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A New Analytical Technique to Solve System of Fractional-Order Partial Differential Equations

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ABSTRACT In this research article, a new analytical technique is implemented to solve system of fractional-order partial differential equations. The fractional derivatives are carried out with the help of Caputo fractional derivative operator. The direct implementation of Mohand and its inverse transformation provide sufficient easy less and reliability of the proposed method. Decomposition method along with Mohand transformation is proceeded to attain the analytical solution of the targeted problems. The applicability of the suggested method is analyzed through illustrative examples. The solutions graph has the best contact with the graphs of exact solutions in paper. Moreover, the convergence of the present technique is sufficiently fast, so that it can be considered the best technique to solve system of nonlinear fractional-order partial differential equations.

INDEX TERMS Mohand transform, Adomian decomposition, analytical solution, fractional-order system of partial differential equations, Caputo derivatives.

I. INTRODUCTION

In a few decades, it has been observed that fractional analysis has tremendous applications in many branches of science. It is on the basses that in many physical phenomena, experiments have proved that fractional order derivatives have good agreements with experimental data or real phenomena as compared to integer order derivatives. For instance, the non-integer order derivative more effectively delineates memory, heredity effects properties of different materials and process the internal friction as well [1]–[4]. The fractional calculus has nowadays is an essential tool that many phenomena in engineering, physics, chemistry, other sciences from $1 - 3D$ can be described very successfully. The recent applications of fractional calculus in different filed attract the whole concentration of researchers and from which many results are concluded. These results have contributed in many fields of science, numerous applications in various fields of science, such as fractional diffusion and fractional Buck master's equation [5], fractional-order time delay system [59] space-fractional telegraph equation [6], fractional KdV-Burger-Kuramoto equation [7], fractal vehicular traffic flow [8], fractional Drinfeld-Sokolov-Wilson equation [9], fractional calculus and dynamic system [10], [11], time fractional modified anomalous sub-diffusion equation [12], fractional model for the dynamics of Hepatitis B virus [13], fractional model for tuberculosis [14], design of optimal lighting control strategy based on multi-variable fractional-order extremum seeking method. [16], fractional-order sliding mode based extremum seeking control of a class of nonlinear systems [17], fractional order pine wilt disease model [15], fractional diabetes model [18], anomalous transport in disordered systems [19], percolation in porous media [20], the diffusion of biological population [21], fractional diffusion-reaction equation [22]and time-fractional Klein-Gordon equations [23], [24] etc .

Fractional partial differential equations (FPDEs) are the major mathematical tools that are used to model various physical phenomena in different branches of applied science such as physics, engineering and other social science. The modeling in the form of FPDEs system appear in many applications of science and engineering such as material sciences,

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biology, chemistry, fluid dynamics, chemical kinetics and many other physical processes [25]–[35].

The numerical as well as analytical solution of PDEs system has been drawn a lot of attention of researchers and scholars in recent years. Many numerical and analytical algorithm have been use for solving fractional FPDEs such as double Laplace transform method [36], first integral method [37], generalized two-dimensional differential transform method [38], conformable fractional Laplace transform method [39], modified variational iteration method [40], He's Variational iteration method [41], variational iteration method [42], [43], Fourier transform [44] and fractional complex differential transformation method [45], Laplace-Adomian decomposition method [46].

Thus, the aim of this study is to propose an analytic solution for the one dimensional time fractional system of PDEs by using new integral transform called Mohand transform. The Mohand Transform is one of the new integral transform use for the analytical treatment of different physical phenomena molded by Ordinary Differential Equations (ODEs), Partial Differential Equations (PDEs) or Fractional Partial Differential Equations (FPDEs). Recently, Kumar et al. used Mohand transform and solved the mechanics and electrical circuit problems [47]. Sudhanshu Aggarwal have Comparatively Studied Mohand and Aboodh transforms for the solution of differential equations. The numerical applications reflect that both the transforms (Mohand and Aboodh transforms) are closely connected to each other [48]. Sudhanshu Aggarwal have also discussed the comparative study of Mohand and Laplace transforms, Mohand and Sumudu transforms, Mohand and Mahgoub transforms [49]–[51]. Sudhanshu Aggarwal have successfully discussed the Mohand transform of Bessel's functions of zero, one and two orders, which is very useful for solving many equations in cylindrical or spherical coordinates such as heat equation, wave equation etc [52]. The exact solution of linear Volterra integral equations of second kind get by using Mohand transform. It is claimed that Mohand transform take very little time and has no large computational work [53]. Mohand transform have also used the for solution of Abel's integral equation. The obtained results show that Mohand transform is a powerful integral transform for handling Abel's integral equation [54]. For purpose of implementation of Mohand transform, we considered a time fractional system of PDEs in the form

$$\begin{aligned} D^{\vartheta}_{\tau^*}\psi_\alpha(\xi,\tau^*)+\Re_{\alpha_1}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)+\aleph_{\beta_1}&\\ \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)&=\delta_1(\xi,\tau^*),\\ D^{\vartheta}_{\tau^*}\varphi_\beta(\xi,\tau^*)+\Re_{\alpha_2}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)+\aleph_{\beta_2}&\\ \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)&=\delta_2(\xi,\tau^*), \end{aligned} \tag{1}$$

with initial condition

$$\psi_\alpha(\xi,0)=w_1(\xi),\quad \varphi_\beta(\xi,0)=w_2(\xi)$$

where, $0<\vartheta\leq 1$ and $\Re_{\alpha_1}$, $\Re_{\alpha_2}$, $\aleph_{\beta_1}$, $\aleph_{\beta_2}$ are linear and non-linear operators. The rest of the paper is structured as follows: In the section (2), we present the basic definitions of fractional calculus and basic concepts of Mohand transform. In section (3) presents the implementation the proposed transform. In section (4) we represent different system of PDEs are examined separately and plotted. Finally, we depict our conclusions.

II. PRELIMINARIES CONCEPTS

In this section, we present some basic necessary definitions and preliminaries concepts related to fractional calculus and Mohand transform.

A. MOHAND TRANSFORM

Mohand transform first time was defined by Mohand and Mahgoub of the function $\psi_\alpha(\tau^*)$ for $\tau^*\geq 0$ in the year 2017. A new transformation called Mohand transform specified for the function of the exponential order that we supposed to be variables in Set A specified by:

$$A=\{f(t):\exists M,k_1,k_2>0.|f(t)|<Me^{\frac{|t|}{k_j}},\\ if t\varepsilon(-1)^j\times[0,\infty)\}$$

In the case of a given function in set A, the constant M must be a finite number, k_1, k_2 may be finite or infinite. The Mohand transform which is represented by $M(.)$ for a function $\psi_\alpha(\tau^*)$ is define as [55], [56]

$$M\{\psi_\alpha(\tau^*)\}=R(\upsilon)=\upsilon^2\int_0^\infty \psi_\alpha(\tau^*)e^{-\upsilon\tau^*}d\tau^*,\\ k_1\leq\upsilon\leq k_2 \tag{2}$$

The Mohand transform of a function $\psi_\alpha(\tau^*)$ is $R(\upsilon)$ then $\psi_\alpha(\tau^*)$ is called the inverse of $R(\upsilon)$ which is expressed as

$$M^{-1}\{R(\upsilon)\}=\psi_\alpha(\tau^*),\quad M^{-1} \tag{3}$$

is inverse Mohand operator.

