&\qquad + 4\left(q' + \epsilon\right)\epsilon \int_\mu^1 \left|G(t,s)\right| ds \le 4(q+\epsilon)\epsilon \frac{2+\alpha+T_0}{\Gamma(\alpha)} \int_0^\lambda (1-s)^{\alpha-2}\, ds \\
&\qquad + \frac{\epsilon(2+\alpha+T_0)}{\Gamma(\alpha)} \sum_{i=1}^4 \int_\lambda^\mu a_i(s)(1-s)^{\alpha-2}\, ds + 4\left(q'+\epsilon\right)\epsilon \frac{2+\alpha+T_0}{\Gamma(\alpha)} \int_\mu^1 (1-s)^{\alpha-2}\, ds \\
&\quad= \epsilon \frac{2+\alpha+T_0}{\Gamma(\alpha)} \Bigg[4(q+\epsilon)\cdot \frac{1}{\alpha-1}\left(1-(1-\lambda)^{\alpha-1}\right) \\
&\qquad + \sum_{i=1}^4 \|\hat{a}_i\|_{[\lambda,\mu]} + 4\left(q'+\epsilon\right)\cdot \frac{1}{\alpha-1}(1-\mu)^{\alpha-1}\Bigg].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|T_{x_1} - T_{x_2}\| \le \epsilon \frac{2+\alpha+T_0}{\Gamma(\alpha)} \Bigg[&4(q+\epsilon)\cdot \frac{1}{\alpha-1}\left(1-(1-\lambda)^{\alpha-1}\right) \\
&+ \sum_{i=1}^4 \|\hat{a}_i\|_{[\lambda,\mu]} + 4\left(q'+\epsilon\right)\cdot \frac{1}{\alpha-1}(1-\mu)^{\alpha-1}\Bigg].
\end{aligned}$$

In a similar way, we get

$$\begin{aligned}
&\left|T'_{x_1}(t) - T'_{x_2}(t)\right| \\
&\quad\le \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| \bigg| f\left(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\, d\xi\right) \\
&\qquad - f\left(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\, d\xi\right)\bigg|\, ds \\
&\quad\le \int_0^\lambda \left|\frac{\partial G}{\partial t}(t,s)\right| \bigg| f_1\left(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)\, d\xi\right) \\
&\qquad - f_1\left(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)\, d\xi\right)\bigg|\, ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\lambda}^{\mu} \left| \frac{\partial G}{\partial t}(t,s) \right| \bigg| f_2\left(s, x_1(s), x_1'(s), D^{\beta} x_1(s), \int_0^s h(\xi) x_1(\xi)\, d\xi \right) \\
&- f_2\left(s, x_2(s), x_2'(s), D^{\beta} x_2(s), \int_0^s h(\xi) x_2(\xi)\, d\xi \right) \bigg|\, ds \\
&+ \int_{\mu}^{1} \left| \frac{\partial G}{\partial t}(t,s) \right| \bigg| f_3\left(s, x_1(s), x_1'(s), D^{\beta} x_1(s), \int_0^s h(\xi) x_1(\xi)\, d\xi \right) \\
&- f_3\left(s, x_2(s), x_2'(s), D^{\beta} x_2(s), \int_0^s h(\xi) x_2(\xi)\, d\xi \right) \bigg|\, ds \\
\le{}& 4(q+\epsilon)\epsilon \int_0^{\lambda} \left| \frac{\partial G}{\partial t}(t,s) \right| ds + \epsilon \int_{\lambda}^{\mu} \big[a_1(s) + \cdots + a_4(s) \big] \big| G(t,s) \big|\, ds \\
&+ 4\big(q'+\epsilon\big)\epsilon \int_{\mu}^{1} \left| \frac{\partial G}{\partial t}(t,s) \right| ds \le 4(q+\epsilon) \frac{3\epsilon\alpha}{\Gamma(\alpha)} \int_0^{\lambda} (1-s)^{\alpha-2}\, ds \\
&+ \frac{3\epsilon\alpha}{\Gamma(\alpha)} \sum_{i=1}^{4} \int_{\lambda}^{\mu} a_i(s)(1-s)^{\alpha-2}\, ds + 4\big(q'+\epsilon\big) \frac{3\epsilon\alpha}{\Gamma(\alpha)} \int_{\mu}^{1} (1-s)^{\alpha-2}\, ds \\
={}& \frac{3\epsilon\alpha}{\Gamma(\alpha)} \Bigg[4(q+\epsilon) \cdot \frac{1}{\alpha-1} \big(1 - (1-\lambda)^{\alpha-1} \big) \\
&+ \sum_{i=1}^{4} \|\hat{a}_i\|_{[\lambda,\mu]} + 4\big(q'+\epsilon\big) \cdot \frac{1}{\alpha-1} (1-\mu)^{\alpha-1} \Bigg]
\end{aligned}
$$

and so

$$
\begin{aligned}
\big\| T'_{x_1} - T'_{x_2} \big\| \le{}& \frac{3\epsilon\alpha}{\Gamma(\alpha)} \Bigg[4(q+\epsilon) \cdot \frac{1}{\alpha-1} \big(1 - (1-\lambda)^{\alpha-1} \big) \\
&+ \sum_{i=1}^{4} \|\hat{a}_i\|_{[\lambda,\mu]} + 4\big(q'+\epsilon\big) \cdot \frac{1}{\alpha-1} (1-\mu)^{\alpha-1} \Bigg].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|T_{x_1} - T_{x_2}\|_* \le{}& \epsilon \Bigg[4(q+\epsilon) \cdot \frac{1}{\alpha-1} \big(1 - (1-\lambda)^{\alpha-1} \big) + \sum_{i=1}^{4} \|\hat{a}_i\|_{[\lambda,\mu]} \\
&+ 4\big(q'+\epsilon\big) \cdot \frac{1}{\alpha-1} (1-\mu)^{\alpha-1} \Bigg] \max \left\{ \frac{2+\alpha+T_0}{\Gamma(\alpha)}, \frac{3\alpha}{\Gamma(\alpha)} \right\}.
\end{aligned}
$$

