Certain Hermite–Hadamard Inequalities for Logarithmically Convex Functions with Applications

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Abstract: In this paper, we discuss various estimates to the right-hand (resp. left-hand) side of the Hermite–Hadamard inequality for functions whose absolute values of the second (resp. first) derivatives to positive real powers are log-convex. As an application, we derive certain inequalities involving the q-digamma and q-polygamma functions, respectively. As a consequence, new inequalities for the q-analogue of the harmonic numbers in terms of the q-polygamma functions are derived. Moreover, several inequalities for special means are also considered.

Keywords: Hermite–Hadamard inequality; log-convex function; q-digamma; q-polygamma function; harmonic number; special means

1. Introduction

In recent years, because of the importance of the relationship between inequalities and convexity, the study of the Hermite–Hadamard's inequality has attracted much attention due to applications of convex functions (for details see [1,2]).

A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is convex. Then, for $a, b \in I$, $a < b$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

A function $f : [a, b] \subseteq \mathbb{R} \to (0, \infty)$ is log-convex. Then, for $x, y \in [a, b]$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}.$$

If the above inequality is reversed then f is called a log-concave function. It is also known that if g is differentiable, then f is log-convex (log-concave) if and only if f'/f is increasing (decreasing). It is easy to see this by the arithmetic-mean–geometric-mean (AM–GM) inequality

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t)f(y), \tag{2}$$

for all $x, y \in I$ and $t \in [0, 1]$, i.e., the set of log-convex functions is a proper subset of the class of convex functions.

The main idea of this paper is motivated by the above results. The paper is organized as follows. In Section 2, we first present some lemmas which are used in the main results. Then, we present some estimates to the left-hand side and right-hand side of Hermite–Hadamard inequalities for functions whose absolute values of the first and second derivatives to positive real powers are log-convex. In Section 3, we derive some inequalities for q-digamma and q-polygamma functions by applying the main results of Section 2 and deducing some of them for the q-analogue harmonic numbers. Moreover, several inequalities for special means are derived. In Section 3, our aim is to apply the idea taken in the previous section for the q-digamma and q-polygamma functions and the classical Hermite–Hadamard inequalities (1). As applications, we show new lower and upper bounds for the q-analogue of the harmonic numbers in terms of the q-trigamma function. Moreover, some inequalities for the harmonic numbers and Euler–Mascheroni constant are presented.

2. Main Results

In this section, we establish our main results, but first we briefly recall some basic well-known results which will be used in the sequel.

Lemma 1 ([3]). *Let $f : I = [a, b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^0, with $a < b$, then*

$$\frac{f(b) + f(a)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t)f''(ta + (1-t)b)dt. \tag{3}$$

Lemma 2 ([4]). *Let $f : I = [a, b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^0, with $a < b$, then*

$$\frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = (b-a)\left[\int_0^{1/2} tf'(b + (a-b)t)dt + \int_{1/2}^1 (t-1)f'(b + (a-b)t)dt\right]. \tag{4}$$

Theorem 1. *Let $f : I^0 = [a, b] \longrightarrow \mathbb{R}$ be a twice differentiable mapping on I^0 with $a < b$. If $|f''|^{q'}$, $q' \geq 1$ is log-convex on $[a, b]$, such that $|f''(a)| \neq 1$, and $|f''(b)| \neq 1$, then the inequality holds*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q'}}\left(\frac{1}{q'^3}\right)^{\frac{1}{q'}}$$

$$\times \left[\frac{2(1 - |f''(a)|^{\alpha q'}) + \alpha q' \log|f''(a)|\left(1 + |f''(a)|^{\alpha q'}\right)}{\alpha^4 \log^3 |f''(a)|}\right. \tag{5}$$

$$\left. + \frac{2(1 - |f''(b)|^{\beta q'}) + \beta q' \log|f''(b)|\left(1 + |f''(b)|^{\beta q'}\right)}{\beta^4 \log^3 |f''(b)|}\right]^{\frac{1}{q'}},$$

where $\alpha, \beta > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. Let $q' = 1$ and by means of Lemma 1 and known result [5] (p. 4), we have

$$xy \leq \frac{x^\alpha}{\alpha} + \frac{y^\beta}{\beta}, \ x, y \geq 0 \tag{6}$$

where $\alpha, \beta > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we find that

$$
\begin{aligned}
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| &\le \frac{(b-a)^2}{2} \int_0^1 t(1-t)|f''(ta + (1-t)b)|dt \\
&\le \frac{(b-a)^2}{2} \int_0^1 t(1-t)|f''(a)|^t |f''(b)|^{1-t}dt \\
&\le \frac{(b-a)^2}{2} \int_0^1 t(1-t)\left(\frac{|f''(a)|^{\alpha t}}{\alpha} + \frac{|f''(b)|^{\beta(1-t)}}{\beta} \right) dt \\
&= \frac{(b-a)^2}{2} \left(\int_0^1 \frac{t(1-t)|f''(a)|^{\alpha t}}{\alpha}dt + \int_0^1 \frac{t(1-t)|f''(b)|^{\beta(1-t)}}{\beta}dt \right) \\
&= \frac{I_{11}}{\alpha} + |f''(b)|^\beta \frac{I_{12}}{\beta}.
\end{aligned}
\tag{7}
$$