B. MOHAND TRANSFORM FOR NTH DERIVATIVES

$$M\{\psi^n_\alpha(\tau^*)\}=\upsilon^nR(\upsilon)-\upsilon^{n+1}\psi_\alpha(0)-\upsilon^n\psi'_\alpha(0)\\ -\cdots-\upsilon^2\psi^{n-1}_\alpha(0) \tag{4}$$

C. DEFINITION

Caputo operator of fractional partial derivative [57]

$$D^{\beta}_{\tau^*}\psi_\alpha(\xi,\tau^*)=\frac{\partial^\beta\psi_\alpha(\xi,\tau^*)}{\partial\tau^{*\beta}}\\ =\begin{cases} I^{n-\beta}[\frac{\partial^\beta\psi_\alpha(\xi,\tau^*)}{\partial\tau^{*\beta}}], & n-1<\beta<n,\\ & n\in N\\ \frac{\partial^\beta\psi_\alpha(\xi,\tau^*)}{\partial\tau^{*\beta}}, & n=\beta\end{cases} \tag{5}$$

D. DEFINITION

Function of Mittag-Leffler, $E_\kappa(b)$ for $\kappa > 0$ is defined as

$$E_\kappa(b) = \sum_{\tilde{m}=0}^{\infty} \frac{b^{\tilde{m}}}{\Gamma(\kappa\tilde{m}+1)} \qquad \kappa > 0 \quad b \in \mathbb{C},$$

III. IMPLEMENTATION OF MOHAND TRANSFORM

For implementation of the proposed technique we have taken the time fractional system of PDEs in the form

$$\begin{aligned} &D^{\vartheta}_{\tau^*}\psi_\alpha(\xi,\tau^*)+\Re_{\alpha_1}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)+\aleph_{\beta_1}\\ &\quad\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)=\delta_1(\xi,\tau^*),\\ &D^{\vartheta}_{\tau^*}\varphi_\beta(\xi,\tau^*)+\Re_{\alpha_2}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)+\aleph_{\beta_2}\\ &\quad\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)=\delta_2(\xi,\tau^*), \end{aligned} \tag{6}$$

with initial condition

$$\psi_\alpha(\xi,0)=w_1(\xi), \quad \varphi_\beta(\xi,0)=w_2(\xi)$$

where, $0<\vartheta\leq 1$ *and* $\Re_{\alpha_1}, \Re_{\alpha_2}, \aleph_{\beta_1}, \aleph_{\beta_2}$ are linear and non-linear operators.

Applying Mohand transform

$$\begin{aligned} &M[D^{\vartheta}_{\tau^*}\psi_\alpha(\xi,\tau^*)+\Re_{\alpha_1}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)+\aleph_{\beta_1}\\ &\quad\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)]=M\left\{\delta_1(\xi,\tau^*)\right\},\\ &M[D^{\vartheta}_{\tau^*}\varphi_\beta(\xi,\tau^*)+\Re_{\alpha_2}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)+\aleph_{\beta_2}\\ &\quad\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)]=M\left\{\delta_2(\xi,\tau^*)\right\}, 0<\vartheta\leq 1 \end{aligned} \tag{7}$$

By using the transform property

$$\begin{aligned} &\upsilon^{\vartheta}\{R(\upsilon)-\upsilon u(0)\}+\Re_{\alpha_1}(\psi_\alpha(\xi,\tau^*)\\ &\quad\varphi_\beta(\xi,\tau^*))+\aleph_{\beta_1}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)\}=M\left\{\delta_1(\xi,\tau^*)\right\}\\ &\upsilon^{\vartheta}\{R(\upsilon)-\upsilon v(0)\}+\Re_{\alpha_2}(\psi_\alpha(\xi,\tau^*)\\ &\quad\varphi_\beta(\xi,\tau^*))+\aleph_{\beta_2}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)\}=M\left\{\delta_2(\xi,\tau^*)\right\}, \end{aligned} \tag{8}$$

After some evaluation, equation (8) simplified as

$$\begin{aligned} R(\upsilon)&=\upsilon u(0)-\frac{1}{\upsilon^{\vartheta}}M\{\Re_{\alpha_1}(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*))\\ &\quad+\aleph_{\beta_1}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)\}+\frac{1}{\upsilon^{\vartheta}}M\left\{\delta_1(\xi,\tau^*)\right\}\\ R(\upsilon)&=\upsilon v(0)-\frac{1}{\upsilon^{\vartheta}}M\{\Re_{\alpha_2}(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*))\\ &\quad 0+\aleph_{\beta_2}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)\}+\frac{1}{\upsilon^{\vartheta}}M\left\{\delta_2(\xi,\tau^*)\right\}, \end{aligned} \tag{9}$$

By applying inverse Mohand transform

$$\begin{aligned} \psi_\alpha(\xi,\tau^*)&=\psi_\alpha(\xi,0)-M^{-1}\{\frac{1}{\upsilon^{\vartheta}}M\{\Re_{\alpha_1}\left(\psi_\alpha(\xi,\tau^*)\right.\\ &\quad\left.\varphi_\beta(\xi,\tau^*)\right)+\aleph_{\beta_1}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)\}\}\\ &\quad+M^{-1}\left\{\frac{1}{\upsilon^{\vartheta}}M\left\{\delta_1(\xi,\tau^*)\right\}\right\} \end{aligned}$$

$$\begin{aligned} \varphi_\beta(\xi,\tau^*)&=\varphi_\beta(\xi,0)-M^{-1}\{\frac{1}{\upsilon^{\vartheta}}M\{\Re_{\alpha_2}\\ &\quad\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)\\ &\quad+\aleph_{\beta_2}\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)\}\}\\ &\quad+M^{-1}\left\{\frac{1}{\upsilon^{\vartheta}}M\left\{\delta_2(\xi,\tau^*)\right\}\right\}, \end{aligned} \tag{10}$$

Thus, the first term $\psi_\alpha(\xi,\tau^*), \varphi_\beta(\xi,\tau^*)$ will be obtain by using initial conditions

$$\begin{cases} \psi_{0\alpha}(\xi,\tau^*)=\psi_\alpha(0)+M^{-1}\left\{\frac{1}{\upsilon^{\vartheta}}M\left\{\delta_1(\xi,\tau^*)\right\}\right\}\\ \varphi_{0\beta}(\xi,\tau^*)=\varphi_\beta(0)+M^{-1}\left\{\frac{1}{\upsilon^{\vartheta}}M\left\{\delta_2(\xi,\tau^*)\right\}\right\} \end{cases} \quad m=0. \tag{11}$$

Finally we obtain the recursive general relation form as

$$\begin{aligned} \psi_{(m+1)\alpha}(\xi,\tau^*)&=-M^{-1}\{\frac{1}{\upsilon^{\vartheta}}M\{D^{\vartheta}_{\tau^*}\psi_{m\alpha}(\xi,\tau^*)\\ &\quad+\Re_{\alpha_1}\left(\psi_\alpha(\xi,\tau^*)\varphi_{m\beta}(\xi,\tau^*)\right)\\ &\quad+\aleph_{\beta_1}(\psi_{m\alpha}(\xi,\tau^*)\\ &\quad\varphi_{m\beta}(\xi,\tau^*))\}\}\\ m&\geq 0. \end{aligned} \tag{12}$$

The nonlinear terms $\aleph_{\beta_1}, \aleph_{\beta_2}$ is evaluated by using the procedure of Adomian polynomial given by

$$\aleph_{\beta_1}(u,v)=\sum_{m=0}^{\infty}A_m, \quad \aleph_{\beta_2}(u,v)=\sum_{m=0}^{\infty}B_m \tag{13}$$

where,

$$\begin{aligned} A_m&=\frac{1}{m}\left[\frac{d^m}{d\lambda^m}\left[\aleph_{\beta_1}\left(\sum_{i=0}^{\infty}\lambda^i u_i,\sum_{i=0}^{\infty}\lambda^i v_i\right)\right]\right]_{\lambda=0}, \\ m&=0,1,\cdots \end{aligned} \tag{14}$$

$$\begin{aligned} B_m&=\frac{1}{m}\left[\frac{d^m}{d\lambda^m}\left[\aleph_{\beta_2}\left(\sum_{i=0}^{\infty}\lambda^i u_i,\sum_{i=0}^{\infty}\lambda^i v_i\right)\right]\right]_{\lambda=0}, \\ m&=0,1,\cdots \end{aligned} \tag{15}$$

IV. APPLICATIONS AND DISCUSSION

Here, we have implemented the Mohand transform on some time fractional system of PDEs.