This implies that $\|T_{x_1} - T_{x_2}\|_* \to 0$ as $x_1 \to x_2$. Hence, T is continuous. Since $\lim_{z\to 0^+} \frac{\Lambda(z)}{z} = q$ and Λ is nondecreasing, for each $\epsilon > 0$ there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that $\frac{\Lambda(lz)}{lz} < q + \epsilon$ for all $z \in (0, \delta_1]$. Thus, $\Lambda(lz) < (q+\epsilon)lz$. By using similar reason, there exists $\delta_2(\epsilon) > 0$ such that $\Lambda'(lz) < (q'+\epsilon)lz$ for all $z \in (0, \delta_2]$. Put $\delta = \delta(\epsilon) := \min\{\delta_1(\epsilon), \delta_2(\epsilon)\}$. Then $\Lambda(lz) < (q+\epsilon)lz$ and $\Lambda'(lz) < (q'+\epsilon)lz$ for all $z \in (0, \delta]$. In particular, $\Lambda(l\delta) < (q+\epsilon)l\delta$ and $\Lambda'(l\delta) < (q'+\epsilon)l\delta$. On other hand, we have $\frac{4q}{\alpha-1}(1-(1-\lambda)^{\alpha-1}) + \sum_{i=1}^{4} \|\hat{a}_i\|_{[\lambda,\mu]} + \frac{4q'}{\alpha-1}(1-\mu)^{\alpha-1} < \frac{\Gamma(\alpha)}{l\theta_0}$. Choose $\epsilon_0 > 0$ such that $\frac{4(q+\epsilon_0)}{\alpha-1}(1-(1-\lambda)^{\alpha-1}) + \sum_{i=1}^{4} \|\hat{a}_i\|_{[\lambda,\mu]} + \frac{4(q'+\epsilon_0)}{\alpha-1}(1-\mu)^{\alpha-1} < \frac{\Gamma(\alpha)}{l\theta_0}$ and put $\delta_0 = \delta(\epsilon_0)$. Then $\Lambda(l\delta_0) < (q+\epsilon)l\delta_0$ and $\Lambda'(l\delta_0) < (q'+\epsilon)l\delta_0$. Now, assume that $E = \{x \in X :$

$\|x\|_* < \delta_0\}$, $x \in E$ and $t \in [0,1]$. Then we have

$$\begin{aligned}
\left|T_x(t)\right| &\le \int_0^1 \left|G(t,s)\right| \left| f\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \\
&= \int_0^\lambda \left|G(t,s)\right| \left| f_1\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \\
&\quad + \int_\lambda^\mu \left|G(t,s)\right| \left| f_2\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \\
&\quad + \int_\mu^1 \left|G(t,s)\right| \left| f_3\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \\
&= \int_0^\lambda \left|G(t,s)\right| \left| f_1\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right) - f_1(s,0,0,0,0,0)\right| ds \\
&\quad + \int_\lambda^\mu \left|G(t,s)\right| \left| f_2\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right) - f_2(s,0,0,0,0,0)\right| ds \\
&\quad + \int_\mu^1 \left|G(t,s)\right| \left| f_3\left(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right) - f_3(s,0,0,0,0,0)\right| ds \\
&\le \int_0^\lambda \left|G(t,s)\right| \Big[\Lambda\left(\left|x(s)\right|\right) + \Lambda\left(\left|x'(s)\right|\right) + \Lambda\left(\left|D^\beta x(s)\right|\right) \\
&\quad + \Lambda\left(\int_0^s \left|h(\xi)\right|\left|x(\xi)\right| d\xi\right)\Big] ds \\
&\quad + \int_\lambda^\mu \left|G(t,s)\right| \Big[a_1(s)\left|x(s)\right| + a_2(s)\left|x'(s)\right| + a_3(s)\left|D^\beta x(s)\right| \\
&\quad + a_4(s)\int_0^s \left|h(\xi)\right|\left|x(\xi)\right| d\xi\Big] ds \\
&\quad + \int_\mu^1 \left|G(t,s)\right| \Big[\Lambda'\left(\left|x(s)\right|\right) + \Lambda'\left(\left|x'(s)\right|\right) + \Lambda'\left(\left|D^\beta x(s)\right|\right) \\
&\quad + \Lambda'\left(\int_0^s \left|h(\xi)\right|\left|x(\xi)\right| d\xi\right)\Big] ds \\
&\le \int_0^\lambda \left|G(t,s)\right| \left[\Lambda\left(\|x\|\right) + \Lambda\left(\|x'\|\right) + \Lambda\left(\frac{\|x'\|}{\Gamma(2-\beta)}\right) + \Lambda\left(m_0\|x\|\right)\right] ds \\
&\quad + \int_\lambda^\mu \left|G(t,s)\right| \left[a_1(s)\|x\| + a_2(s)\|x'\| + a_3(s)\frac{\|x'\|}{\Gamma(2-\beta)} + a_4(s)m_0\|x\|\right] ds \\
&\quad + \int_\mu^1 \left|G(t,s)\right| \left[\Lambda'\left(\|x\|\right) + \Lambda'\left(\|x'\|\right) + \Lambda'\left(\frac{\|x'\|}{\Gamma(2-\beta)}\right) + \Lambda'\left(m_0\|x\|\right)\right] ds \\
&\le 4\Lambda\left(l\|x\|_*\right) \int_0^\lambda \left|G(t,s)\right| ds \\
&\quad + l\|x\|_* \int_\lambda^\mu \left|G(t,s)\right| \sum_{i=1}^4 a_i(s)\,ds + 4\Lambda\left(l\|x\|_*\right)\int_\mu^1 \left|G(t,s)\right| ds \\
&\le 4\Lambda(l\delta_0)\frac{2+\alpha+T_0}{\Gamma(\alpha)} \int_0^\lambda (1-s)^{\alpha-2}\,ds \\
&\quad + l\delta_0 \frac{2+\alpha+T_0}{\Gamma(\alpha)} \sum_{i=1}^4 \int_\lambda^\mu a_i(s)(1-s)^{\alpha-2}\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ 4\Lambda'(l\delta_0)\frac{2+\alpha+T_0}{\Gamma(\alpha)}\int_\mu^1 (1-s)^{\alpha-2}\,ds \\
&\le 4(q+\epsilon_0)l\delta_0\frac{(2+\alpha+T_0)}{\Gamma(\alpha)}\cdot\frac{1}{\alpha-1}\bigl[1-(1-\lambda)^{\alpha-1}\bigr] \\
&+ l\delta_0\frac{(2+\alpha+T_0)}{\Gamma(\alpha)}\sum_{i=1}^4 \|\hat{a}_i\|_{[\lambda,\mu]} \\
&+ 4(q+\epsilon_0)l\delta_0\frac{(2+\alpha+T_0)}{\Gamma(\alpha)}\cdot\frac{1}{\alpha-1}\bigl[1-(1-\lambda)^{\alpha-1}\bigr] \\
&= \delta_0\frac{(2+\alpha+T_0)l}{\Gamma(\alpha)}\Bigl[\frac{4(q+\epsilon_0)}{\alpha-1}\bigl(1-(1-\lambda)^{\alpha-1}\bigr) \\
&+ \sum_{i=1}^4 \|\hat{a}_i\|_{[\lambda,\mu]} + \frac{4(q'+\epsilon_0)}{\alpha-1}(1-\mu)^{\alpha-1} < \delta_0.
\end{aligned}
$$

