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Research Paper

A Novel Approach for Korteweg-de Vries Equation of Fractional Order

Hassan Kamil Jassim[1], Dumitru Baleanu[2, 3]

[1] Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar
Nasiriyah, Iraq, hassan.kamil@yahoo.com

[2] Department of Mathematics and Computer Sciences, Faculty of Art and Sciences, Çankaya University
Balgat 0630, Ankara, Turkey, dumitru@cankaya.edu.tr

[3] Institute of Space Sciences, Magurele-Bucharest, Romania

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Corresponding author: H.K. Jassim, hassan.kamil28@yahoo.com

Abstract. In this study, the local fractional variational iteration method (LFVIM) and the local fractional series expansion method (LFSEM) are utilized to obtain approximate solutions for Korteweg-de Vries equation (KdVE) within local fractional derivative operators (LFDOs). The efficiency of the considered methods is illustrated by some examples. The results reveal that the suggested algorithms are very effective and simple and can be applied for linear and nonlinear problems in mathematical physics.

Keywords: Local fractional operators; Local fractional variational iteration method; Local fractional series expansion method; Korteweg-de Vries.

1. Introduction

The local fractional calculus was successfully utilized to describe the PDEs arising in mathematical physics, such as the diffusion equations [1-4], the gas dynamic equation [5], the telegraph equation [6], the wave equation [7], the Fokker Planck equation [8,9], the Laplace equation [10], the Klein-Gordon equations [11,12], the Helmholtz equation, [13,14] and the Goursat problem [15] on Cantor sets. Recently, the KdVE with LFDOs was given by [16, 17]:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}+\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}}+\frac{\partial^{3\alpha} u(x,t)}{\partial x^{3\alpha}}=0,\ 0<\alpha\leq 1. \tag{1}$$

subjected to initial condition as:

$$u(x,0)=\phi(x). \tag{2}$$

The local fractional VIM [18–20] and local fractional SEM [21, 22] are powerful approximate techniques for various kinds of partial and integral differential equations with LFDOs. Jafari et al. [18, 19] utilized LFVIM to solve linear and nonlinear PDEs, Yang, et al. [20] solved fractal heat conduction problem using LFVIM, Jassim [21] found the approximate analytical solutions of Fractal Vehicular Traffic Flow by LFSEM, and Yang, et al. [22] applied LFSEM to the wave equation.

It is worth mentioning that entropy plays an important role in the analysis of anomalous diffusion processes and fractional diffusion equations. These fractional novel entropy indices and fractional operators allow their implementation in complex dynamical systems [23-25]. Another application is related to local fractional wave equations under fixed entropy arising in fractal hydrodynamics [26].

In the present study, the local fractional variational iteration method and the local fractional series expansion method were

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applied to solve the local fractional Korteweg-de Vries equation. The advantage of these methods with respect to other numerical methods is that they don't need discretization.

2. Basic Definitions of Local Fractional Calculus

In this section, some basic definitions and properties of the fractional calculus theory [10-13] are provided.

Definition 1. A function $f(x)$ is local fractional continuous at $x = x_0$, if it holds

$$|f(x) - f(x_0)| < \varepsilon^{\alpha}, 0 < \alpha \leq 1 \tag{3}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$, and $\varepsilon, \delta \in R$. For $x \in (a,b)$, it is so called local fractional continuous on (a,b), denoted by $f(x) \in C_{\alpha}(a,b)$.

Definition 2. Setting $f(x) \in C_{\alpha}(a,b)$, the local fractional derivative of $f(x)$ at $x = x_0$ is defined as:

$$f^{(\alpha)}(x) = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}}, \tag{4}$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha + 1)(f(x) - f(x_0))$.

Not that the local fractional derivative of high order is written in the following form:

$$D_x^{k\alpha} f(x) = f^{(k\alpha)}(x) = \overbrace{D_x^{\alpha} D_x^{\alpha} \ldots D_x^{\alpha}}^{k \text{ times}} f(x), \tag{5}$$

and the local fractional partial derivative of high order as:

$$\frac{\partial^{k\alpha} f(x,y)}{x^{k\alpha}} = \overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \cdots \frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k \text{ times}} f(x,y). \tag{6}$$

Definition 3. Let's denote a partition of the interval $[a,b]$ as (t_j, t_{j+1}), $j = 0, \ldots, N-1$, and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots\}$. The local fractional integral of $f(x)$ in the interval $[a,b]$ is given by:

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha}. \tag{7}$$

Definition 4. In fractal space, the Mittage Leffler function, sine function, and cosine function are defined as:

$$E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, \ 0 < \alpha \leq 1 \tag{8}$$

$$\sin_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]}, \ 0 < \alpha \leq 1 \tag{9}$$

$$\cos_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k\alpha}}{\Gamma[1+2k\alpha]}, \ 0 < \alpha \leq 1 \tag{10}$$

The following results are valid:

$$\frac{d^{\alpha} x^{k\alpha}}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma\left(1+(k-1)\alpha\right)} x^{(k-1)\alpha} \tag{11}$$

$$\frac{d^{\alpha} E_{\alpha}(x^{\alpha})}{dx^{\alpha}} = E_{\alpha}(x^{\alpha}) \tag{12}$$

$$\frac{d^{\alpha} E_{\alpha}(kx^{\alpha})}{dx^{\alpha}} = kE_{\alpha}(kx^{\alpha}) \tag{13}$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b E_{\alpha}(x^{\alpha})(dx)^{\alpha} = E_{\alpha}(b^{\alpha}) - E_{\alpha}(a^{\alpha}) \tag{14}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b \sin_\alpha(x^\alpha)(dx)^\alpha = \cos_\alpha(a^\alpha)-\cos_\alpha(b^\alpha) \tag{15}$$

3. Analysis of the Methods

A general nonlinear local fractional partial differential equation is considered as follows:

$$L_\alpha u(x,t)+R_\alpha u(x,t)+N_\alpha u(x,t)=f(x,t)\ , \tag{16}$$

where $L_\alpha = \partial^\alpha / \partial t^\alpha$ and R_α are linear local fractional derivative operators of order, N_α denotes nonlinear local fractional operator, and $f(x,t)$ is the source term.

3.1. Analysis of the LFVIM

According to the rule of local fractional variational iteration method, the correction local fractional functional for (3.1) is constructed as:

$$u_{n+1}(t)=u_n(t)+{}_0I_t^{(\alpha)}\left(\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\left[L_\alpha u_n(\xi)+R_\alpha \tilde{u}_n(\xi)+N_\alpha \tilde{u}_n(\xi)-f(\xi)\right]\right) \tag{17}$$

where $\lambda^\alpha / \Gamma(1+\alpha)$ is a fractal Lagrange multiplier. Making the local fractional variation of (17) yields:

$$\delta^\alpha u_{n+1}(t)=\delta^\alpha u_n(t)+{}_0I_t^{(\alpha)}\delta^\alpha\left(\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left[L_\alpha u_n(\xi)+R_\alpha \tilde{u}_n(\xi)+N_\alpha \tilde{u}_n(\xi)-f(\xi)\right]\right) \tag{18}$$

The extremum condition of u_{n+1} is given by:

$$\delta^\alpha u_{n+1}(t)=0 \tag{19}$$

This yields the stationary conditions as:

$$1+\left.\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\right|_{\xi=t}=0, \qquad \left.\left(\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\right)^{(\alpha)}\right|_{\xi=t}=0. \tag{20}$$

This in turn gives Lagrange multiplier as:

$$\frac{\lambda^\alpha}{\Gamma(1+\alpha)}=-1. \tag{21}$$

By substituting this value of the Lagrange multiplier into Eq. (17), one gives the iteration formula as:

$$u_{n+1}(t)=u_n(t)-{}_0I_t^{(\alpha)}\left(\left[L_\alpha u_n(\xi)+R_\alpha u_n(\xi)+N_\alpha u_n(\xi)-f(\xi)\right]\right). \tag{22}$$

Finally, from Eq. (22), the solution of Eq. (16) is obtained as follows:

$$u(x,t)=\lim_{n\to\infty} u_n(x,t). \tag{23}$$

3.2. Local Fractional SEM.

Equation (16) can be written in the following form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\mathrm{P}_\alpha u(x,t) \tag{24}$$

where P_α is a linear local fractional derivative operator with respect to x and $u(x,t)$ is a local fractional continuous function. Suppose that:

$$u(x,t)=\sum_{i=0}^{\infty} T_i(t)U_i(x) \tag{25}$$

where $T_i(t)$ and $U_i(x)$ are local fractional continuous functions. Regarding Eq. (25), it is considered that:

$$T_i(t)=\frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \tag{26}$$

which reduces to the following form:

$$u(x,t)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}U_i(x) \tag{27}$$

Making use of Eq. (27), we get

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}U_{i+1}(x) \tag{28}$$

$$\mathrm{P}_{\alpha}u(x,t)=\mathrm{P}_{\alpha}\left(\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}U_{i}(x)\right)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}(\mathrm{P}_{\alpha}U_{i})(x) \tag{29}$$