Now, by using the following formula

$$
\int_0^1 t(1-t)e^{\sigma t}dt = \begin{cases} \frac{e^\sigma + 1}{\sigma^2} + \frac{2(1 - e^\sigma)}{\sigma^3}, & \text{if } \sigma \ne 0 \\ \frac{1}{6}, & \text{if } \sigma = 0 \end{cases}
\tag{8}
$$

with $\sigma = \alpha \log |f''(a)|$ and $\sigma = -\beta \log |f''(b)|$ in I_{11} and I_{12}, respectively, we get

$$
I_{11} = \frac{2(1 - |f''(a)|^\alpha) + \alpha \log |f''(a)| \, (1 + |f''(a)|^\alpha)}{\alpha^3 \log^3 |f''(a)|},
$$

and

$$
I_{12} = \frac{2(1 - |f''(b)|^\beta) + \beta \log |f''(b)| \, (1 + |f''(b)|^\beta)}{\beta^3 |f''(b)|^\beta \log^3 |f''(b)|}.
$$

Now, suppose that $q' > 1$.

Next by considering the Hölder–Rogers inequality,

$$
\int_a^b |f(t)g(t)|dt \le \left[\int_a^b |f(t)|^p dt \right]^{\frac{1}{p}} \left[\int_a^b |g(t)|^q dt \right]^{\frac{1}{q}},
\tag{9}
$$

for $q', p = \frac{q'}{q'-1}$, we get

$$
\begin{aligned}
\int_0^1 t(1-t)|f''(a)|^t |f''(b)|^{1-t}dt &= \int_0^1 [t(1-t)]^{1-\frac{1}{q'}} [t(1-t)]^{\frac{1}{q'}} |f''(a)|^t |f''(b)|^{1-t}dt \\
&\le \left[\int_0^1 t(1-t) \right]^{1-\frac{1}{q'}} \left[\int_0^1 t(1-t)|f''(a)|^{q't} |f''(b)|^{q'(1-t)}dt \right]^{\frac{1}{q'}} \\
&\le \left(\frac{1}{6} \right)^{1-\frac{1}{q'}} \left[\int_0^1 t(1-t) \left(\frac{|f''(a)|^{\alpha q t}}{\alpha} + \frac{|f''(b)|^{\beta q'(1-t)}}{\beta} \right) dt \right]^{\frac{1}{q'}} \\
&= \left(\frac{1}{6} \right)^{1-\frac{1}{q'}} \left(\frac{I_{q1}}{\alpha} + |f''(b)|^{\beta q'} \frac{I_{q2}}{\beta} \right)^{\frac{1}{q'}}.
\end{aligned}
\tag{10}
$$

Again, by using the formula (8), with $\alpha = \alpha q' \log |f''(a)|$ and $\alpha = -\beta q' \log |f''(b)|$, respectively, we obtain

$$
I_{q1} = \frac{2(1 - |f''(a)|^{\alpha q'}) + \alpha q' \log |f''(a)| \left(1 + |f''(a)|^{\alpha q'} \right)}{(\alpha q')^3 \log^3 |f''(a)|}
$$

and

$$
I_{q2} = \frac{2(1 - |f''(b)|^{\beta q'}) + \beta q' \log |f''(b)| \left(1 + |f''(b)|^{\beta q'} \right)}{(\beta q')^3 |f''(b)|^{\beta q'} \log^3 |f''(b)|},
$$

hence, we get (5). □

Remark 1. *In view of the inequalities (5) and (8) when* $|f''(a)| = 1$, *we deduced that the inequality (5) reduces to*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q'}} \left(\frac{1}{q'^3} \right)^{\frac{1}{q'}}$$

$$\times \left[\frac{1}{6\alpha} + \frac{2(1 - |f''(b)|^{\beta q'}) + \beta q' \log |f''(b)| \left(1 + |f''(b)|^{\beta q'} \right)}{\beta^4 \log^3 |f''(b)|} \right]^{\frac{1}{q'}}. \tag{11}$$

In the case when $|f''(b)| = 1$, *the inequality (5) reduce to*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q'}} \left(\frac{1}{q'^3} \right)^{\frac{1}{q'}}$$

$$\times \left[\frac{1}{6\beta} + \frac{2(1 - |f''(a)|^{\alpha q'}) + \alpha q' \log |f''(a)| \left(1 + |f''(a)|^{\alpha q'} \right)}{\alpha^4 \log^3 |f''(a)|} \right]^{\frac{1}{q'}} \tag{12}$$

A similar result is embodied in this theorem. The beta function and gamma function are defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \; x, y > 0 \; \text{ and } \; \gamma(x) = \int_0^\infty t^{x-1}e^{-x}dt, \; x > 0.$$

The beta function satisfies the following properties:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \; B(x, x) = 2^{1-2x}B\left(\frac{1}{2}, x \right).$$

In particular,

$$B(p+1, p+1) = \frac{\sqrt{\pi}\Gamma(p+1)}{2^{2p+1}\Gamma\left(p + \frac{3}{2} \right)}.$$