Hence, $\|Tx\| \le \delta_0$. By using a similar method, we get

$$
\begin{aligned}
\bigl|T_x'(t)\bigr| &\le \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right|\left|f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)\right|ds \\
&= \int_0^\lambda \left|\frac{\partial G}{\partial t}(t,s)\right|\left|f_1\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)\right|ds \\
&+ \int_\lambda^\mu \left|\frac{\partial G}{\partial t}(t,s)\right|\left|f_2\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)\right|ds \\
&+ \int_\mu^1 \left|\frac{\partial G}{\partial t}(t,s)\right|\left|f_3\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)\right|ds \\
&\le \int_0^\lambda \left|\frac{\partial G}{\partial t}(t,s)\right|\Bigl[\Lambda\bigl(|x(s)|\bigr)+\Lambda\bigl(|x'(s)|\bigr)+\Lambda\bigl(|D^\beta x(s)|\bigr) \\
&+ \Lambda\left(\int_0^s |h(\xi)||x(\xi)|\,d\xi\right)\Bigr]ds \\
&+ \int_\lambda^\mu \left|\frac{\partial G}{\partial t}(t,s)\right|\Bigl[a_1(s)|x(s)|+a_2(s)|x'(s)|+a_3(s)|D^\beta x(s)| \\
&+ a_4(s)\int_0^s |h(\xi)||x(\xi)|\,d\xi\Bigr]ds \\
&+ \int_\mu^1 \left|\frac{\partial G}{\partial t}(t,s)\right|\Bigl[\Lambda'\bigl(|x(s)|\bigr)+\Lambda'\bigl(|x'(s)|\bigr)+\Lambda'\bigl(|D^\beta x(s)|\bigr) \\
&+ \Lambda'\left(\int_0^s |h(\xi)||x(\xi)|\,d\xi\right)\Bigr]ds \\
&\le \int_0^\lambda \left|\frac{\partial G}{\partial t}(t,s)\right|\left[\Lambda\bigl(\|x\|\bigr)+\Lambda\bigl(\|x'\|\bigr)+\Lambda\left(\frac{\|x'\|}{\Gamma(2-\beta)}\right)+\Lambda\bigl(m_0\|x\|\bigr)\right]ds \\
&+ \int_\lambda^\mu \left|\frac{\partial G}{\partial t}(t,s)\right|\left[a_1(s)\|x\|+a_2(s)\|x'\|+a_3(s)\frac{\|x'\|}{\Gamma(2-\beta)}+a_4(s)m_0\|x\|\right]ds \\
&+ \int_\mu^1 \left|\frac{\partial G}{\partial t}(t,s)\right|\left[\Lambda'\bigl(\|x\|\bigr)+\Lambda'\bigl(\|x'\|\bigr)+\Lambda'\left(\frac{\|x'\|}{\Gamma(2-\beta)}\right)+\Lambda'\bigl(m_0\|x\|\bigr)\right]ds
\end{aligned}
$$

$$
\begin{aligned}
&\le 4\Lambda\big(l\|x\|_*\big)\int_0^\lambda\left|\frac{\partial G}{\partial t}(t,s)\right|ds + l\|x\|_*\int_\lambda^\mu\left|\frac{\partial G}{\partial t}(t,s)\right|\sum_{i=1}^4 a_i(s)\,ds + 4\Lambda\big(l\|x\|_*\big)\\
&\quad\times\int_\mu^1\big|G(t,s)\big|\,ds \le 4\Lambda(l\delta_0)\frac{3\alpha}{\Gamma(\alpha)}\int_0^\lambda(1-s)^{\alpha-2}\,ds\\
&\quad + l\delta_0\frac{3\alpha}{\Gamma(\alpha)}\sum_{i=1}^4\int_\lambda^\mu a_i(s)(1-s)^{\alpha-2}\,ds + 4\Lambda'(l\delta_0)\frac{3\alpha}{\Gamma(\alpha)}\int_\mu^1(1-s)^{\alpha-2}\,ds\\
&\le 4(q+\epsilon_0)l\delta_0\frac{3\alpha}{\Gamma(\alpha)}\cdot\frac{1}{\alpha-1}\big[1-(1-\lambda)^{\alpha-1}\big] + l\delta_0\frac{3\alpha}{\Gamma(\alpha)}\sum_{i=1}^4\|\hat{a}_i\|_{[\lambda,\mu]}\\
&\quad + 4(q+\epsilon_0)l\delta_0\frac{3\alpha}{\Gamma(\alpha)}\cdot\frac{1}{\alpha-1}\big[1-(1-\lambda)^{\alpha-1}\big]\\
&= \delta_0\frac{3\alpha l}{\Gamma(\alpha)}\left[\frac{4(q+\epsilon_0)}{\alpha-1}\big(1-(1-\lambda)^{\alpha-1}\big) + \sum_{i=1}^4\|\hat{a}_i\|_{[\lambda,\mu]} + \frac{4(q'+\epsilon_0)}{\alpha-1}(1-\mu)^{\alpha-1}\right]\\
&< \delta_0
\end{aligned}
$$

for all $x \in E$ and $t \in [0,1]$. Hence, $\|Tx\| \le \delta_0$, and so $\|Tx\|_* \le \delta_0$. Thus, T maps E into E. It is easy to check that T maps bounded sets into bounded sets. Assume that $t_1, t_2 \in [0,1]$ and $x \in E$. Since $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t}$ are continuous with respect to t, we get

$$
\begin{aligned}
\lim_{t_2\to t_1} T'x(t_2) &= \lim_{t_2\to t_1}\int_0^1\left|\frac{\partial G}{\partial t}(t_2,s)\right|\left|f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)\right|ds\\
&= \int_0^1 \lim_{t_2\to t_1}\frac{\partial G}{\partial t}(t_2,s)f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)ds\\
&= \int_0^1 \frac{\partial G}{\partial t}(t_1,s)f\left(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)ds\\
&= T'x(t_1).
\end{aligned}
$$

Hence, T is equi-continuous on E and so $T: E \to E$ is completely continuous. Now by using Lemma 2, T has a fixed point on E and so the problem (1) has a solution. □

Example 1 Consider the pointwise defined equation

$$
D^{\frac{7}{2}}x(t) + f\left(t,x(t),x'(t),D^{\frac{1}{2}}x(t),\int_0^t sx(s)\,ds\right) = 0
$$

with the boundary conditions in the last result, where

$$
f(t,x_1,x_2,x_3,x_4,x_5) = \begin{cases} t\sum_{i=1}^4 x_i, & 0 \le t < 0.2,\\ d(t)\sum_{i=1}^4 x_i, & 0.2 \le t \le 0.8,\\ t^2\sum_{i=1}^4 x_i, & 0.8 < t \le 1,\end{cases}
$$

and $d(t) = 0$ whenever $t \in [0.1, 0.8] \cap Q$ and $d(t) = 0.1$ whenever $t \in [0.1, 0.8] \cap Q^c$. Now, put $f_1(t,x_1,x_2,x_3,x_4,x_5) = \frac{1}{2}t\sum_{i=1}^4 x_i$, $f_2(t,x_1,x_2,x_3,x_4,x_5) = d(t)\sum_{i=1}^4 x_i$ and $f_3(t,x_1,x_2,x_3,x_4,$