Regarding Eqs. (28) and (29), we have

$$\sum_{i=0}^{\infty}\frac{1}{\Gamma(1+i\alpha)}t^{i\alpha}U_{i+1}(x)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}(\mathrm{P}_{\alpha}U_{i})(x) \tag{30}$$

Therefore, from Eq. (30), a recursion is obtained as

$$U_{i+1}(x)=(\mathrm{P}_{\alpha}U_{i})(x) \tag{31}$$

By using the recursion formula of Eq. (31), the solution of Eq. (24) is obtained as:

$$u(x,t)=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}U_{i}(x) \tag{32}$$

4. Application

In this section, to give a clear overview of the LFVIM and LFSEM for KdVE within LFDOs, the following example is presented.

Example: Let us consider the following KdV equation involving LFDOs:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}+\frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}}+\frac{\partial^{3\alpha}u(x,t)}{\partial x^{3\alpha}}=0,\ \ 0<\alpha\leq 1. \tag{33}$$

subjected to the initial condition

$$u(x,0)=E_{\alpha}(x^{\alpha}). \tag{34}$$

I. Busing LFVIM:

From Eqs. (22) and (33), the local fractional iteration algorithm can be written as follows:

$$u_{n+1}(t)=u_{n}(t)-\frac{1}{\Gamma(1+\alpha)}\int_{0}^{t}\left(\frac{\partial^{\alpha}u_{n}(\vartheta)}{\partial\vartheta^{\alpha}}+\frac{\partial^{\alpha}u_{n}(\vartheta)}{\partial x^{\alpha}}+\frac{\partial^{3\alpha}u_{n}(\vartheta)}{\partial x^{3\alpha}}\right)(d\vartheta)^{\alpha} \tag{35}$$

Therefore, from Eqs. (34) and (35), the components are provided as follows:

$$\begin{aligned}
u_{1}(t)&=u_{0}(t)-\frac{1}{\Gamma(1+\alpha)}\int_{0}^{t}\left(\frac{\partial^{\alpha}u_{0}(\vartheta)}{\partial\vartheta^{\alpha}}+\frac{\partial^{\alpha}u_{0}(\vartheta)}{\partial x^{\alpha}}+\frac{\partial^{3\alpha}u_{0}(\vartheta)}{\partial x^{3\alpha}}\right)(d\vartheta)^{\alpha}\\
&=E_{\alpha}(x^{\alpha})-\frac{1}{\Gamma(1+\alpha)}\int_{0}^{t}2E_{\alpha}(x^{\alpha})(d\vartheta)^{\alpha}\\
&=E_{\alpha}(x^{\alpha})-\frac{2t^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(x)^{\alpha}\\
&=E_{\alpha}(x^{\alpha})\left[1-\frac{2t^{\alpha}}{\Gamma(1+\alpha)}\right],\\
u_{2}(t)&=u_{1}(t)-\frac{1}{\Gamma(1+\alpha)}\int_{0}^{t}\left(\frac{\partial^{\alpha}u_{1}(\vartheta)}{\partial\vartheta^{\alpha}}+\frac{\partial^{\alpha}u_{1}(\vartheta)}{\partial x^{\alpha}}+\frac{\partial^{3\alpha}u_{1}(\vartheta)}{\partial x^{3\alpha}}\right)(d\vartheta)^{\alpha}\\
&=E_{\alpha}(x^{\alpha})-\frac{2t^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(x^{\alpha})+\frac{1}{\Gamma(1+\alpha)}\int_{0}^{t}\frac{4\vartheta^{\alpha}}{\Gamma(1+\vartheta)}E_{\alpha}(x^{\alpha})(d\vartheta)^{\alpha}\\
&=E_{\alpha}(x^{\alpha})-\frac{2t^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(x)^{\alpha}+\frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}E_{\alpha}(x^{\alpha})\\
&=E_{\alpha}(x^{\alpha})\left[1-\frac{2t^{\alpha}}{\Gamma(1+\alpha)}+\frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}\right],
\end{aligned}$$

$$u_3(t)=u_2(t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\frac{\partial^\alpha u_2(\vartheta)}{\partial\vartheta^\alpha}+\frac{\partial^\alpha u_2(\vartheta)}{\partial x^\alpha}+\frac{\partial^{3\alpha} u_2(\vartheta)}{\partial x^{3\alpha}}\right)(d\vartheta)^\alpha$$
$$=E_\alpha(x^\alpha)\left[1-\frac{2t^\alpha}{\Gamma(1+\alpha)}+\frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}\right],$$