Theorem 2. *Let* $f : I^0 = [a, b] \longrightarrow \mathbb{R}$ *be a twice differentiable mapping on* I^0 *with* $a < b$. *If* $|f''|^{q'}$, $q' > 1$ *is log-convex on* $[a, b]$, *such that* $|f''(a)| \neq 1$, *and* $|f''(b)| \neq 1$, *then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \left[\frac{\sqrt{\pi}\Gamma(p+1)}{2^{2p+1}\Gamma\left(p + \frac{3}{2} \right)} \right]^{\frac{1}{p}} \left[\frac{|f''(a)|^{\alpha q'} - 1}{\alpha^2 q' \log |f''(a)|} + \frac{|f''(b)|^{\beta q'} - 1}{\beta^2 q' \log |f''(b)|} \right]^{\frac{1}{q'}}, \tag{13}$$

where $p = \frac{q'}{q'-1}$ *and* $\alpha, \beta > 1$ *such that* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. By Lemma 1, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \int_0^1 t(1-t)|f''(ta + (1-t)b)|dt. \tag{14}$$

Since $|f''|^{q'}$ is convex on $[a, b]$ for all $q' > 1$, we know that for any $t \in [0, 1]$,

$$|f''(ta + (1-t)b)|^{q'} \leq |f''(a)|^{tq'}|f''(b)|^{q'(1-t)} \tag{15}$$

So, by using the Hölder–Rogers inequality (9) and the arithmetic-mean–geometric-mean (AM–GM) inequality (6), we have

$$
\begin{aligned}
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| &\le \frac{(b-a)^2}{2}\left[\int_0^1 [t(1-t)]^p dt\right]^{\frac{1}{p}}\left[\int_0^1 |f''(ta+(1-t)b)|^{q'}dt\right]^{\frac{1}{q'}} \\
&\le \frac{(b-a)^2}{2}\left[\int_0^1 [t(1-t)]^p dt\right]^{\frac{1}{p}}\left[\int_0^1 |f''(a)|^{qt}|f''(b)|^{q'(1-t)}dt\right]^{\frac{1}{q'}} \\
&\le \frac{(b-a)^2}{2}\left[\frac{\Gamma^2(p+1)}{\Gamma(2p+2)}\right]^{\frac{1}{p}}\left[\int_0^1\left(\frac{|f''(a)|^{\alpha q't}}{\alpha}+\frac{|f''(b)|^{\beta q'(1-t)}}{\beta}\right)dt\right]^{\frac{1}{q'}} \\
&= \frac{(b-a)^2}{2}\left[\frac{\Gamma^2(p+1)}{\Gamma(2p+2)}\right]^{\frac{1}{p}}\left[\int_0^1\frac{|f''(a)|^{\alpha q't}}{\alpha}dt+\int_0^1\frac{|f''(b)|^{\beta q'(1-t)}}{\beta}dt\right]^{\frac{1}{q'}} \\
&= \frac{(b-a)^2}{2}\left[\frac{\Gamma^2(p+1)}{\Gamma(2p+2)}\right]^{\frac{1}{p}}\left[\frac{|f''(a)|^{\alpha q'}-1}{\alpha^2 q'\log|f''(a)|}+\frac{|f''(b)|^{\beta q'}-1}{\beta^2 q'\log|f''(b)|}\right]^{\frac{1}{q'}} \\
&= \frac{(b-a)^2}{2}\left[\frac{\sqrt{\pi}\Gamma(p+1)}{2^{2p+1}\Gamma\left(p+\frac{3}{2}\right)}\right]^{\frac{1}{p}}\left[\frac{|f''(a)|^{\alpha q'}-1}{\alpha^2 q'\log|f''(a)|}+\frac{|f''(b)|^{\beta q'}-1}{\beta^2 q'\log|f''(b)|}\right]^{\frac{1}{q'}}.
\end{aligned}
\tag{16}
$$

The desired inequality (13) is thus established. □

Remark 2. *If* $|f''(x)| > 1,$ *for all* $x \in [a,b].$ *Assume that the hypotheses of the Theorem 2 are satisfied. Then in view of*

$$
\frac{x-1}{\log(x)} < x, \ x > 1,
\tag{17}
$$

and (13), *we have*

$$
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{(b-a)^2}{2}\left[\frac{\sqrt{\pi}\Gamma(p+1)}{2^{2p+1}\Gamma\left(p+\frac{3}{2}\right)}\right]^{\frac{1}{p}}\left[\frac{|f''(a)|^{\alpha q'}}{\alpha}+\frac{|f''(b)|^{\beta q'}}{\beta}\right]^{\frac{1}{q'}}, \tag{18}
$$

where $p = \frac{q'}{q'-1}$ *and* $\alpha, \beta > 1$ *such that* $\frac{1}{\alpha}+\frac{1}{\beta}=1.$