EXAMPLE 4.1:

Considering the time fractional system of PDE in the form [45], [58]

$$\begin{aligned} &D^{\vartheta}_{\tau^*}\psi_\alpha(\xi,\tau^*)-\varphi_{\beta\xi}(\xi,\tau^*)+\psi_\alpha(\xi,\tau^*)+\varphi_\beta(\xi,\tau^*)=0\\ &D^{\vartheta}_{\tau^*}\varphi_\beta(\xi,\tau^*)-\psi_{\alpha\xi}(\xi,\tau^*)+\psi_\alpha(\xi,\tau^*)+\varphi_\beta(\xi,\tau^*)=0,\\ &\qquad\qquad 0<\vartheta\leq 1 \end{aligned} \tag{16}$$

with initial source

$$\psi_\alpha(\xi,0)=\sinh(\xi), \quad \varphi_\beta(\xi,0)=\cosh(\xi)$$

The exact solution of equation (16) at $\vartheta = 1$

$$\psi_\alpha(\xi,\tau^*) = \sinh(\xi-\tau^*), \quad \varphi_\beta(\xi,\tau^*) = \cosh(\xi-\tau^*)$$

Taking Mohand transform of equation (16), we get

$$\begin{aligned} \upsilon^\vartheta\{R(\upsilon)-\upsilon u(0)\} &= M\left\{\varphi_{\beta\xi}(\xi,\tau^*)-\psi_\alpha(\xi,\tau^*)-\varphi_\beta(\xi,\tau^*)\right\} \\ \upsilon^\vartheta\{R(\upsilon)-\upsilon v(0)\} &= M\left\{\psi_{\alpha\xi}(\xi,\tau^*)-\psi_\alpha(\xi,\tau^*)-\varphi_\beta(\xi,\tau^*)\right\} \end{aligned} \tag{17}$$

After some evaluation, equation (17) is simplified as

$$\begin{aligned} R(\upsilon) &= \upsilon u(0)+\frac{1}{\upsilon^\vartheta}M\left\{\varphi_{\beta\xi}(\xi,\tau^*)-\psi_\alpha(\xi,\tau^*)-\varphi_\beta(\xi,\tau^*)\right\} \\ R(\upsilon) &= \upsilon v(0)+\frac{1}{\upsilon^\vartheta}M\left\{\psi_{\alpha\xi}(\xi,\tau^*)-\psi_\alpha(\xi,\tau^*)-\varphi_\beta(\xi,\tau^*)\right\} \end{aligned} \tag{18}$$

By applying inverse Mohand transform, we get

$$\begin{aligned} \psi_{m\alpha}(\xi,\tau^*) &= u(\xi,0)+M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\varphi_{\beta\xi}(\xi,\tau^*) \\ &\quad -\psi_\alpha(\xi,\tau^*)-\varphi_\beta(\xi,\tau^*)\}\} \\ \varphi_\beta(\xi,\tau^*) &= \varphi_\beta(\xi,0)+M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\psi_{\alpha\xi}(\xi,\tau^*) \\ &\quad -\psi_\alpha(\xi,\tau^*)-\varphi_\beta(\xi,\tau^*)\}\} \end{aligned} \tag{19}$$

Thus, by using the define recursive scheme of equation (11), (12), we get

$$\begin{aligned} \psi_{0\alpha}(\xi,\tau^*) &= \psi_\alpha(\xi,0), \quad \varphi_{0\beta}(\xi,\tau^*) = \varphi_\beta(\xi,0) \\ &= \cosh(\xi) \end{aligned} \tag{20}$$

$$\begin{aligned} \psi_{(m+1)\alpha}(\xi,\tau^*) &= M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\varphi_{m\beta\xi}(\xi,\tau^*) \\ &\quad -\psi_{m\alpha}(\xi,\tau^*)-\varphi_{m\beta}(\xi,\tau^*)\}\} \\ \varphi_{(m+1)\beta}(\xi,\tau^*) &= M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\psi_{m\alpha\xi}(\xi,\tau^*) \\ &\quad -\psi_{m\alpha}(\xi,\tau^*)-\varphi_{m\beta}(\xi,\tau^*)\}\} \end{aligned} \tag{21}$$

From the recursive formula (17), for $m = 0$

$$\begin{aligned} \psi_{1\alpha}(\xi,\tau^*) &= M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\varphi_{0\beta\xi}(\xi,\tau^*)-\psi_{0\alpha}(\xi,\tau^*) \\ &\quad -\varphi_{0\beta}(\xi,\tau^*)\}\} \\ \varphi_{1\beta}(\xi,\tau^*) &= M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\psi_{0\alpha\xi}(\xi,\tau^*)-\psi_{0\alpha}(\xi,\tau^*) \\ &\quad -\varphi_{0\beta}(\xi,\tau^*)\}\} \end{aligned} \tag{22}$$

$$\begin{aligned} \psi_{1\alpha}(\xi,\tau^*) &= +M^{-1}\left\{\frac{1}{\upsilon^\vartheta}M\left\{\sinh(\xi)-\sinh(\xi)-\cosh(\xi)\right\}\right\} \\ \varphi_{1\beta}(\xi,\tau^*) &= M^{-1}\left\{\frac{1}{\upsilon^\vartheta}M\left\{\cosh(\xi)-\sinh(\xi)-\cosh(\xi)\right\}\right\} \end{aligned} \tag{23}$$

By using transform property

$$\begin{aligned} \psi_{1\alpha}(\xi,\tau^*) &= M^{-1}\left\{\frac{-\cosh(\xi)}{\upsilon^{\vartheta-1}}\right\} \\ \varphi_{1\beta}(\xi,\tau^*) &= M^{-1}\left\{\frac{-\sinh(\xi)}{\upsilon^{\vartheta-1}}\right\} \end{aligned} \tag{24}$$

By using inverse transform, we get

$$\begin{aligned} \psi_{1\alpha}(\xi,\tau^*) &= \frac{-\cosh(\xi)\tau^{*\vartheta}}{\vartheta!} \\ \varphi_{1\beta}(\xi,\tau^*) &= \frac{-\sinh(\xi)\tau^{*\vartheta}}{\vartheta!} \end{aligned} \tag{25}$$

similarly for $m = 1$

$$\begin{aligned} \psi_{2\alpha}(\xi,\tau^*) &= \frac{\sinh(\xi)\tau^{*2\vartheta}}{2\vartheta!} \\ \varphi_{2\beta}(\xi,\tau^*) &= \frac{\cosh(\xi)\tau^{*2\vartheta}}{2\vartheta!} \end{aligned} \tag{26}$$

for $m = 2$

$$\begin{aligned} \psi_{3\alpha}(\xi,\tau^*) &= \frac{-\cosh(\xi)\tau^{*3\vartheta}}{3\vartheta!} \\ \varphi_{3\beta}(\xi,\tau^*) &= \frac{-\sinh(\xi)\tau^{*3\vartheta}}{3\vartheta!} \end{aligned} \tag{27}$$

for $m = 3$

$$\begin{aligned} \psi_{4\alpha}(\xi,\tau^*) &= \frac{\sinh(\xi)\tau^{*4\vartheta}}{4\vartheta!} \\ \varphi_{4\beta}(\xi,\tau^*) &= \frac{\cosh(\xi)\tau^{*4\vartheta}}{4\vartheta!} \end{aligned} \tag{28}$$

for $m = 4$

$$\begin{aligned} \psi_{5\alpha}(\xi,\tau^*) &= \frac{-\cosh(\xi)\tau^{*5\vartheta}}{5\vartheta!} \\ \varphi_{5\beta}(\xi,\tau^*) &= \frac{-\sinh(\xi)\tau^{*5\vartheta}}{5\vartheta!} \\ &\vdots \end{aligned} \tag{29}$$

The Mohand transform solution for example 4.1 is

$$\begin{aligned} \psi_\alpha(\xi,\tau^*) &= \psi_{0\alpha}(\xi,\tau^*)+\psi_{1\alpha}(\xi,\tau^*)+\psi_{2\alpha}(\xi,\tau^*) \\ &\quad +\psi_{3\alpha}(\xi,\tau^*)+\psi_{4\alpha}(\xi,\tau^*)+\cdots \\ \varphi_\beta(\xi,\tau^*) &= \varphi_{0\beta}(\xi,\tau^*)+\varphi_{1\beta}(\xi,\tau^*)+\varphi_{2\beta}(\xi,\tau^*) \\ &\quad +\varphi_{3\beta}(\xi,\tau^*)+\varphi_{4\beta}(\xi,\tau^*)+\cdots \end{aligned} \tag{30}$$

$$\begin{aligned} \psi_\alpha(\xi,\tau^*) &= \sinh(x^*)-\frac{\cosh(\xi)\tau^{*\vartheta}}{\vartheta!}+\frac{\sinh(\xi)\tau^{*2\vartheta}}{2\vartheta!} \\ &\quad -\frac{\cosh(\xi)\tau^{*3\vartheta}}{3\vartheta!}+\frac{\sinh(\xi)\tau^{*4\vartheta}}{4\vartheta!} \\ &\quad -\frac{\cosh(\xi)\tau^{*5\vartheta}}{5\vartheta!}\cdots \\ \varphi_\beta(\xi,\tau^*) &= \cosh(\xi)-\frac{\sinh(\xi)\tau^{*\vartheta}}{\vartheta!}+\frac{\cosh(\xi)\tau^{*2\vartheta}}{2\vartheta!} \\ &\quad -\frac{\sinh(\xi)\tau^{*3\vartheta}}{3\vartheta!}+\frac{\cosh(\xi)\tau^{*4\vartheta}}{4\vartheta!} \\ &\quad -\frac{\sinh(x)\tau^{5\vartheta}}{5\vartheta}\cdots \end{aligned} \tag{31}$$