$$u_4(t)=u_3(t)-\frac{1}{\Gamma(1+\alpha)}\int_0^t\left(\frac{\partial^\alpha u_3(\vartheta)}{\partial\vartheta^\alpha}+\frac{\partial^\alpha u_3(\vartheta)}{\partial x^\alpha}+\frac{\partial^{3\alpha} u_3(\vartheta)}{\partial x^{3\alpha}}\right)(d\vartheta)^\alpha$$
$$=E_\alpha(x^\alpha)\left[1-\frac{2t^\alpha}{\Gamma(1+\alpha)}+\frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{16t^{4\alpha}}{\Gamma(1+4\alpha)}\right],$$
$$\vdots$$
$$u_n(t)=E_\alpha(x^\alpha)\sum_{r=0}^{n}(-2)^r\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}.$$

Consequently, it yields

$$\begin{aligned}u(x,t)&=\lim_{n\to\infty}u_n(x,t)\\&=E_\alpha(x^\alpha)\sum_{r=0}^{\infty}(-2)^r\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\\&=E_\alpha(x^\alpha)E_\alpha(-2t^\alpha).\end{aligned}\tag{36}$$

II. By using LFSEM:

From Eq. (31), the following recursive formula is obtained:

$$U_{i+1}(x)=-\left(\frac{\partial^\alpha u_i(x,t)}{\partial x^\alpha}+\frac{\partial^\alpha u_i(x,t)}{\partial x^{3\alpha}}\right)\tag{37}$$

$$U_0(x)=E_\alpha(x^\alpha).\tag{38}$$

Regarding Eqs. (37) and (38), the following equations are obtained:

$$U_1(x)=-\left(\frac{\partial^\alpha u_0(x,t)}{\partial x^\alpha}+\frac{\partial^\alpha u_0(x,t)}{\partial x^{3\alpha}}\right)=-2E_\alpha(x^\alpha),$$
$$U_2(x)=-\left(\frac{\partial^\alpha u_1(x,t)}{\partial x^\alpha}+\frac{\partial^\alpha u_1(x,t)}{\partial x^{3\alpha}}\right)=4E_\alpha(x^\alpha),$$
$$U_3(x)=-\left(\frac{\partial^\alpha u_2(x,t)}{\partial x^\alpha}+\frac{\partial^\alpha u_2(x,t)}{\partial x^{3\alpha}}\right)=-8E_\alpha(x^\alpha),$$
$$U_4(x)=-\left(\frac{\partial^\alpha u_3(x,t)}{\partial x^\alpha}+\frac{\partial^\alpha u_3(x,t)}{\partial x^{3\alpha}}\right)=16E_\alpha(x^\alpha),$$
$$U_5(x)=-\left(\frac{\partial^\alpha u_4(x,t)}{\partial x^\alpha}+\frac{\partial^\alpha u_4(x,t)}{\partial x^{3\alpha}}\right)=-32E_\alpha(x^\alpha),$$
$$\vdots$$

Therefore, we have:

$$\begin{aligned}u(x,t)&=\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)}U_i(x)\\&=E_\alpha(x^\alpha)\left[1-\frac{2t^\alpha}{\Gamma(1+\alpha)}+\frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}-\frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots\right],\\&=E_\alpha(x^\alpha)\sum_{r=0}^{\infty}(-2)^r\frac{t^{r\alpha}}{\Gamma(1+r\alpha)}\\&=E_\alpha(x^\alpha)E_\alpha(-2t^\alpha).\end{aligned}\tag{39}$$

In Figs. 1, 2, 3 and 4, the approximate solutions of Eq. (33) along with initial condition of Eq. (34) are shown for different values of $\alpha=1/2,\ \ln(2)/\ln(3),\ \ln(3)/\ln(4),\ 1$ respectively.