Theorem 3. *With the assumptions of Theorem 2, we have*

$$
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{(b-a)^2}{2q'^{1+\frac{1}{q'}}}\frac{\gamma^{\frac{1}{q'}}(q'+1,1)}{(p+1)^{\frac{1}{p}}}\left[\frac{|f''(a)|^{\alpha q'}}{\alpha^{q'+2}\log^{q'+1}|f''(a)|}+\frac{(-1)^{q'+1}}{\beta^{q'+2}\log^{q'+1}|f''(b)|}\right]^{\frac{1}{q'}}, \tag{19}
$$

where $\frac{1}{p}+\frac{1}{q'}=1,$ $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *and* $\gamma(a,z)$ *is the incomplete gamma function*

$$
\gamma(a,z) = \int_0^z t^{a-1}e^{-t}dt.
$$

Proof. From Lemma 1, Hölder's inequality, and the AM–GM inequality (6), we get

$$
\begin{aligned}
\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| &\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)|f''(ta+(1-t)b)|dt \\
&\le \frac{(b-a)^2}{2}\left[\int_0^1 t^p dt\right]^{\frac{1}{p}}\left[\int_0^1 (1-t)^{q'}|f''(ta+(1-t)b)|^{q'}dt\right]^{\frac{1}{q'}} \\
&\le \frac{(b-a)^2}{2}\left[\int_0^1 t^p dt\right]^{\frac{1}{p}}\left[\int_0^1 (1-t)^{q'}|f''(a)|^{q't}|f''(b)|^{q'(1-t)}dt\right]^{\frac{1}{q'}} \\
&\le \frac{(b-a)^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\int_0^1 (1-t)^{q'}\left(\frac{|f''(a)|^{\alpha q't}|}{\alpha}+\frac{f''(b)|^{\beta q'(1-t)}}{\beta}\right)\right]^{\frac{1}{q'}}.
\end{aligned}
\tag{20}
$$

On the other hand, by using the fact

$$\int_0^1 (1-t)^q e^{\sigma t} dt = \begin{cases} \frac{\gamma(q+1,1)e^\sigma}{\sigma^{q+1}} & if\ \sigma \neq 0, \\ \frac{1}{q+1} & if\ \sigma = 0, \end{cases} \qquad (21)$$

we obtain

$$\int_0^1 (1-t)^{q'} |f''(a)|^{\alpha q' t} dt = \frac{|f''(a)|^{\alpha q'} \gamma(q'+1,1)}{[\alpha q' \log |f''(a)|]^{q'+1}}, \qquad (22)$$

and

$$\int_0^1 (1-t)^{q'} |f''(b)|^{\beta q'(1-t)} dt = \frac{\gamma(q'+1,1)}{[-\beta q' \log |f''(b)|]^{q'+1}}. \qquad (23)$$

Finally, in view of (20), (22) and (23), we deduce the inequality (19). □

Remark 3. *If $|f''(a)| = 1$, then the inequality (19) reduces to the following inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{1}{\alpha(q'+1)} + \frac{(-1)^{q'+1}\gamma(q'+1,1)}{(\beta q' \log |f''(b)|)^{q'+1}} \right]^{\frac{1}{q'}} \quad (24)$$

Moreover, in the case when $|f''(b)| = 1$, then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{|f''(a)|^{\alpha q'} \gamma(q'+1,1)}{[\alpha q' \log |f''(a)|]^{q'+1}} + \frac{1}{\beta(q'+1)} \right]^{\frac{1}{q'}}, \quad (25)$$

is true.

Theorem 4. *Let $f : I^0 = [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $a < b$. If $|f'|^{q'}$, $q' > 1$ is log-convex on $[a,b]$, such that $|f'(a)| \neq 1$, and $|f'(b)| \neq 1$, then*

$$\left| \frac{1}{b-a} \int_a^b f(t) - f\left(\frac{a+b}{2} \right) \right| \leq (b-a) \left(\frac{1}{2^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[\left\{ \frac{|f'(a)|^{\frac{\alpha q'}{2}} - 1}{\alpha^2 q' \log |f'(a)|} + \frac{|f'(b)|^{\frac{\beta q'}{2}} (|f'(b)|^{\frac{\beta q'}{2}} - 1)}{\beta^2 q' \log |f'(b)|} \right\}^{\frac{1}{q'}} \right.$$

$$\left. + \left\{ \frac{|f'(a)|^{\frac{\alpha q'}{2}} \left(|f'(a)|^{\frac{\alpha q'}{2}} - 1 \right)}{\alpha^2 q' \log |f'(a)|} + \frac{|f'(b)|^{\frac{\beta q'}{2}} - 1}{\beta^2 q' \log |f'(b)|} \right\}^{\frac{1}{q'}} \right] \qquad (26)$$

holds for all $p, q', \alpha, \beta > 1$ such that $\frac{1}{p} + \frac{1}{q'} = 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. From Lemma 2, we get

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2} \right) \right| \leq (b-a) \left[\int_0^{1/2} t|f'(b+(a-b)t)|dt + \int_{1/2}^1 |(t-1)f'(b+(a-b)t)|dt \right]$$

$$\leq (b-a) \left[\int_0^{1/2} t|f'(ta+(1-t)b)|dt + \int_{1/2}^1 (1-t)|f'(ta+(1-t)b)|dt \right].$$