By simplifying

$$\begin{aligned} \psi_\alpha(\xi,\tau^*) &= \sinh(\xi)\left\{1+\frac{\tau^{*2\vartheta}}{2\vartheta!}+\frac{4\tau^{*\vartheta}}{4\vartheta!}+\cdots\right\} \\ &\quad -\cosh(\xi)\left\{\frac{\tau^{*\vartheta}}{\vartheta!}+\frac{\tau^{*3\vartheta}}{3\vartheta!}+\frac{\tau^{*5\vartheta}}{5\vartheta!}+\cdots\right\} \end{aligned}$$

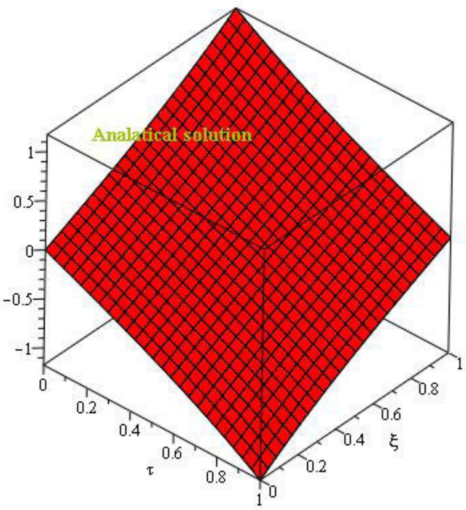

FIGURE 1. $\psi_\alpha(\xi,\tau^*)$ **Represents the analytical solution of example 4.1.**

$$\varphi_\beta(\xi,\tau^*) = \cosh(\xi)\left\{1+\frac{\tau^{*2\vartheta}}{2\vartheta!}+\frac{4\tau^{*\vartheta}}{4\vartheta!}+\cdots\right\} - \sinh(\xi)\left\{\frac{\tau^{*\vartheta}}{\vartheta!}+\frac{\tau^{*3\vartheta}}{3\vartheta!}+\frac{\tau^{*5\vartheta}}{5\vartheta!}+\cdots\right\} \quad (32)$$

For particular case $\vartheta = 1$, the Mohand transform solution become as

$$\psi_\alpha(\xi,\tau^*) = \sinh(\xi)\left\{1+\frac{\tau^{*2}}{2!}+\frac{\tau^{*4}}{4!}+\cdots\right\} - \cosh(\xi)\left\{\tau^*+\frac{\tau^{*3}}{3!}+\frac{\tau^{*5}}{5!}+\cdots\right\}$$
$$\varphi_\beta(\xi,\tau^*) = \cosh(\xi)\left\{1+\frac{\tau^{*2}}{2!}+\frac{\tau^{*4}}{4!}+\cdots\right\} - \sinh(\xi)\left\{\tau^*+\frac{\tau^{*3}}{3!}+\frac{\tau^{*5}}{5!}+\cdots\right\} \quad (33)$$

The calculated result provide the exact solution in the close form

$$\psi_\alpha(\xi,\tau^*) = \sinh(\xi-\tau^*),\quad \varphi_\beta(\xi,\tau^*) = \cosh(\xi-\tau^*) \quad (34)$$

EXAMPLE 4.2

Consider the time fractional PDE system the form [45], [58]

$$D_\tau^{*\vartheta}\psi_\alpha(\xi,\tau^*) - \varphi_{\beta\xi}(\xi,\tau^*) - \psi_\alpha(\xi,\tau^*) + \varphi_\beta(\xi,\tau^*) = -2$$
$$D_{\tau^*}^{\vartheta}\varphi_\beta(\xi,\tau^*) + \psi_{\alpha\xi}(\xi,\tau^*) - \psi_\alpha(\xi,\tau^*) + \varphi_\beta(\xi,\tau^*) = -2,\quad 0<\vartheta\le 1 \quad (35)$$

with initial condition

$$\psi_\alpha(\xi,0) = 1+e^{\xi},\quad \varphi_\beta(\xi,0) = -1+e^{\xi}$$

The exact solution of equation (35) is

$$\psi_\alpha(\xi,\tau^*) = 1+e^{\xi+\tau^*},\quad \varphi_\beta(\xi,\tau^*) = -1+e^{\xi-\tau^*}$$

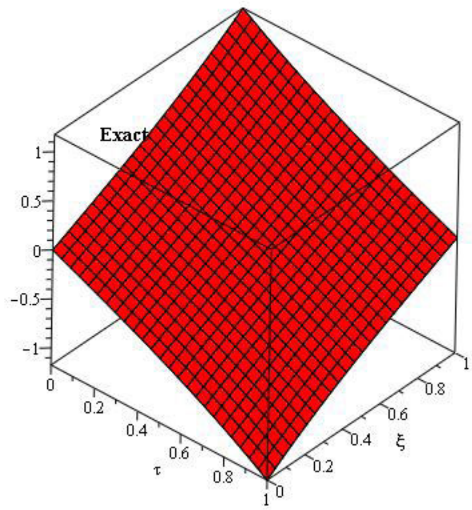

FIGURE 2. $\psi_\alpha(\xi,\tau^*)$ **Represents the exact solution of example 4.1.**

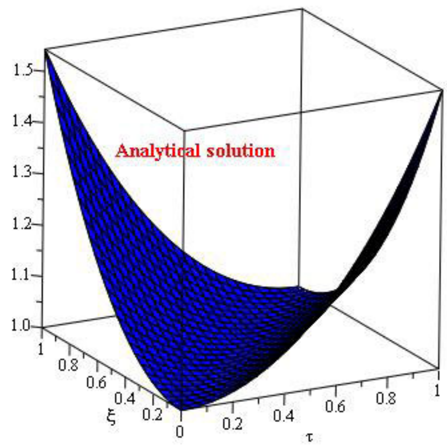

FIGURE 3. $\psi_\beta(\xi,\tau^*)$ **Represents the analytical solution of example 4.1.**

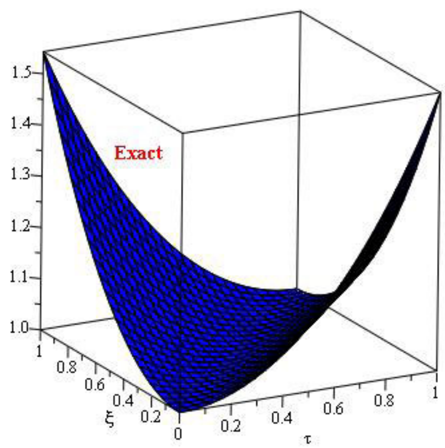

FIGURE 4. $\psi_\beta(\xi,\tau^*)$ **Represents the exact solution of example 4.1.**

Taking Mohand transform of equation (35)

$$R(\upsilon) - \upsilon u(0) + M\left\{-\varphi_{\beta\xi}(\xi,\tau^*) + \psi_\alpha(\xi,\tau^*) - \varphi_\beta(\xi,\tau^*)\right\} = M\{-2\}$$
$$R(\upsilon) - \upsilon v(0) + M\left\{-\psi_{\alpha\xi}(\xi,\tau^*) + \psi_\alpha(\xi,\tau^*) - \varphi_\beta(\xi,\tau^*)\right\} = M\{-2\}, \quad (36)$$

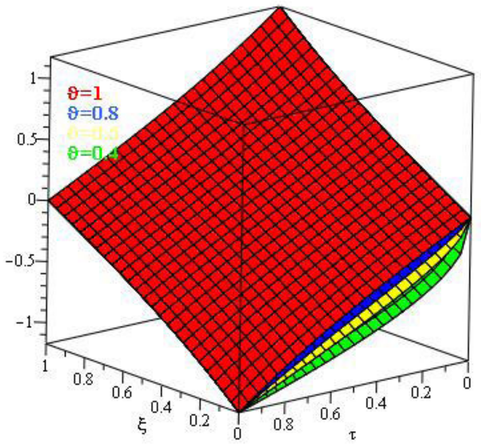

FIGURE 5. $\psi_\alpha(\xi,\tau^*)$ **Represents the solution at different fractional order of example 4.1.**