$x_5) = t^2 \sum_{i=1}^4 x_i$. Then we have $f_1(t,0,0,0,0) = f_2(t,0,0,0,0) = f_3(t,0,0,0,0) = 0$,

$$\begin{aligned}
\bigl|f_1(t,x_1,x_2,x_3,x_4) - f_1(t,y_1,y_2,y_3,y_4)\bigr| &\le t \sum_{i=1}^4 \bigl||x_i| - |y_i|\bigr| \\
&\le t \sum_{i=1}^4 |x_i - y_i| \le 0.1 \sum_{i=1}^4 \Lambda\bigl(|x_i - y_i|\bigr), \\
\bigl|f_2(t,x_1,x_2,x_3,x_4) - f_2(t,y_1,y_2,y_3,y_4)\bigr| &\le d(t) \sum_{i=1}^4 \bigl||x_i| - |y_i|\bigr| \\
&\le d(t) \sum_{i=1}^4 |x_i - y_i|,
\end{aligned}$$

and

$$\begin{aligned}
\bigl|f_3(t,x_1,x_2,x_3,x_4) - f_3(t,y_1,y_2,y_3,y_4)\bigr| &\le \frac{1}{2} t \sum_{i=1}^4 \bigl||x_i| - |y_i|\bigr| \\
&\le \frac{1}{2} t \sum_{i=1}^4 |x_i - y_i| \le \frac{1}{2} \sum_{i=1}^4 \Lambda\bigl(|x_i - y_i|\bigr),
\end{aligned}$$

where $\Lambda(x) = |x|$ and $\Lambda'(x) = \frac{1}{2}|x|$. Hence, $\lim_{z\to 0^+} \frac{\Lambda'(z)}{z} = 0.1 := q$, $\lim_{z\to 0^+} \frac{\Lambda'(z)}{z} = \frac{1}{2} := q'$, $\hat{a}_i = \hat{d} \in L^1[0.1, 0.8]$, $\sum_{i=1}^4 \|\hat{a}_i\|_{[\lambda,\mu]} < 0.092$ and

$$\begin{aligned}
&\left[\frac{4q(1-(1-\lambda)^{\alpha-1})}{\alpha - 1} + \sum_{i=1}^4 \|\hat{a}_i\| + \frac{4q'}{\alpha-1}(1-\mu)^{\alpha-1}\right] \\
&\quad < \left[\frac{4\times 0.1(1-(1-0.1)^{\frac{5}{2}}}{\frac{5}{2}} + 0.092 + \frac{4\times 0.5(1-0.9)^{\frac{5}{2}}}{\frac{5}{2}}\right] < \frac{\Gamma(\alpha)}{3l\alpha}.
\end{aligned}$$

Now by using Theorem 5, the problem has a solution.

Now, we present our second result by using different conditions.

Theorem 6 *Suppose that* $f = [f_1, f_2, f_3, \lambda, \mu]$, f *is nonnegative on* $[0,1]$ *and there exist nonnegative functions* $a_1, a_2, a_3, a_4 : [0,\lambda] \to \mathbb{R}^+$, *maps* $b_1, \dots, b_{k_0} : [\lambda,\mu] \to \mathbb{R}^+$ *for some* $k_0 \ge 1$, *and functions* $c_1, c_2, c_3, c_4 : [\mu, 1] \to \mathbb{R}^+$ *such that* $\hat{a}_i \in L^1[0,\lambda]$, $\hat{b}_j \in L^1[\lambda,\mu]$, $\hat{c}_i \in L^1[\mu,1]$ *and* $\hat{a}_1(s) = (1-s)^{\alpha-2} a_i(s)$. *Assuming that there are nonnegative and nondecreasing functions* $\phi_i, \Phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $H_j : {\mathbb{R}_+}^4 \to \mathbb{R}_+$ *such that* $\lim_{z\to 0^+} \frac{\phi_i(z)}{z^{\mu_i}} := l_{\mu_i} < \infty$, $\lim_{z\to 0^+} \frac{\Phi_i(z)}{z^{\gamma_i}} := l_{\gamma_i} < \infty$ *and* $\lim_{z\to 0^+} \frac{H_j(z,z,z,z)}{z^m} := q_j < \infty$ *for some* $\mu_i, \gamma_i, m \in [1,\infty)$ *and* H_j *are nonnegative and nondecreasing with respect to all their components* $(1 \le i \le 4,\ 1 \le j \le k_0)$,

$$\begin{aligned}
\bigl|f_1(t,x_1,\dots,x_4) - f_1(t,y_1,\dots,y_4)\bigr| &\le \sum_{i=1}^4 a_i(t)\phi_i\bigl(|x_i - y_i|\bigr), \\
\bigl|f_2(t,x_1,\dots,x_4) - f_2(t,y_1,\dots,y_4)\bigr| &\le \sum_{i=1}^5 b_j(t) H_j\bigl(|x_1 - y_1|, \dots, |x_4 - y_4|\bigr)
\end{aligned}$$

and $|f_3(t,x_1,\dots,x_4)-f_3(t,y_1,\dots,y_4)| \le \sum_{i=1}^{4} c_i(t)\Phi_i(|x_i-y_i|)$. *Suppose that* $|f_2(t,x_1,\dots,x_4)| \le \Theta(t)\Lambda(x_1,\dots,x_4)$, *where* Λ *are nonnegative and nondecreasing with respect to all their components*, $\lim_{x\to 0^+}\frac{\Lambda(x,x,x,x)}{x} := P_2, < \infty$, $\hat{\Theta}\in L^1[\lambda,\mu]$, $\lim_{\max|x_i|\to 0}\frac{|f_1(t,x_1,\dots,x_4)|}{\max|x_i|} = P_1(t)$ *and* $\lim_{\max|x_i|\to 0}\frac{|f_3(t,x_1,\dots,x_4)|}{\max|x_i|} = P_3(t)$, *where* $\hat{P}_1\in L^1[0,\lambda]$, $\hat{P}_3\in L^1[\mu,1]$. *If*

$$\max\left\{\frac{3\alpha}{\Gamma(\alpha)},\frac{2+\alpha+T_0}{\Gamma(\alpha)}\right\}\max\left\{\sum_{i=1}^{5}\|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i})+\sum_{j=1}^{k_0}\|\hat{b}_j\|_{[\lambda,\mu]}q_j\right.$$
$$\left.+\sum_{i=1}^{4}\|\hat{c}_i\|_{[\mu,1]}](l_{\gamma_i}),\max\left\{1,\frac{1}{\Gamma(2-\beta)},m_0\right\}\left[\|\hat{P}_1\|_{[0,\lambda]}+P_2\|\Theta\|_{[\lambda,\mu]}+\|\hat{P}_3\|_{[\mu,1]}\right]\right\}<1,$$

then the pointwise defined equation (1) *with boundary conditions has a solution.*