On the other hand, by using the Hölder–Rogers inequality (9) and the AM–GM inequality (6), we obtain

$$
\begin{aligned}
\int_0^{1/2} t|f'(ta+(1-t)b)|dt &\le \left[\int_0^{1/2} t^p dt\right]^{\frac{1}{p}}\left[\int_0^{1/2}|f'(ta+(1-t)b)|^{q'}dt\right]^{\frac{1}{q'}} \\
&\le \left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\int_0^{1/2}|f'(a)|^{tq'}|f'(b)|^{(1-t)q'}dt\right]^{\frac{1}{q'}} \\
&\le \left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\int_0^{1/2}\left(\frac{|f'(a)|^{\alpha tq'}}{\alpha}+\frac{|f'(b)|^{\beta(1-t)q'}}{\beta}\right)dt\right]^{\frac{1}{q'}} \\
&= \left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^{\frac{\alpha q'}{2}}-1}{\alpha^2 q'\log|f'(a)|}+\frac{|f'(b)|^{\frac{\beta q'}{2}}(|f'(b)|^{\frac{\beta q'}{2}}-1)}{\beta^2 q'\log|f'(b)|}\right]^{\frac{1}{q'}}
\end{aligned}
\tag{27}
$$

Similarly, we get

$$
\begin{aligned}
\int_{1/2}^1 (1-t)|f'(ta+(1-t)b)|dt &\le \left[\int_{1/2}^1 (1-t)^p dt\right]^{\frac{1}{p}}\left[\int_{1/2}^1|f'(ta+(1-t)b)|^{q'}dt\right]^{\frac{1}{q'}} \\
&\le \left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\int_{1/2}^1|f'(a)|^{tq'}|f'(b)|^{(1-t)q'}dt\right]^{\frac{1}{q'}} \\
&\le \left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\int_{1/2}^1\left(\frac{|f'(a)|^{\alpha tq'}}{\alpha}+\frac{|f'(b)|^{\beta(1-t)q'}}{\beta}\right)dt\right]^{\frac{1}{q'}} \\
&= \left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^{\frac{\alpha q'}{2}}\left(|f'(a)|^{\frac{\alpha q'}{2}}-1\right)}{\alpha^2 q'\log|f'(a)|}+\frac{|f'(b)|^{\frac{\beta q'}{2}}-1}{\beta^2 q'\log|f'(b)|}\right]^{\frac{1}{q'}},
\end{aligned}
\tag{28}
$$

which implies (19). □

Remark 4. *Let the assumptions of Theorem 4 be satisfied, with $|f'(x)|>1$ for all $x\in[a,b]$. In view of (26) and (17), we get*

$$
\left|\frac{1}{b-a}\int_a^b f(t)-f\left(\frac{a+b}{2}\right)\right|\le \frac{(b-a)}{2^{\frac{1}{q'}}}\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left\{\left[\frac{|f'(a)|^{\frac{\alpha q'}{2}}}{\alpha}+\frac{|f'(b)|^{\beta q'}}{\beta}\right]^{\frac{1}{q'}}+\left[\frac{|f'(a)|^{\alpha q'}}{\alpha}+\frac{|f'(b)|^{\frac{\beta q'}{2}}}{\beta}\right]^{\frac{1}{q'}}\right\},
\tag{29}
$$

with $\frac{1}{p}+\frac{1}{q'}=1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Theorem 5. *Let $f:I^0=[a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on I^0 with $a<b$. If $|f'|$ is log-convex on $[a,b]$, such that $|f'(a)|\ne 1$ and $|f'(b)|\ne 1$, then*

$$
\left|\frac{1}{b-a}\int_a^b f(t)-f\left(\frac{a+b}{2}\right)\right|\le (b-a)\left[\left(\frac{|f'(a)|^{\frac{\alpha}{2}}-1}{\alpha^{\frac{3}{2}}\log|f'(a)|}\right)^2+\left(\frac{|f'(b)|^{\frac{\beta}{2}}-1}{\beta^{\frac{3}{2}}\log|f'(b)|}\right)^2\right].
\tag{30}
$$

Proof. Since $|f'|$ is, in fact, log-convex and due to the AM–GM inequality (6), we successively get

$$\int_0^{\frac{1}{2}} t|f'(ta + (1 - t)b)|dt \le \int_0^{\frac{1}{2}} t|f'(a)|^t|f'(b)|^{1-t}dt$$

$$\le \int_0^{\frac{1}{2}} t\left(\frac{|f'(a)|^{\alpha t}}{\alpha} + \frac{|f'(b)|^{\beta(1-t)}}{\beta}\right) dt \tag{31}$$

$$= \frac{1}{4}\left(\int_0^1 t\frac{|f'(a)|^{\frac{\alpha t}{2}}}{\alpha}dt + \frac{|f'(b)|^{\beta}}{\beta}\int_0^1 t|f'(b)|^{-\frac{\beta t}{2}}dt\right).$$

So, along with the identity

$$\int_0^1 te^{\sigma t}dt = \frac{\sigma e^{\sigma} - e^{\sigma} + 1}{\sigma^2}, \ \sigma \ne 0, \tag{32}$$

we get

$$\int_0^1 t\frac{|f'(a)|^{\frac{\alpha t}{2}}}{\alpha}dt = \frac{2|f'(a)|^{\frac{\alpha}{2}}}{\alpha^2\log|f'(a)|} - \frac{4(|f'(a)|^{\frac{\alpha}{2}} - 1)}{\alpha^3\log^2|f'(a)|} \tag{33}$$

and

$$\int_0^1 t|f'(b)|^{-\frac{\beta t}{2}}dt = \frac{2}{\beta|f'(b)|^{\frac{\beta}{2}}\log|f'(b)|}\left(\frac{2\left(|f'(b)|^{\frac{\beta}{2}} - 1\right)}{\beta\log|f'(b)|} - 1\right). \tag{34}$$