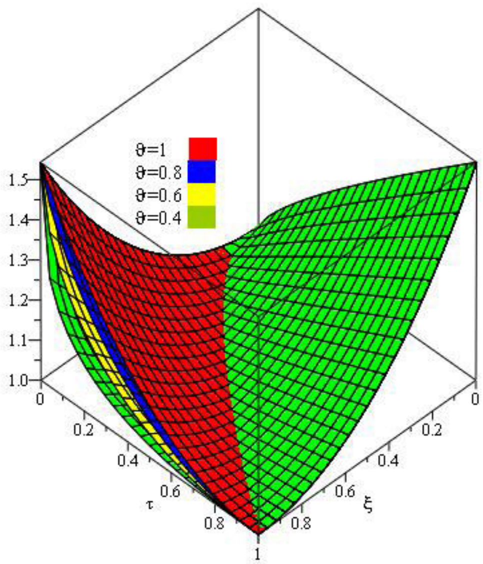

FIGURE 6. $\varphi_\beta(\xi,\tau^*)$ **Represents the solution at different fractional order of example 4.1.**

After some evaluation, equation (27) is simplified as

$$
\begin{aligned}
R(\upsilon) &= \upsilon u(0) + \frac{1}{\vartheta} M\left\{\varphi_{\beta\xi}(\xi,\tau^*) + \psi_\alpha(\xi,\tau^*) - \varphi_\beta(\xi,\tau^*)\right\} \\
&\quad + \frac{1}{\vartheta}M\{-2\} \\
R(\upsilon) &= \upsilon \nu(0) + \frac{1}{\vartheta} M\left\{-\psi_{\alpha\xi}(\xi,\tau^*) - \psi_\alpha(\xi,\tau^*)\right. \\
&\qquad \left. + \varphi_\beta(\xi,\tau^*)\right\} + \frac{1}{\vartheta}M\{-2\},
\end{aligned} \tag{37}
$$

Taking inverse Mohand transform of equation (37)

$$
\begin{aligned}
\psi_\alpha(\xi,\tau^*) &= \psi_\alpha(\xi,0) + M^{-1}\{\frac{1}{\vartheta}M\{\varphi_{\beta\xi}(\xi,\tau^*) + \psi_\alpha(\xi,\tau^*) \\
&\qquad - \varphi_\beta(\xi,\tau^*)\}\} + M^{-1}\left\{\frac{1}{\vartheta}M\{-2\}\right\} \\
\varphi_\beta(\xi,\tau^*) &= \varphi_\beta(\xi,0) + M^{-1}\{\frac{1}{\vartheta}M\{-\psi_{\alpha\xi}(\xi,\tau^*) + \psi_\alpha(\xi,\tau^*) \\
&\qquad - \varphi_\beta(\xi,\tau^*)\}\} + M^{-1}\left\{\frac{1}{\vartheta}M\{-2\}\right\},
\end{aligned} \tag{38}
$$

By using the recursive scheme equation (11), we get

$$
\begin{aligned}
\psi_\alpha(\xi,\tau^*) &= \psi_\alpha(\xi,0) + M^{-1}\left\{\frac{1}{\vartheta}M\{-2\}\right\} \\
&= 1 + e^\xi - 2\frac{\tau^{*\vartheta}}{\vartheta!} \\
\varphi_\beta(\xi,\tau^*) &= \varphi_\beta(\xi,0) + M^{-1}\left\{\frac{1}{\vartheta}M\{-2\}\right\} \\
&= -1 + e^\xi - 2\frac{\tau^{*\vartheta}}{\vartheta!},
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
\psi_{(m+1)\alpha}(\xi,\tau^*) &= M^{-1}\{\frac{1}{\vartheta}M\{\varphi_{m\beta\xi}(\xi,\tau^*) - \varphi_{m\beta}(\xi,\tau^*) \\
&\quad + \quad \psi_{m\alpha}(\xi,\tau^*)\}\} \\
\varphi_{(m+1)\beta}(\xi,\tau^*) &= M^{-1}\{\frac{1}{\vartheta}M\{-\psi_{m\alpha\xi}(\xi,\tau^*) + \psi_{m\alpha}(\xi,\tau^*) \\
&\quad - \quad \varphi_{m\beta}(\xi,\tau^*)\}\}
\end{aligned} \tag{40}
$$

From the recursive formula (40),
for $m = 0$

$$
\begin{aligned}
\psi_{1\alpha}(\xi,\tau^*) &= M^{-1}\{\frac{1}{\vartheta}M\{\varphi_{0\beta\xi}(\xi,\tau^*) - \varphi_{0\beta}(\xi,\tau^*) \\
&\quad + \quad \psi_{0\alpha}(\xi,\tau^*)\}\} \\
\varphi_{1\beta}(\xi,\tau^*) &= M^{-1}\{\frac{1}{\vartheta}M\{-\psi_{0\alpha\xi}(\xi,\tau^*) + \psi_{0\alpha}(\xi,\tau^*) \\
&\quad - \quad \varphi_{0\beta}(\xi,\tau^*)\}\}
\end{aligned} \tag{41}
$$

we get

$$
\begin{aligned}
\psi_{1\alpha}(\xi,\tau^*) &= (e^\xi - 2)\frac{\tau^{*\vartheta}}{\vartheta!} \\
\varphi_{1\beta}(\xi,\tau^*) &= (e^\xi - 2)\frac{\tau^{*\vartheta}}{\vartheta!}
\end{aligned} \tag{42}
$$

for $m = 1$

$$
\begin{aligned}
\psi_{2\alpha}(\xi,\tau^*) &= e^\xi \frac{\tau^{*2\vartheta}}{2\vartheta!} \\
\varphi_{2\beta}(\xi,\tau^*) &= e^{x\xi} \frac{\tau^{*2\vartheta}}{2\vartheta!}
\end{aligned} \tag{43}
$$

for $m = 2$

$$
\begin{aligned}
\psi_{3\alpha}(\xi,\tau^*) &= e^\xi \frac{\tau^{*3\vartheta}}{3\vartheta!} \\
\varphi_{3\beta}(\xi,\tau^*) &= e^\xi \frac{\tau^{*3\vartheta}}{3\vartheta!} \\
&\vdots
\end{aligned} \tag{44}
$$

The Mohand transform solution for example 4.2 is

$$
\begin{aligned}
\psi_\alpha(\xi,\tau^*) &= \psi_{0\alpha}(\xi,\tau^*) + \psi_{1\alpha}(\xi,\tau^*) + \psi_{2\alpha}(\xi,\tau^*) \\
&\quad + \psi_{3\alpha}(\xi,\tau^*) + \cdots \\
\varphi_\beta(\xi,\tau^*) &= \varphi_{0\beta}(\xi,\tau^*) + \varphi_{1\beta}(\xi,\tau^*) + \varphi_{2\beta}(\xi,\tau^*)^a \\
&\quad + \varphi_{3\beta}(\xi,\tau^*) + \cdots
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\psi_\alpha(\xi,\tau^*) &= 1 + e^\xi - 2\frac{\tau^{*\vartheta}}{\vartheta!} + (e^\xi + 2)\frac{\tau^{*\vartheta}}{\vartheta!} + e^\xi\frac{\tau^{*2\vartheta}}{2\vartheta!} \\
&\quad + \frac{\tau^{*3\vartheta}}{3\vartheta!} + \cdots
\end{aligned}
$$

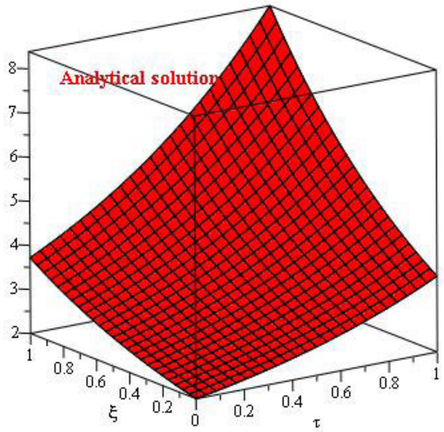

FIGURE 7. $\psi_\alpha(\xi,\tau^*)$ Represents the analytical solution of example 4.2.