Proof Let $x,y\in X$ and $t\in[0,1]$. Then we have

$$\begin{aligned}
&\left|F_x(t)-F_y(t)\right|\\
&\le\int_0^{\lambda}\left|G(t,s)\right|\left|f_1\left(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)\right.\\
&\quad\left.-f_1\left(s,y(s),y'(s),D^{\beta}y(s),\int_0^s h(\xi)y(\xi)\,d\xi\right)\right|ds\\
&\quad+\int_{\lambda}^{\mu}\left|G(t,s)\right|\left|f_2\left(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)\right.\\
&\quad\left.-f_2\left(s,y(s),y'(s),D^{\beta}y(s),\int_0^s h(\xi)y(\xi)\,d\xi\right)\right|ds\\
&\quad+\int_{\mu}^{1}\left|G(t,s)\right|\left|f_3\left(s,x(s),x'(s),D^{\beta}x(s),\int_0^s h(\xi)x(\xi)\,d\xi\right)\right.\\
&\quad\left.-f_3\left(s,y(s),y'(s),D^{\beta}y(s),\int_0^s h(\xi)y(\xi)\,d\xi\right)\right|ds\\
&\le\int_0^{\lambda}\left|G(t,s)\right|\left[a_1(s)\phi\left(\left|x(s)-y(s)\right|\right)+a_2(s)\phi\left(\left|x'(s)-y'(s)\right|\right)\right.\\
&\quad\left.+a_3(s)\phi\left(\left|D^{\beta}x(s)-D^{\beta}y(s)\right|\right)+a_4(s)\phi\left(\left|\int_0^s h(\xi)\left(x(\xi)-y(\xi)\right)d\xi\right|\right)\right]ds\\
&\quad+\int_{\lambda}^{\mu}\left|G(t,s)\right|\sum_{i=1}^{k_0}b_i(s)H_i(\left|x(s)-y(s)\right|,\left|x'(s)-y'(s)\right|,\\
&\quad\left|D^{\beta}x(s)-D^{\beta}y(s)\right|,\left|\int_0^s h(\xi)\left(x(\xi)-y(\xi)\,d\xi\right)\right|ds\\
&\quad+\int_{\mu}^{1}\left|G(t,s)\right|\left[c_1(s)\Phi\left(\left|x(s)-y(s)\right|\right)+c_2(s)\Phi\left(\left|x'(s)-y'(s)\right|\right)\right.\\
&\quad\left.+c_3(s)\Phi\left(\left|D^{\beta}x(s)-D^{\beta}y(s)\right|\right)+c_4(s)\Phi(\left|\int_0^s h(\xi)\left(x(\xi)-y(\xi)\right)d\xi\right|\right]ds\\
&\le\frac{2+\alpha+T_0}{\Gamma(\alpha)}\int_0^{\lambda}(1-s)^{\alpha-2}\left[a_1(s)\phi\left(\|x-y\|\right)+a_2(s)\phi\left(\|x'-y'\|\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+ a_3(s)\phi\left(\frac{\|x'-y'\|}{\Gamma(2-\beta)}\right) + a_4(s)\phi\bigl(m_0\|x-y\|\bigr)\,ds \\
&+ \frac{2+\alpha+T_0}{\Gamma(\alpha)}\int_\lambda^\mu (1-s)^{\alpha-2} \\
&\times \sum_{i=1}^{k_0} b_i(s)H_i\left(\|x-y\|, \bigl\|x'-y'\bigr\|, \frac{\|x'-y'\|}{\Gamma(2-\beta)}, m_0\|x-y\|\right)ds \\
&+ \frac{2+\alpha+T_0}{\Gamma(\alpha)}\int_\mu^1 (1-s)^{\alpha-2}\Bigl[c_1(s)\Phi\bigl(\|x-y\|\bigr) + c_2(s)\Phi\bigl(\bigl\|x'-y'\bigr\|\bigr) \\
&+ c_3(s)\Phi\left(\frac{\|x'-y'\|}{\Gamma(2-\beta)}\right) + c_4(s)\Phi\bigl(m_0\|x-y\|\bigr)\Bigr]\,ds \\
\le{}& \frac{2+\alpha+T_0}{\Gamma(\alpha)}\Biggl[\sum_{i=1}^{4}\phi\bigl(l\|x-y\|_*\bigr)\int_0^\lambda (1-s)^{\alpha-2}a_i(s)\,ds \\
&+ \sum_{i=1}^{k_0} H_i\bigl(l\|x-y\|_*,\dots,l\|x-y\|_*\bigr)\int_\lambda^\mu (1-s)^{\alpha-2}b_i(s)\,ds \\
&+ \sum_{i=1}^{4}\Phi\bigl(l\|x-y\|_*\bigr)\int_\mu^1 (1-s)^{\alpha-2}c_i(s)\,ds\Biggr] \\
={}& \frac{2+\alpha+T_0}{\Gamma(\alpha)}\Biggl[\sum_{i=1}^{5}\phi\bigl(l\|x-y\|_*\bigr)\|\hat{a}_i\|_{[0,\lambda]} \\
&+ \sum_{i=1}^{k_0} H_i\bigl(l\|x-y\|_*,\dots,l\|x-y\|_*\bigr)\|\hat{b}_i\|_{[\lambda,\mu]} + \sum_{i=1}^{4}\Phi\bigl(l\|x-y\|_*\bigr)\|\hat{c}_i\|_{[\mu,1]}\Biggr].
\end{aligned}
$$

On the other hand, $\lim_{z\to 0^+}\frac{\phi_i(z)}{z^{\mu_i}} = l_{\mu_i}$ for $1\le i\le 4$. This implies that for each $\epsilon > 0$ there exists $0 < \delta_i = \delta_i(\epsilon) < \epsilon$ such that $\frac{\phi_i(z)}{z^{\mu_i}} < l_{\mu_i} + \epsilon$ for all $z \in (0,\delta_i]$. Hence, $\phi_i(\delta_i) < (l_{\mu_i}+\epsilon)\delta_i^{\mu_i} < (l_{\mu_i}+\epsilon)\epsilon^{\mu_i}$. By using a similar method, we conclude that there exists $0 < \delta_i' = \delta_i'(\epsilon) < \epsilon$ such that $\Phi_i(\delta_i') < (l_{\gamma_i}+\epsilon)(\delta_i')^{\gamma_i} < (l_{\gamma_i}+\epsilon)\epsilon^{\gamma_i}$. Also, we have $\lim_{z\to 0^+}\frac{H_j(z,z,z,z)}{z^m} = q_j$ for $1\le j\le k_0$ and so there exists $0 < \delta_{q_j} < \epsilon$ such that $\frac{H_j(z,z,z,z)}{z^m} < q_j + \epsilon$ for all $z\in(0,\delta_{q_j}]$ and $1\le j\le k_0$. Hence, $H_j(z,z,z,z) < (q_j+\epsilon)z^m$ for all $z\in(0,\delta_{q_j}]$ and so $H_j(\delta_{q_j},\delta_{q_j},\delta_{q_j},\delta_{q_j}) < (q_j+\epsilon)\delta_{q_j}^m < (q_j+\epsilon)\epsilon^m$. Let $x\to y$ in X. If $l\|x-y\|_* < \delta := \min\{\delta_1,\dots,\delta_4,\delta_1',\dots,\delta_4',\delta_{q_1},\dots,\delta_{q_{k_0}}\}$, then $\phi_i(\delta) < \phi_i(\delta_i) < (l_{\mu_i}+\epsilon)\epsilon^{\mu_i}$, $\Phi_i(\delta) < \Phi_i(\delta_i') < (l_{\gamma_i}+\epsilon)\epsilon^{\gamma_i}$ and $H_j(\delta,\dots,\delta) < H_j(\delta_{q_j},\dots,\delta_{q_j}) < (q_j+\epsilon)\epsilon^m$ for $1\le i\le 4$ and $1\le j\le k_0$. If $l\|x-y\|_* < \delta$, then $|F_x(t)-F_y(t)| \le \frac{2+\alpha+T_0}{\Gamma(\alpha)}[\sum_{i=1}^5\|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon)\epsilon^{\mu_i} + \sum_{j=1}^{k_0}\|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon)\epsilon^m + \sum_{i=1}^4\|\hat{c}_i\|_{[\mu,1]}](l_{\gamma_i}+\epsilon)\epsilon^{\gamma_i}]$ and so $|F_x - F_y\| \le \frac{2+\alpha+T_0}{\Gamma(\alpha)} \times [\sum_{i=1}^5\|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon)\epsilon^{\mu_i} + \sum_{j=1}^{k_0}\|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon)\epsilon^m + \sum_{i=1}^4\|\hat{c}_i\|_{[\mu,1]}](l_{\gamma_i}+\epsilon)\epsilon^{\gamma_i}]$. By a similar way, we get