Therefore,

$$\int_0^{\frac{1}{2}} t|f'(ta + (1-t)b)|dt \le \frac{1}{4}\left[\frac{2|f'(a)|^{\frac{\alpha}{2}}}{\alpha^2\log|f'(a)|} - \frac{4(|f'(a)|^{\frac{\alpha}{2}} - 1)}{\alpha^3\log^2|f'(a)|} + \frac{2|f'(b)|^{\frac{\beta}{2}}}{\beta^2\log|f'(b)|}\left(\frac{2\left(|f'(b)|^{\frac{\beta}{2}} - 1\right)}{\beta\log|f'(b)|} - 1\right)\right]. \tag{35}$$

Also from the AM–GM inequality (6), and since the function $|f'|$ is log-convex, we get

$$\int_{\frac{1}{2}}^1 |(t-1)f'(ta + (1-t)b)|dt \le \int_{\frac{1}{2}}^1 (1-t)|f'(a)|^t|f'(b)|^{1-t}dt$$

$$\le \int_{\frac{1}{2}}^1 (1-t)\left(\frac{|f'(a)|^{\alpha t}}{\alpha} + \frac{|f'(b)|^{\beta(1-t)}}{\beta}\right) dt. \tag{36}$$

Taking the values $\sigma = \alpha\log|f'(a)|$ and $\sigma = -\beta\log|f'(b)|$ respectively, in the following formula

$$\int_{\frac{1}{2}}^1 (1-t)e^{\sigma t} = \frac{2e^{\sigma} - (2+\sigma)e^{\frac{\sigma}{2}}}{2\sigma^2} \ \text{if } \sigma \ne 0, \tag{37}$$

we deduce

$$\int_{\frac{1}{2}}^1 (1-t)|f'(a)|^{\alpha t}dt = \frac{|f'(a)|^{\alpha} - |f'(a)|^{\frac{\alpha}{2}}}{\alpha^2\log^2|f'(a)|} - \frac{|f'(a)|^{\frac{\alpha}{2}}}{2\alpha\log|f'(a)|}, \tag{38}$$

and

$$\int_{\frac{1}{2}}^1 (1-t)|f'(b)|^{\beta(1-t)}dt = \frac{1 - |f'(b)|^{\frac{\beta}{2}}}{\beta^2\log^2|f'(b)|} + \frac{|f'(b)|^{\frac{\beta}{2}}}{2\beta\log|f'(b)|}. \tag{39}$$

In view of (36), (38) and (39), we can easily get the following inequality

$$\int_{\frac{1}{2}}^1 |(t-1)f'(ta + (1-t)b)|dt \le \left[\frac{|f'(a)|^{\alpha} - |f'(a)|^{\frac{\alpha}{2}}}{\alpha^3\log^2|f'(a)|} - \frac{|f'(a)|^{\frac{\alpha}{2}}}{2\alpha^2\log|f'(a)|} + \frac{1 - |f'(b)|^{\frac{\beta}{2}}}{\beta^3\log^2|f'(b)|} + \frac{|f'(b)|^{\frac{\beta}{2}}}{2\beta^2\log|f'(b)|}\right]. \tag{40}$$

Taking into account Lemma 2 along with (35) and (40), we obtain (30). $\quad\square$

Remark 5. *Suppose that* $|f'(x)| > 1$, *for all* $x \in [a, b]$. *By combining the inequalities* (17) *and* (30), *we obtain*

$$\left| \frac{1}{b-a} \int_a^b f(t) - f\left(\frac{a+b}{2} \right) \right| \leq \frac{(b-a)}{4} \left(\frac{|f'(a)|^\alpha}{\alpha} + \frac{|f'(b)|^\beta}{\beta} \right). \tag{41}$$

3. Applications

3.1. Applications to Special Means

Here, we demonstrate new inequalities connecting the above means for arbitrary real numbers.

1. The arithmetic mean:

$$A = A(a, b) = \frac{a+b}{2}; \ a, b \in \mathbb{R}, \ \text{with} \ a, b > 0.$$

2. The geometric mean:

$$G = G(a, b) = \sqrt{ab}; \ a, b \in \mathbb{R}, \ \text{with} \ a, b > 0.$$

3. The logarithmic mean:

$$L(a, b) = \frac{b-a}{\log b - \log a}.$$

4. The generalized logarithmic mean:

$$L_n(a, b) = \left[\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)} \right]^{\frac{1}{n}}; \ n \in \mathbb{Z} \setminus \{-1, 0\}, \ a, b \in \mathbb{R}, \ \text{with} \ a, b > 0.$$

We use the following notations throughout this section:

$$\alpha = \beta = \frac{1}{2}, \ \mathbb{N} = \{1, 2, 3, ...\}, \ \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}, \mathbb{Z}^- = \{..., -2, -1, 0\}.$$