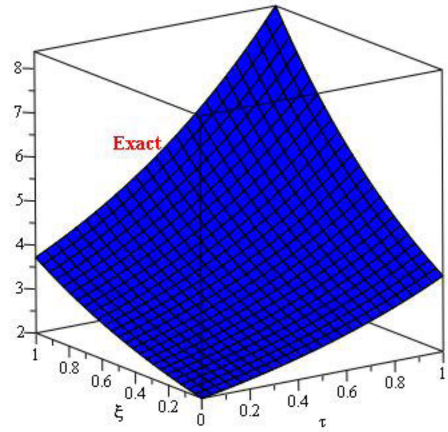

FIGURE 8. $\psi_\alpha(\xi,\tau^*)$ Represents the exact solution of example 4.2.

$$\varphi_\beta(\xi,\tau^*) = -1 + e^{\xi} - 2\frac{\tau^{*\vartheta}}{\vartheta!} + (e^{\xi}+2)\frac{\tau^{*\vartheta}}{\vartheta!} + e^{\xi}\frac{\tau^{*2\vartheta}}{2\vartheta!} + \frac{\tau^{*3\vartheta}}{3\vartheta!} + \cdots \tag{46}$$

By simplify, we get

$$\begin{aligned}\psi_\alpha(\xi,\tau^*) &= 1 + e^{\xi}\left\{1 + \frac{\tau^{*\vartheta}}{\vartheta!} + \frac{\tau^{*2\vartheta}}{2\vartheta!} + \frac{\tau^{*3\vartheta}}{3\vartheta!} + \cdots\right\}\\ \varphi_\beta(\xi,\tau^*) &= -1 + e^{\xi}\left\{1 + \frac{\tau^{*\vartheta}}{\vartheta!} + \frac{\tau^{*2\vartheta}}{2\vartheta!} + \frac{\tau^{*3\vartheta}}{3\vartheta!} + \cdots\right\}\end{aligned} \tag{47}$$

For particular case $\vartheta = 1$, the Mohand transform solution become as

$$\begin{aligned}\psi_\alpha(\xi,\tau^*) &= 1 + e^{\xi}\left\{1 + \tau^* + \frac{\tau^{*2}}{2!} + \frac{\tau^{*3}}{3!} + \cdots\right\}\\ \varphi_\beta(\xi,\tau^*) &= -1 + e^{\xi}\left\{1 + \tau^* + \frac{\tau^{*2}}{2!} + \frac{\tau^{*3}}{3!} + \cdots\right\}\end{aligned} \tag{48}$$

The calculated result provide the exact solution in the close form

$$\begin{aligned}\psi_\alpha(\xi,\tau^*) &= 1 + e^{\xi+\tau^*}\\ \varphi_\beta(\xi,\tau^*) &= -1 + e^{\xi+\tau^*}\end{aligned} \tag{49}$$

EXAMPLE 4.3

Consider the third order time fractional PDE system the form [33]

$$\begin{aligned}&D_\tau^{*\vartheta}\psi_\alpha(\xi,\tau^*) - \psi_{\alpha\xi\xi}(\xi,\tau^*) - 2\psi_\alpha(\xi,\tau^*)\psi_{\alpha\xi}(\xi,\tau^*) + \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi = 0\\ &D_\tau^{*\vartheta}\varphi_\beta(\xi,\tau^*) - \varphi_{\beta\xi\xi}(\xi,\tau^*) - 2\varphi_\beta(\xi,\tau^*)\varphi_{\beta\xi}(\xi,\tau^*) + \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi = 0\\ &0 < \vartheta \le 1\end{aligned} \tag{50}$$

with initial condition

$$\psi_\alpha(\xi,0) = \sin(\xi), \quad \varphi_\beta(\xi,0) = \sin(\xi)$$

The exact solution of equation (50) is

$$\psi_\alpha(\xi,\tau^*) = \sin(\xi)e^{-\tau^*}, \quad \varphi_\beta(\xi,\tau^*) = \sin(\xi)e^{-\tau^*}$$

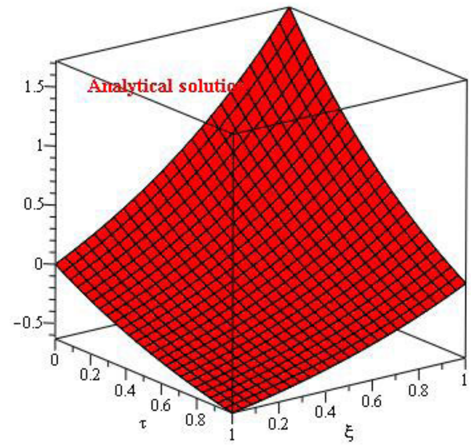

FIGURE 9. $\psi_\beta(\xi,\tau^*)$ Represents the analytical solution of example 4.2.

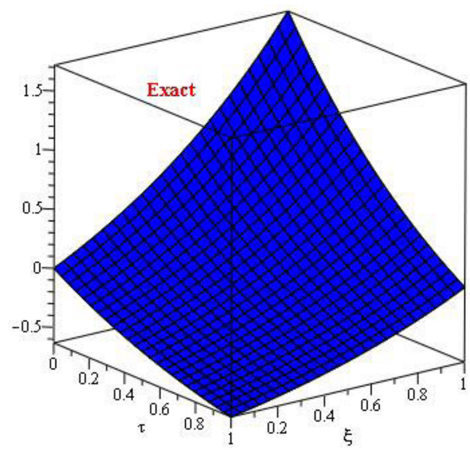

FIGURE 10. $\psi_\beta(\xi,\tau^*)$ Represents the exact solution of example 4.2.

Taking Mohand transform of equation (50), we get

$$\begin{aligned}\upsilon^{\vartheta}\{R(\upsilon) - \upsilon u(0)\} = M\{&\psi_{\alpha\xi\xi}(\xi,\tau^*)\\ &- 2\psi_\alpha(\xi,\tau^*)\psi_{\alpha\xi}(\xi,\tau^*)\\ &+ \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi\}\end{aligned}$$

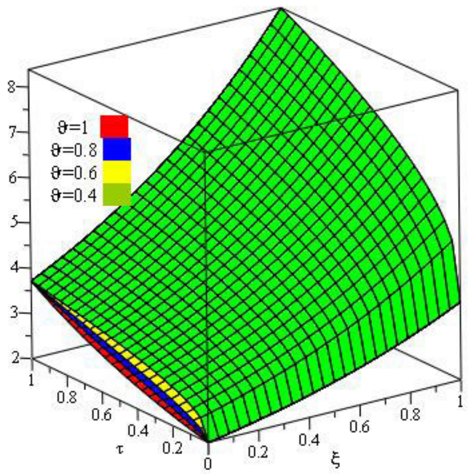

FIGURE 11. $\psi_\alpha(\xi,\tau^*)$ **Represents the solution at different fractional order of example 4.2.**

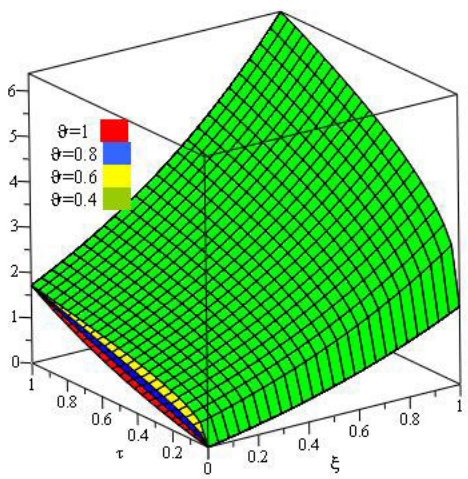

FIGURE 12. $\varphi_\beta(\xi,\tau^*)$ **Represents the solution at different fractional order of example 4.2.**

$$\upsilon^\vartheta\{R(\upsilon)-\upsilon\nu(0)\} = M\{\varphi_{\beta\xi\xi}(\xi,\tau^*) - 2\varphi_\beta(\xi,\tau^*)\varphi_{\beta\xi}(\xi,\tau^*) + \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi\} \quad (51)$$

After some evaluation, equation (51) is simplified as

$$R(\upsilon) = \upsilon u(0)+\frac{1}{\upsilon^\vartheta}M\{\psi_{\alpha\xi\xi}(\xi,\tau^*)-2\psi_\alpha(\xi,\tau^*)\psi_{\alpha\xi}(\xi,\tau^*) + \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi\}$$
$$R(\upsilon) = \upsilon\nu(0)+\frac{1}{\upsilon^\vartheta}M\{\varphi_{\beta\xi\xi}(\xi,\tau^*)-2\varphi_\beta(\xi,\tau^*)\varphi_{\beta\xi}(\xi,\tau^*) + \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi\} \quad (52)$$