$$
\begin{aligned}
\bigl\|F_x'-F_y'\bigr\| \le \frac{3\alpha}{\Gamma(\alpha)}\Biggl[&\sum_{i=1}^{5}\|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon)\epsilon^{\mu_i} + \sum_{j=1}^{k_0}\|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon)\epsilon^m \\
&+ \sum_{i=1}^{4}\|\hat{c}_i\|_{[\mu,1]}\Biggr](l_{\gamma_i}+\epsilon)\epsilon^{\gamma_i}]
\end{aligned}
$$

and so $\|F_x-F_y\|_* \le \max\{\frac{3\alpha}{\Gamma(\alpha)},\frac{2+\alpha+T_0}{\Gamma(\alpha)}\}[\sum_{i=1}^5\|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon)\epsilon^{\mu_i} + \sum_{j=1}^{k_0}\|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon)\epsilon^m + \sum_{i=1}^4\|\hat{c}_i\|_{[\mu,1]}](l_{\gamma_i}+\epsilon)\epsilon^{\gamma_i}]$. Since $\epsilon > 0$ was arbitrary, we conclude that $\|F_x-F_y\|_* \to 0$ as $x \to$

y. This implies that F is continuous on X. Since $\lim_{x\to 0^+}\frac{\Lambda(x,x,x,x)}{x}=P_2$, $\lim_{x\to 0^+}\frac{\Lambda(lx,lx,lx,lx)}{lx}=P_2$, where $l=\max\{1,\frac{1}{\Gamma(2-\beta)},m_0\}$. Thus for each $\epsilon>0$ there exists $\delta_1=\delta_1(\epsilon)$ such that $\frac{\Lambda(lx,lx,lx,lx)}{lx}<P_2+\epsilon$ for all $x\in(0,\delta_1]$. Hence,

$$\Lambda(lx,lx,lx,lx)<(P_2+\epsilon)lx \tag{4}$$

for $x\in(0,\delta_1]$. Also, $\lim_{|x_i|\to 0}\frac{|f_1(t,x_1,\ldots,x_4)|}{\min|x_i|}=P_1(t)$. Thus, there exists $\delta_2=\delta_2(\epsilon)$ such that

$$\bigl|f_1(t,x_1,\ldots,x_4)\bigr|<\bigl(P_1(t)+\epsilon\bigr)\min|x_i| \tag{5}$$

for all $t\in[0,1]$ and $|x_i|\in(0,\delta_2]$ for $1\le i\le 4$. Similarly, there exists $\delta_3=\delta_3(\epsilon)$ such that

$$\bigl|f_3(t,x_1,\ldots,x_4)\bigr|<\bigl(P_3(t)+\epsilon\bigr)\min|x_i| \tag{6}$$

for all $t\in[0,1]$ and $|x_i|\in(0,\delta_3]$ for $1\le i\le 4$. Since $\|\hat{P}_1\|_{[0,\lambda]}+P_2\|\Theta\|_{[\lambda,\mu]}+\|\hat{P}_3\|_{[\mu,1]}<\frac{\Gamma(\alpha)}{l\theta_0}$, we can choose $\epsilon_0>0$ such that $\|\hat{P}_1\|_{[0,\lambda]}+\frac{\epsilon_0}{\alpha-1}(1-(1-\lambda)^{\alpha-1})+(P_2+\epsilon_0)\|\Theta\|_{[\lambda,\mu]}+\|\hat{P}_3\|_{[\mu,1]}+\frac{\epsilon_0}{\alpha-1}(1-\mu)^{\alpha-1}<\frac{\Gamma(\alpha)}{l\theta_0}$. Since

$$\max\biggl\{\frac{3\alpha}{\Gamma(\alpha)},\frac{2+\alpha+T_0}{\Gamma(\alpha)}\biggr\}\Biggl[\sum_{i=1}^{5}\|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i})+\sum_{j=1}^{k_0}\|\hat{b}_j\|_{[\lambda,\mu]}q_j+\sum_{i=1}^{4}\|\hat{c}_i\|_{[\mu,1]}\Biggr](l_{\gamma_i})]<1,$$

pick $\epsilon_1\in(0,1)$ such that

$$\begin{aligned}&\max\biggl\{\frac{3\alpha}{\Gamma(\alpha)},\frac{2+\alpha+T_0}{\Gamma(\alpha)}\biggr\}\Biggl[\sum_{i=1}^{5}\|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon_1)\\&\quad+\sum_{j=1}^{k_0}\|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon_1)+\sum_{i=1}^{4}\|\hat{c}_i\|_{[\mu,1]}\Biggr](l_{\gamma_i}+\epsilon_1)]<1.\end{aligned} \tag{7}$$

Let $r_0=\min\{\delta_1(\epsilon_0),\delta_2(\epsilon_0),\delta_3(\epsilon_0),\frac{\epsilon_1}{2}\}$, and $C=\{x\in X:\|x\|_*<r_0\}$. Define the map α on $X\times X$ by