Proposition 1. *Let* a, b, q *be real numbers such that* $0 < a < b$, $q' > 1$ *and* $n \in \mathbb{Z}^- \setminus \{-1, 0\}$, *then the inequality holds*

$$|A(a^n, b^n) - L_n^n(a, b)| \leq \min(K_1, K_2) \frac{(b-a)^2}{4}, \tag{42}$$

where

$$K_1 = \left[\frac{\sqrt{\pi} \Gamma(p+1)}{4^p \Gamma\left(p + \frac{3}{2}\right)} \right]^{\frac{1}{p}} \left\{ \frac{n(n-1)a^{2(n-2)q'} - 1}{\log(n(n-1)) + (n-2)\log(a)} + \frac{n(n-1)b^{2(n-2)q'} - 1}{\log(n(n-1)) + (n-2)\log(b)} \right\}^{\frac{1}{q}}$$

$$K_2 = \left(\frac{\gamma(q'+1, 1)}{2^{2(q'+1)} q^{2(q'+1)}} \right)^{\frac{1}{q'}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \frac{n(n-1)a^{n-2}}{[\log(n(n-1)) + (n-2)\log(a)]^{q'+1}} + \frac{(-1)^{q'+1}}{[\log(n(n-1)) + (n-2)\log(b)]^{q+1}} \right\}^{\frac{1}{q'}},$$

with $p = \frac{q'}{q'-1}$.

Proof. Apply $f(x) = x^n$, $x \in [a, b]$ and $n \in \mathbb{Z}^- \setminus \{-1, 0\}$ in Theorems 2 and 3. □

Proposition 2. *Let* a, b, q' *be a real number such that* $0 < a < b$, *and* $q' > 1$. *Then, the inequality holds:*

$$\left| A(a^{-1}, b^{-1} - L^{-1}(a, b) \right| \leq \min(K_3, K_4) \frac{(b-a)^2}{4}, \tag{43}$$

where

$$K_3 = \left[\frac{\sqrt{\pi}\Gamma(p+1)}{4^p \Gamma\left(p+\frac{3}{2}\right)} \right]^{\frac{1}{p}} \left\{ \frac{\left(2a^{-3}\right)^{2q} - 1}{\log(2a^{-3})} + \frac{\left(2b^{-3}\right)^{2q} - 1}{\log(2b^{-3})} \right\}^{\frac{1}{q}}$$

$$K_4 = \left(\frac{\gamma(q+1,1)}{2^{2(q+1)}q^{2(q+1)}} \right)^{\frac{1}{q}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \frac{\left(2a^{-3}\right)^{2q}}{[\log(2a^{-3})]^{q+1}} + \frac{(-1)^{q+1}}{[\log(2b^{-3})]^{q+1}} \right\}^{\frac{1}{q}},$$

with $p = \frac{q}{q-1}$.

Proof. Apply $f(x) = \frac{1}{x}$, $x \in [a,b]$ in Theorems 2 and 3. □

3.2. Inequalities for Some Special Functions

Let $0 < q < 1$, the q-digamma function ψ_q, is the q-analogue of the psi or digamma function ψ defined by

$$\psi_q(x) = -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1 - q^{k+x}}$$
$$= -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kx}}{1 - q^k}. \tag{44}$$

For $q > 1$ and $x > 0$, the q-digamma function ψ_q is defined by

$$\psi_q(x) = -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+x)}}{1 - q^{-(k+x)}} \right]$$
$$= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-kx}}{1 - q^{-kx}} \right].$$

In [6], it was shown that $\lim_{q \to 1^+} \psi_q(x) = \lim_{q \to 1^-} \psi_q(x) = \psi(x)$.

The nth derivative of the q-digamma function is called the q-polygamma function and can be defined as

$$\psi_q^{(n)}(x) = \frac{d^n}{dx^n} \psi_q(x), \ x > 0, \ 0 < q < 1.$$

We recall that a function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \geq 0$, for all $x \in I$ and all integers $n \geq 0$. These functions have important applications in probability and numerical analysis.

Proposition 3. *Let* a, b, q, q' *be real numbers such that* $0 < a < b$, $q' > 1$ *and* $0 < q < 1$. *Then the following inequality*

$$\left| \frac{\psi_q'(b) + \psi_q'(a)}{2} - \left(\frac{\psi_q(b) - \psi_q(a)}{b-a} \right) \right| \leq \frac{(b-a)^2}{16} \left[\frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)} \right]^{\frac{1}{p}} \left[\frac{|\psi_q^{(3)}(a)|^{2q'} - 1}{2q' \log |\psi_q^{(3)}(a)|} + \frac{|\psi_q^{(3)}(b)|^{2q'} - 1}{2q' \log |\psi_q^{(3)}(b)|} \right]^{\frac{1}{q'}}, \tag{45}$$

holds true for all $x > 0$ *and* $p = \frac{q'}{q'-1}$.