By applying inverse Mohand transform, we get

$$u(x^*,\tau^*) = u(x^*,0)+M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\psi_{\alpha\xi\xi}(\xi,\tau^*) - 2\psi_\alpha(\xi,\tau^*)\psi_{\alpha\xi}(\xi,\tau^*) + \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi\}\}$$
$$\nu(x^*,\tau^*) = \nu(x^*,0)+M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\varphi_{\beta\xi\xi}(\xi,\tau^*) - 2\varphi_\beta(\xi,\tau^*)\varphi_{\beta\xi}(\xi,\tau^*) + \left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi\}\} \quad (53)$$

Thus, by using the define recursive scheme of equation (11), (12), we get

$$\psi_{0\alpha}(\xi,\tau^*) = \psi_\alpha(\xi,0)=\sin(\xi),\quad \varphi_{0\beta}(\xi,\tau^*)=\varphi_\beta(\xi,0) = \sin \quad (54)$$

$$\psi_{(m+1)\alpha}(\xi,\tau^*) = M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\psi_{\alpha\xi\xi}(\xi,\tau^*)-2\psi_\alpha(\xi,\tau^*) \psi_{\alpha\xi}(\xi,\tau^*)+\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi\}\}$$
$$\varphi_{(m+1)\beta}(\xi,\tau^*) = M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\varphi_{\beta\xi\xi}(\xi,\tau^*)-2\varphi_\beta(\xi,\tau^*) \varphi_{\beta\xi}(\xi,\tau^*)\left(\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)\right)_\xi\}\} \quad (55)$$

The non-linear term is obtained by using equation (13), the recursive scheme (55) become as

$$\psi_{(m+1)\alpha}(\xi,\tau^*) = M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\psi_{\alpha\xi\xi}(\xi,\tau^*)+2\sum_{m=0}^{\infty}A_m - \sum_{m=0}^{\infty}B_m\}\}$$
$$\varphi_{(m+1)\beta}(\xi,\tau^*) = M^{-1}\{\frac{1}{\upsilon^\vartheta}M\{\varphi_{\beta\xi\xi}(\xi,\tau^*)+2\sum_{m=0}^{\infty}A_m - \sum_{m=0}^{\infty}B_m\}\} \quad (56)$$

From the recursive formula (56),
for $m = 0$

$$\psi_{1\alpha}(\xi,\tau^*) = M^{-1}\left\{\frac{1}{\upsilon^\vartheta}M\left\{\psi_{0\alpha\xi\xi}(\xi,\tau^*)+2A_0-B_0\right\}\right\}$$
$$\varphi_{1\beta}(\xi,\tau^*) = M^{-1}\left\{\frac{1}{\upsilon^\vartheta}M\left\{\varphi_{0\beta\xi\xi}(\xi,\tau^*)+2A_0-B_0\right\}\right\} \quad (57)$$

We get

$$\psi_{1\alpha}(\xi,\tau^*) = M^{-1}\frac{1}{\upsilon^\vartheta}M\{\psi_{\alpha\xi\xi}(\xi,\tau^*) + 2\psi_\alpha(\xi,\tau^*)\psi_{\alpha\xi}(\xi,\tau^*)\} - \{\psi_\alpha(\xi,\tau^*)\varphi_\beta(\xi,\tau^*)_\xi\}$$
$$\varphi_\beta(\xi,\tau^*) = M^{-1}\frac{1}{\upsilon^\vartheta}M\varphi_{0\beta}(\xi,\tau^*) + 2\varphi_\beta(\xi,\tau^*)\{\varphi_{0\beta\xi}(\xi,\tau^*) - \{\psi_\alpha(\xi,\tau^*)\varphi_{0\beta}(\xi,\tau^*)_\xi\}\} \quad (58)$$

By putting the values and using transform property, we get

$$\psi_{1\alpha}(\xi,\tau^*) = -\sin(\xi)\frac{\tau^{*\vartheta}}{\vartheta!}$$
$$\varphi_{1\beta}(\xi,\tau^*) = -\sin(\xi)\frac{\tau^{*\vartheta}}{\vartheta!} \quad (59)$$

for $m = 1$, we will find the non-linear term by using equation (13)

$$\psi_{2\alpha}(\xi,\tau^*) = M^{-1}\left\{\frac{1}{\upsilon^{\vartheta}}M\left\{\psi_{1\alpha\xi\xi}(\xi,\tau^*)+2A_1-B_1\right\}\right\}$$
$$\varphi_{2\beta}(\xi,\tau^*) = M^{-1}\left\{\frac{1}{\upsilon^{\vartheta}}M\left\{\varphi_{1\beta\xi\xi}(\xi,\tau^*)+2A_1-B_1\right\}\right\} \quad (60)$$

$$\psi_{2\alpha}(\xi,\tau^*) = \sin(\xi)\frac{\tau^{*2\vartheta}}{2\vartheta!}$$
$$\varphi_{2\beta}(\xi,\tau^*) = \sin(\xi)\frac{\tau^{*2\vartheta}}{2\vartheta!} \quad (61)$$

for $m = 2$

$$\psi_{3\alpha}(\xi,\tau^*) = -\sin(\xi)\frac{\tau^{*3\vartheta}}{3\vartheta!}$$
$$\varphi_{3\beta}(\xi,\tau^*) = -\sin(\xi)\frac{\tau^{*3\vartheta}}{3\vartheta!} \quad (62)$$

for $m = 3$

$$\psi_{4\alpha}(\xi,\tau^*) = \frac{\sin(\xi)\tau^{*4\vartheta}}{4\vartheta!}$$
$$\varphi_{4\beta}(\xi,\tau^*) = \frac{\sin(\xi)\tau^{*\vartheta}}{4\vartheta!} \quad (63)$$

for $m = 4$

$$\psi_{5\alpha}(\xi,\tau^*) = \frac{-\sin(\xi)\tau^{*5\vartheta}}{5\vartheta!}$$
$$\varphi_{5\beta}(\xi,\tau^*) = \frac{-\sin(\xi)\tau^{*5\vartheta}}{5\vartheta!}$$
$$\vdots \quad (64)$$

The Mohand transform solution for example 4.3 is

$$\psi_{\alpha}(\xi,\tau^*) = \psi_{0\alpha}(\xi,\tau^*)+\psi_{1\alpha}(\xi,\tau^*)+\psi_{2\alpha}(\xi,\tau^*) + \psi_{3\alpha}(\xi,\tau^*)+\psi_{4\alpha}(\xi,\tau^*)+\cdots$$
$$\varphi_{\beta}(\xi,\tau^*) = \varphi_{0\beta}(\xi,\tau^*)+\varphi_{1\beta}(\xi,\tau^*)+\varphi_{2\beta}(\xi,\tau^*) + \varphi_{3\beta}(\xi,\tau^*)+\varphi_{4\beta}(\xi,\tau^*)+\cdots \quad (65)$$

$$\psi_{\alpha}(\xi,\tau^*) = \sin(\xi)\{1-\frac{\tau^{*\vartheta}}{\vartheta!}+\frac{\tau^{*2\vartheta}}{2\vartheta!}-\frac{\tau^{*3\vartheta}}{3\vartheta!}+\frac{\tau^{*4\vartheta}}{4\vartheta!} -\frac{\tau^{*5\vartheta}}{5\vartheta!}\cdots\}$$
$$\varphi_{\beta}(\xi,\tau^*) = \sin(\xi)\{1-\frac{\tau^{*\vartheta}}{\vartheta!}+\frac{\tau*2\vartheta}{2\vartheta!}-\frac{\tau^{*3\vartheta}}{3\vartheta!}+\frac{\tau^{*4\vartheta}}{4\vartheta!} -\frac{\tau^{*5\vartheta}}{5\vartheta!}\cdots\} \quad (66)$$

For particular case $\vartheta = 1$, the Mohand transform solution become as

$$\psi_{\alpha}(\xi,\tau^*) = \sin(\xi)\{1-\tau^*+\frac{\tau^{*2}}{(2)!}-\frac{\tau^{*3}}{(3)!}+\frac{\tau^{*4}}{(4)!} -\frac{\tau^{*5}}{(5)!}+\cdots\}$$

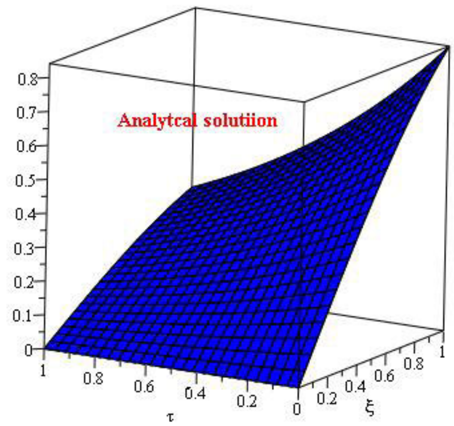

FIGURE 13. $\psi_{\alpha}(\xi,\tau^*)$ **Represents the analytical solution of example 4.3.**