$$\alpha(x,y)=\begin{cases}1, & x,y\in C,\\0, & \text{otherwise.}\end{cases}$$

Let $x,y\in X$ and $\alpha(x,y)\ge 1$. Then $x,y\in C$ and so

$$\begin{aligned}\bigl|F_x(t)\bigr|&\le\int_0^\lambda\bigl|G(t,s)\bigr|\biggl|f_1\biggl(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\biggr)\biggr|\,ds\\&\quad+\int_\lambda^\mu\bigl|G(t,s)\bigr|\biggl|f_2\biggl(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\biggr)\biggr|\,ds\\&\quad+\int_\mu^1\bigl|G(t,s)\bigr|\biggl|f_3\biggl(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\biggr)\biggr|\,ds\\&\le\frac{2+\alpha+T_0}{\Gamma(\alpha)}\biggl[\int_0^\lambda(1-s)^{\alpha-2}\biggl|f_1\biggl(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)\,d\xi\biggr)\biggr|\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\lambda}^{\mu} (1-s)^{\alpha-2} \left| f_2\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \\
&+ \int_{\mu}^{1} (1-s)^{\alpha-2} \left| \mathfrak{Z}_2\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \Bigg] \\
\leq{}& \frac{2+\alpha+T_0}{\Gamma(\alpha)} \Bigg[\int_0^{\lambda} (1-s)^{\alpha-2} \left| f_1\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \\
&+ \int_{\lambda}^{\mu} (1-s)^{\alpha-2} \Theta(s) \Lambda\left(x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right) ds \\
&+ \int_{\mu}^{1} (1-s)^{\alpha-2} \left| \mathfrak{Z}_2\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \Bigg] \\
\leq{}& \frac{2+\alpha+T_0}{\Gamma(\alpha)} \Bigg[\int_0^{\lambda} (1-s)^{\alpha-2} \left| f_1\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \\
&+ \int_{\lambda}^{\mu} (1-s)^{\alpha-2} \Theta(s) \Lambda\left(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, m_0\|x\|\right) ds \\
&+ \int_{\mu}^{1} (1-s)^{\alpha-2} \left| \mathfrak{Z}_2\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \Bigg] \\
\leq{}& \frac{2+\alpha+T_0}{\Gamma(\alpha)} \Bigg[\int_0^{\lambda} (1-s)^{\alpha-2} \left| f_1\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \\
&+ \int_{\lambda}^{\mu} (1-s)^{\alpha-2} \Theta(s) \Lambda\big(l\|x\|_*, l\|x\|_*, l\|x\|_*, l\|x\|_*\big)\, ds \\
&+ \int_{\mu}^{1} (1-s)^{\alpha-2} \left| \mathfrak{Z}_2\left(s, x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right)\right| ds \Bigg]
\end{aligned}
$$

for all $t \in [0,1]$. Since $\|x\|_* < r_0$, $x \in [0, \min\{\delta_1, \delta_2, \delta_3\})$ and so by using (4), (5) and (6) we conclude that

$$
\begin{aligned}
\big|F_x(t)\big| \leq{}& \frac{2+\alpha+T_0}{\Gamma(\alpha)} \\
&\times \Bigg[\int_0^{\lambda} (1-s)^{\alpha-2} \big(P_1(s)+\epsilon\big) \min\left\{x_(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right\} ds \\
&+ (P_2+\epsilon_0) l\|x\|_* \int_{\lambda}^{\mu} (1-s)^{\alpha-2} \Theta(s)\, ds \\
&+ \int_{\mu}^{1} (1-s)^{\alpha-2} |\big(P_3(s)+\epsilon_0\big) \min\left\{x(s), x'(s), D^{\beta}x(s), \int_0^s h(\xi)x(\xi)\,d\xi\right\} ds \Bigg] \\
\leq{}& \frac{2+\alpha+T_0}{\Gamma(\alpha)} \Bigg[\int_0^{\lambda} (1-s)^{\alpha-2} \big(P_1(s)+\epsilon\big) \min\left\{\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, m_0\|x\|\right\} ds \\
&+ (P_2+\epsilon_0) l\|x\|_* \|\hat{\Theta}\|_{[\lambda,\mu]} \\
&+ \int_{\mu}^{1} (1-s)^{\alpha-2} |\big(P_3(s)+\epsilon_0\big) \min\left\{\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, m_0\|x\|\right\} ds \Bigg] \\
\leq{}& \frac{2+\alpha+T_0}{\Gamma(\alpha)} \Bigg[l\|x\|_* \int_0^{\lambda} (1-s)^{\alpha-2} \big(P_1(s)+\epsilon_0\big)\, ds + (P_2+\epsilon) l\|x\|_* \|\hat{\Theta}\|_{[\lambda,\mu]} \\
&+ l\|x\|_* \int_{\mu}^{1} (1-s)^{\alpha-2} \big(P_3(s)+\epsilon_0\big)\, ds \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2+\alpha+T_0}{\Gamma(\alpha)} l\|x\|_* \bigg[\int_0^\lambda (1-s)^{\alpha-2} P_1(s)\,ds + \epsilon_0 \int_0^\lambda (1-s)^{\alpha-2}\,ds \\
&\quad + (P_2+\epsilon_0)\|\hat{\Theta}\|_{[\lambda,\mu]} + \int_\mu^1 (1-s)^{\alpha-2} P_3(s)\,ds + \epsilon_0 \int_\mu^1 (1-s)^{\alpha-2}\,ds\bigg] \\
&= \frac{2+\alpha+T_0}{\Gamma(\alpha)} l\|x\|_* \bigg[\|\hat{P}_1\|_{[0,\lambda]} + \frac{\epsilon_0}{\alpha-1}\big(1-(1-\lambda)^{\alpha-1}\big) \\
&\quad + (P_2+\epsilon_0)\|\hat{\Theta}\|_{[\lambda,\mu]} + \|\hat{P}_3\|_{[\mu,1]} + \frac{\epsilon_0}{\alpha-1}(1-\mu)^{\alpha-1}\bigg] \\
&\le \theta_0 l\bigg[\|\hat{P}_1\|_{[0,\lambda]} + \frac{\epsilon_0}{\alpha-1}\big(1-(1-\lambda)^{\alpha-1}\big) \\
&\quad + (P_2+\epsilon_0)\|\hat{\Theta}\|_{[\lambda,\mu]} + \|\hat{P}_3\|_{[\mu,1]} + \frac{\epsilon_0}{\alpha-1}(1-\mu)^{\alpha-1}\bigg]\|x\|_* \\
&\le \|x\|_*
\end{aligned}
$$

and so $\|Fx\| \le \|x\|_* < r_0$. Also, we can conclude that $\|F'x\| \le \|x\|_* < r_0$. Hence, $\|Fx\| < r_0$ and so $F_x \in C$. For the same reason, $F_y \in C$. Similar to (7), we conclude that

$$
\begin{aligned}
\|F_x - F_y\|_* &\le \max\left\{\frac{3\alpha}{\Gamma(\alpha)}, \frac{2+\alpha+T_0}{\Gamma(\alpha)}\right\}\Bigg[\sum_{i=1}^{5} \|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon_1)\|x-y\|_*^{\mu_i} \\
&\quad + \sum_{j=1}^{k_0} \|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon_1)\|x-y\|_*^{m} + \sum_{i=1}^{4} \|\hat{c}_i\|_{[\mu,1]}\Bigg](l_{\gamma_i}+\epsilon_1)\|x-y\|_*^{\gamma_i}] \\
&\le \max\left\{\frac{3\alpha}{\Gamma(\alpha)}, \frac{2+\alpha+T_0}{\Gamma(\alpha)}\right\}\Bigg[\sum_{i=1}^{5} \|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon_1)\|x-y\|_* \\
&\quad + \sum_{j=1}^{k_0} \|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon_1)\|x-y\|_* + \sum_{i=1}^{4} \|\hat{c}_i\|_{[\mu,1]}\Bigg](l_{\gamma_i}+\epsilon_1)\|x-y\|_*] \\
&= \max\left\{\frac{3\alpha}{\Gamma(\alpha)}, \frac{2+\alpha+T_0}{\Gamma(\alpha)}\right\}\Bigg[\sum_{i=1}^{5} \|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon_1) \\
&\quad + \sum_{j=1}^{k_0} \|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon_1) + \sum_{i=1}^{4} \|\hat{c}_i\|_{[\mu,1]}\Bigg](l_{\gamma_i}+\epsilon_1)]\|x-y\|_*
\end{aligned}
$$