Proof. By using the definition of the q-digamma function, we deduce that the q-trigamma function $f(x) = \psi_q'(x)$ is a completely monotonic function on $(0, \infty)$. This implies that the function $\psi_q^{(3)}(x)$ is also completely monotonic on $(0, \infty)$ for each $q \in (0, 1)$, and consequently is log-convex, since every

completely monotonic function is log-convex, (see [7] (p. 167)). So, Theorem 2 completes the proof of Proposition 3. □

Proposition 4. *Let $0 < a < b$ and $q \in (0,1)$. Then the inequality holds,*

$$\left| \frac{\psi_q(b) - \psi_q(a)}{b - a} - \psi'_q\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{8} \left[\left(\frac{|\psi_q^{(2)}(a)| - 1}{\log |\psi_q^{(2)}(a)|} \right)^2 + \left(\frac{|\psi_q^{(2)}(b)| - 1}{\log |\psi_q^{(2)}(b)|} \right)^2 \right]. \tag{46}$$

Proof. The inequality (46) follows immediately from Theorem 5 when $f = \psi'_q$. □

As another application of inequality (46) we can provide the following inequalities for the q-triagamma and q-polygamma functions and the q-analogue of harmonic numbers H_{nq} defined by

$$H_{nq} = \sum_{k=1}^{n} \frac{q^k}{1 - q^k}, \ n \in \mathbb{N}. \tag{47}$$

So, in view of inequality (46) and using the equation

$$\psi_q(n+1) = \psi_q(1) - \log(q) H_{nq}, \ n \in \mathbb{N}, \tag{48}$$

we obtain the following result.

Corollary 1. *Let $n \in \mathbb{N}$ and $0 < q < 1$. Then, the following inequality*

$$\left| -\frac{\log(q) H_{nq}}{n} - \psi'_q\left(\frac{n}{2}+1\right) \right| \leq \frac{n^2}{8} \left[\left(\frac{|\psi_q^{(2)}(1)| - 1}{\log |\psi_q^{(2)}(1)|} \right)^2 + \left(\frac{|\psi_q^{(2)}(n+1)| - 1}{\log |\psi_q^{(2)}(n+1)|} \right)^2 \right] \tag{49}$$

holds true.

Proposition 5. *Let n be a integer number. Then the inequality holds,*

$$\left| H_n - \psi'\left(\frac{n}{2}+1\right) \right| \leq \frac{n^2}{8} \left[\left(\frac{|\psi^{(2)}(1)| - 1}{\log |\psi^{(2)}(1)|} \right)^2 + \left(\frac{|\psi^{(2)}(n+1)| - 1}{\log |\psi^{(2)}(n+1)|} \right)^2 \right] \tag{50}$$

where H_n is a harmonic number.

Proof. By virtue of inequality (49) when $q \longrightarrow 1$, and using the relation

$$\begin{aligned}
\lim_{q \to 1} \log(q) H_{nq} &= \lim_{q \to 1} \left[\left(\frac{\log(q)}{q-1} \right) ((q-1) H_{nq}) \right] \\
&= \lim_{q \to 1} (q-1) H_{nq} \\
&= -\lim_{q \to 1} \sum_{k=1}^{n} \frac{1-q}{1-q^k} \\
&= -H_n,
\end{aligned} \tag{51}$$

we obtain the desired result. □

Remark 6. *By using the equation*

$$H_n = \gamma + \psi(n+1), \tag{52}$$

where γ is the Euler–Mascheroni constant, the inequality ([50]) can be read as

$$\left| \gamma + \psi(n+1) - \psi'\left(\frac{n}{2}+1\right) \right| \leq \frac{n^2}{8} \left[\left(\frac{|\psi^{(2)}(1)| - 1}{\log|\psi^{(2)}(1)|} \right)^2 + \left(\frac{|\psi^{(2)}(n+1)| - 1}{\log|\psi^{(2)}(n+1)|} \right)^2 \right]. \tag{53}$$

We conclude our investigation by applying the methods which were developed in Section [2], for the classical Hermite–Hadamard inequalities. Since the q-trigamma function $\psi'_q(x)$ is completely monotonic on $(0, \infty)$ for each $q \in (0, 1)$, and consequently is convex on $(0, \infty)$. Then we get

$$\psi'_q\left(\frac{a+b}{2}\right) \leq \frac{\psi_q(b) - \psi_q(a)}{b - a} \leq \frac{\psi'_q(b) + \psi'_q(a)}{2}. \tag{54}$$

In particular, we obtain the following inequalities for the q-analogue of harmonic numbers

$$\psi'_q\left(\frac{n}{2}+1\right) \leq \frac{-\log(q)H_{nq}}{n} \leq \frac{\psi'_q(n+1) + \psi'_q(1)}{2}, \; n \in \mathbb{N}. \tag{55}$$

Letting $q \longrightarrow 1$, in the above inequalities, we obtain the new inequality for the harmonic number H_n,

$$\psi'\left(\frac{n}{2}+1\right) \leq \frac{H_n}{n} \leq \frac{\pi^2 + 6\psi'(n+1)}{12}. \tag{56}$$

Now, combining the inequalities ([56]) and ([52]), we get the two-sided bounding inequalities for the Euler–Mascheroni constant:

$$n\psi'\left(\frac{n}{2}+1\right) - \psi(n+1) \leq \gamma \leq \frac{n(\pi^2 + 6\psi'(n+1)) - 12\psi(n+1)}{12}. \tag{57}$$

Conflicts of Interest: The authors declare no conflict of interest.

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