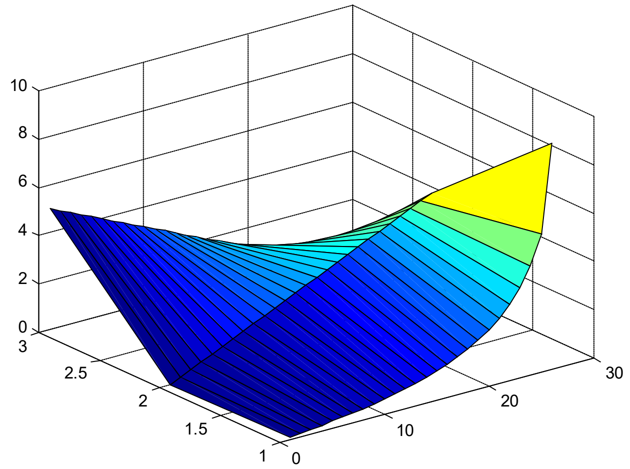

Fig. 1. The plot of solution to local fractional Korteweg-de Vries equation with fractal dimension $\alpha = 1/2$

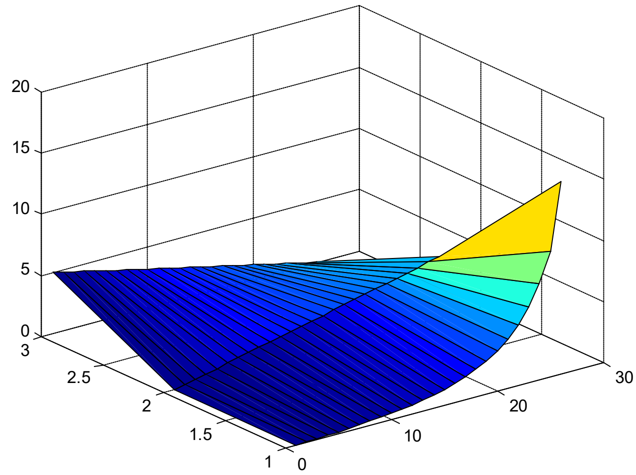

Fig. 2. The plot of solution to local fractional Korteweg-de Vries equation with fractal dimension $\alpha = \ln(2)/\ln(3)$

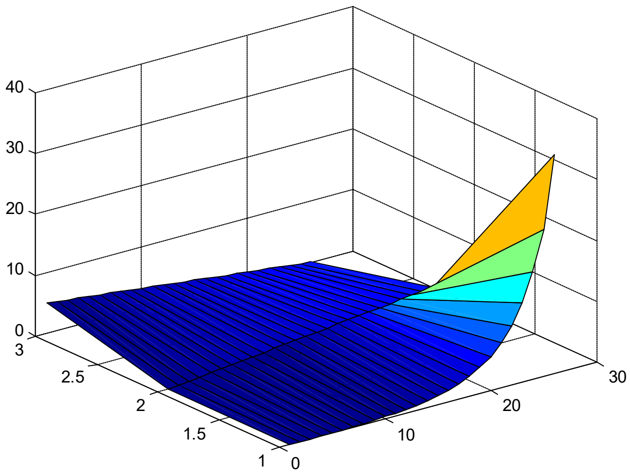

Fig. 3. The plot of solution to local fractional Korteweg-de Vries equation with fractal dimension $\alpha = \ln(3)/\ln(4)$

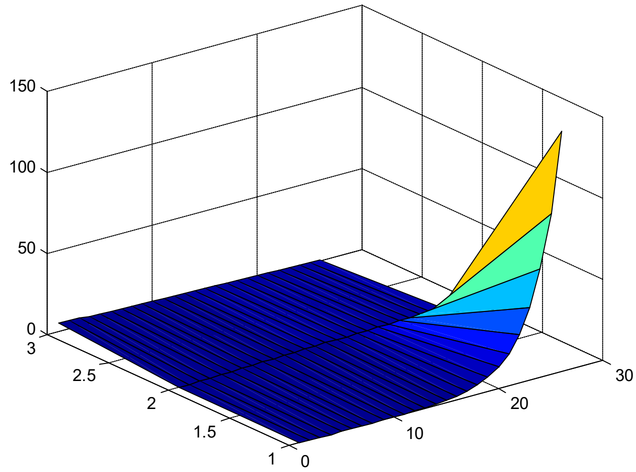

Fig. 4. The plot of solution to local fractional Korteweg-de Vries equation with fractal dimension $\alpha = 1$

5. Conclusions

The LFVIM and LFSEM are successfully applied to find the approximate analytical solutions for KdVE with LFDOs. The approximate analytical solutions for the Korteweg-de Vries equation on Cantor sets involving local fractional derivatives are successfully developed by recurrence relations resulting in convergent series solutions. The suggested methods are a potential tool for developing approximate solutions for local fractional partial differential equations with fractal initial value conditions, which of course, draws new problems beyond the scope of the present study. LFSEM provides us with less computations as compared with LFVIM.

Conflict of Interest

The authors declare no conflict of interest.

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