whenever $\|x-y\|_* < \epsilon_1$. Thus, $\|x-y\|_* \le \|x\|_* + \|y\|_* \le \epsilon_0$ whenever $x, y \in C$. Hence,

$$
\begin{aligned}
\|F_x - F_y\|_* &\le \max\left\{\frac{3\alpha}{\Gamma(\alpha)}, \frac{2+\alpha+T_0}{\Gamma(\alpha)}\right\}\Bigg[\sum_{i=1}^{5} \|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon_1) \\
&\quad + \sum_{j=1}^{k_0} \|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon_1) + \sum_{i=1}^{4} \|\hat{c}_i\|_{[\mu,1]}\Bigg](l_{\gamma_i}+\epsilon_1)]\|x-y\|_* \\
&= \psi\big(\|x-y\|_*\big),
\end{aligned}
$$

where

$$\psi(t) = \max\left\{\frac{3\alpha}{\Gamma(\alpha)}, \frac{2+\alpha+T_0}{\Gamma(\alpha)}\right\}\left[\sum_{i=1}^{5}\|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}+\epsilon_1)\right.$$
$$+\sum_{j=1}^{k_0}\|\hat{b}_j\|_{[\lambda,\mu]}(q_j+\epsilon_1)$$
$$\left.+\sum_{i=1}^{4}\|\hat{c}_i\|_{[\mu,1]}\right](l_{\gamma_i}+\epsilon_1)]t.$$

Note that $\psi \in \Psi$. Now by using Theorem 6, F has a fixed point and so the pointwise defined problem (1) has a solution. □

Example 2 Consider the problem

$$D^{\frac{7}{2}}x(t) + f\left(t, x(t), x'(t), D^{\frac{1}{2}}x(t), \int_0^t x(\xi)\,d\xi\right) = 0$$

with boundary conditions $x(0) = x'(\frac{1}{3})$, $x(1) = x'(\frac{1}{2})$ and $x''(0) = 0$, where

$$f(t,x_1,\ldots,x_4) = \begin{cases} f_1(t,x_1,\ldots,x_4) := t^2(\sum_{i=1}^4 x_i(s)), & t \in [0,0.7), \\ f_2(t,x_1,\ldots,x_4) := \frac{0.1}{p(t)}\sum_{i=1}^4 \frac{|x_i(t)|}{1+|x_i(t)|}, & \frac{t}{3} \in [0.7,0.7], \\ f_3(t,x_1,\ldots,x_4) := t(\sum_{i=1}^4 x_i(s)), & t \in [0.9,1], \end{cases}$$

and

$$p(t) = \begin{cases} 0, & t \in [0.2,0.9] \cap Q, \\ t, & t \in [0.2,0.9] \cap Q^c. \end{cases}$$

Put $a_i(t) = a(t) = t^2$, $b_j(t) = b(t) = \frac{0.1}{p(t)}$ and $c_i(t) = c(t) = t$ (for $1 \le i \le 4$, $k_0 = 1$). Then we have $|f_1(t,x_1,\ldots,x_4) - f_1(t,y_1,\ldots,y_4)| \le t^2\sum_{i=1}^4 |x_i - y_i| = a(t)\sum_{i=1}^4 \phi(|x_i - y_i|)$,

$$\left|f_3(t,x_1,\ldots,x_4) - f_3(t,y_1,\ldots,y_4)\right| \le t\sum_{i=1}^4 |x_i - y_i| = c(t)\sum_{i=1}^4 \Phi\left(|x_i - y_i|\right),$$

and

$$\left|f_2(t,x_1,\ldots,x_4) - f_2(t,y_1,\ldots,y_4)\right| \le t^2\sum_{i=1}^4 |x_i - y_i| = b(t)\sum_{i=1}^4 H\left(|x-y|,\ldots,|x-y|\right),$$

where $\phi(z) = z$, $\Phi(z) = z$ and $H(z_1,\ldots,z_4) = z_1 + \cdots + z_4$. Put $\mu_i = \gamma_i = m = 1$. Then we have $\lim_{z\to 0^+}\frac{\phi(z)}{z} = 1$, $\lim_{z\to 0^+}\frac{\Phi(z)}{z} = 1$ and $\lim_{z\to 0^+}\frac{H_j(z,z,z,z)}{z} = 1$. Also, $\lim_{\max|x_i|\to 0}\frac{|f_1(t,x_1,\ldots,x_4)|}{\max|x_i|} = 4t^2 = P_1(t)$, $\lim_{\max|x_i|\to 0}\frac{|f_3(t,x_1,\ldots,x_4)|}{\max|x_i|} = 4t = P_3(t)$ and $|f_2(t,x_1,\ldots,x_4)| \le \Theta(t)\Lambda(x_1,\ldots,x_4)$, where $\Theta(t) = p(t)$ and $\Lambda(x_1,\ldots,x_4) = \sum_{i=1}^5 \frac{|x_i|}{1+|x_i|}$. It is easy to see that ϕ, Φ, H and Λ satisfy the conditions of Theorem 6 and $\lim_{x\to 0^+} \frac{\Lambda(x,x,x,x)}{x} = 4 := P_2$. Also, we have

$$\max\left\{\frac{3\alpha}{\Gamma(\alpha)}, \frac{2+\alpha+T_0}{\Gamma(\alpha)}\right\} \cdot \max\left\{\sum_{i=1}^{5} \|\hat{a}_i\|_{[0,\lambda]}(l_{\mu_i}) + \sum_{j=1}^{k_0} \|\hat{b}_j\|_{[\lambda,\mu]} q_j \right.$$

$$\left. + \sum_{i=1}^{4} \|\hat{c}_i\|_{[\mu,1]}](l_{\gamma_i}), \max\left\{1, \frac{1}{\Gamma(2-\beta)}, m_0\right\}\left[\|\hat{P}_1\|_{[0,\lambda]} + P_2\|\Theta\|_{[\lambda,\mu]} + \|\hat{P}_3\|_{[\mu,1]}\right]\right\}$$

$$\leq \frac{\frac{21}{2}}{\frac{15\sqrt{\pi}}{8}} \max\{0.19 + 0.005 + 0.9, 1.13[0.19 + 0.02 + 0.9]\} < 1.$$

By using Theorem 6, the pointwise defined problem has a solution.

3 Conclusion

It is very important that we increase our abilities of natural phenomenon modeling. In this way, it is better we investigate different types of high order fractional integro-differential equations or new type model ones. One of the new models is described by the three step crisis fractional integro-differential equations which have been introduced recently. In this work, we reviewed the existence of solutions for a three step crisis fractional integro-differential equation under some boundary conditions.

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