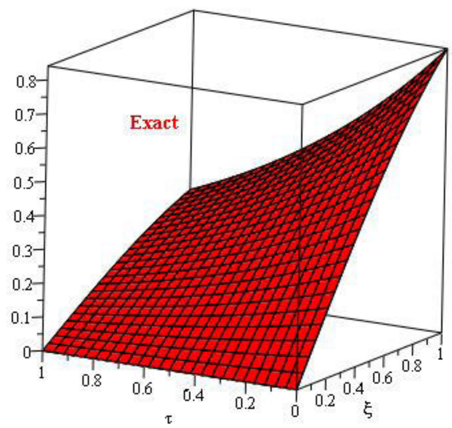

FIGURE 14. $\psi_{\alpha}(\xi,\tau^*)$ **Represents the exact solution of example 4.3.**

$$\varphi_{\beta}(\xi,\tau^*) = \sin(\xi)\{1-\tau^*+\frac{\tau^{*2}}{(2)!}-\frac{\tau^{*3}}{(3)!}+\frac{\tau^{4}}{(4)!} -\frac{\tau^{*5}}{(5)!}+\cdots\} \quad (67)$$

The calculated result provide the exact solution in the close form

$$\psi_{\alpha}(\xi,\tau^*) = e^{-\tau^*}\sin(\xi), \quad \varphi_{\beta}(\xi,\tau^*) = e^{-\tau^*}\sin(\xi) \quad (68)$$

EXAMPLE 4.4

Consider the time fractional PDE system the form [59]

$$D_{\tau}^{*\vartheta}\psi_{\alpha}(\xi,\zeta,\tau^*)-\varphi_{\beta\xi}(\xi,\zeta,\tau^*)\phi_{\gamma\zeta}(\xi,\zeta,\tau^*) = 1$$
$$D_{\tau}^{*\vartheta}\varphi_{\beta}(\xi,\zeta,\tau^*)-\phi_{\gamma\xi}(\xi,\zeta,\tau^*)\psi_{\alpha\zeta}(\xi,\zeta,\tau^*) = 5$$
$$D_{\tau}^{*\vartheta}\phi_{\gamma}(\xi,\zeta,\tau^*)-\psi_{\alpha\xi}(\xi,\zeta,\tau^*)\varphi_{\beta\zeta}(\xi,\zeta,\tau^*) = 5$$
$$0<\vartheta\leq 1 \quad (69)$$

with initial source

$$\psi_{\alpha}(\xi,\zeta,0) = \xi+2\zeta, \quad \varphi_{\beta}(\xi,\zeta,0) = \xi-2\zeta, \quad \phi_{\gamma}(\xi,\zeta,0) = -\xi+2\zeta$$

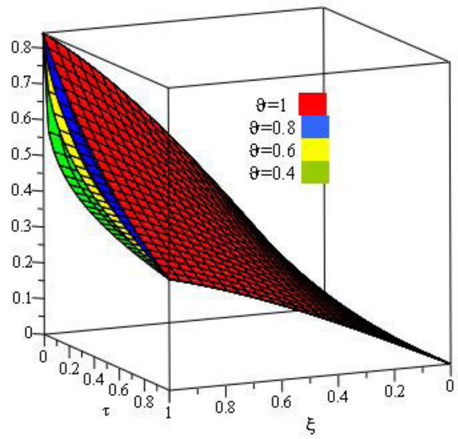

FIGURE 15. $\psi_\alpha(\xi,\tau^*)$ **Represents the solution at different fractional order of example 4.3.**

The exact solution of equation (69) is $\vartheta = 1$

$$\psi_\alpha(\xi,\zeta,\tau^*) = \xi + 2\zeta + 3\tau^*,$$
$$\varphi_\beta(\xi,\zeta,\tau^*) = \xi - 2\zeta + 3\tau^*,$$
$$\phi_\gamma(\xi,\zeta,\tau^*) = -\xi + 2\zeta + 3\tau^*$$

Taking Mohand transform of equation (69)

$$\upsilon^\vartheta\,(R(\upsilon) - \upsilon\psi_\alpha(0)) - M\left\{\varphi_{\beta\xi}(\xi,\zeta,\tau^*)\phi_{\gamma\zeta}(\xi,\zeta,\tau^*)\right\} = M\{1\}$$
$$\upsilon^\vartheta(R(\upsilon) - \upsilon\varphi_\beta(0)) - M\left\{\phi_{\gamma\xi}(\xi,\zeta,\tau^*)\psi_{\alpha\zeta}(\xi,\zeta,\tau^*)\right\} = M\{5\}$$
$$\upsilon^\vartheta(R(\upsilon) - \upsilon\phi_\gamma(0)) - M\left\{\psi_{\alpha\xi}(\xi,\zeta,\tau^*)\varphi_{\beta\zeta}(\xi,\zeta,\tau^*)\right\} = M\{5\}, \tag{70}$$

After some evaluation, equation (70) is simplified as

$$R(\upsilon) = \upsilon\psi_\alpha(0) + \frac{1}{\vartheta}M\left\{\varphi_{\beta\xi}(\xi,\zeta,\tau^*)\phi_{\gamma\zeta}(\xi,\zeta,\tau^*)\right\} + \frac{1}{\vartheta}M\{1\}$$
$$R(\upsilon) = \upsilon\varphi_\beta(0) + \frac{1}{\vartheta}M\left\{\phi_{\gamma\xi}(\xi,\zeta,\tau^*)\psi_{\alpha\zeta}(\xi,\zeta,\tau^*)\right\} + \frac{1}{\vartheta}M\{5\}$$
$$R(\upsilon) = \upsilon\phi_\gamma(0) + \frac{1}{\vartheta}M\left\{\psi_{\alpha\xi}(\xi,\zeta,\tau^*)\varphi_{\beta\zeta}(\xi,\zeta,\tau^*)\right\} + \frac{1}{\vartheta}M\{5\}, \tag{71}$$

Taking inverse Mohand transform of equation (71)

$$\psi_\alpha(\xi,\zeta,\tau^*) = \psi_\alpha(\xi,\zeta,0) + M^{-1}\{\frac{1}{\vartheta}M\{\varphi_{\beta\xi}(\xi,\zeta,\tau^*)\phi_{\gamma\zeta}(\xi,\zeta,\tau^*)\}\} + M^{-1}\left\{\frac{1}{\vartheta}M\{1\}\right\}$$
$$\varphi_\beta(\xi,\zeta,\tau^*) = \varphi_\beta(\xi,\zeta,0) + M^{-1}\{\frac{1}{\vartheta}M\{\phi_{\gamma\xi}(\xi,\zeta,\tau^*)\psi_{\alpha\zeta}(\xi,\zeta,\tau^*)\}\} + M^{-1}\left\{\frac{1}{\vartheta}M\{5\}\right\}$$
$$\phi_\gamma(\xi,\zeta,\tau^*) = \phi_\gamma(\xi,\zeta,0) + M^{-1}\{\frac{1}{\vartheta}M\{\psi_{\alpha\xi}(\xi,\zeta,\tau^*)\varphi_{\beta\zeta}(\xi,\zeta,\tau^*)\}\} + M^{-1}\left\{\frac{1}{\vartheta}M\{5\}\right\}, \tag{72}$$

By using the recursive scheme equation (11), we get

$$x\psi_{0\alpha}(\xi,\zeta,\tau^*) = \psi_\alpha(\xi,\zeta,0) + M^{-1}\left\{\frac{1}{\vartheta}M\{1\}\right\} = \xi + 2\zeta + \frac{\tau^\vartheta}{\vartheta!}$$
$$\varphi_{0\beta}(\xi,\zeta,\tau^*) = \varphi_\beta(\xi,\zeta,0) + M^{-1}\left\{\frac{1}{\vartheta}M\{5\}\right\} = \xi - 2\zeta + \frac{5\tau^{*\vartheta}}{\vartheta!}$$
$$\phi_{0\gamma}(\xi,\zeta,\tau^*) = \phi_\gamma(\xi,\zeta,0) + M^{-1}\left\{\frac{1}{\vartheta}M\{5\}\right\} = -\xi + 2\zeta + \frac{5\tau^{*\vartheta}}{\vartheta!} \tag{73}$$

$$\psi_{(m+1)\alpha\zeta}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{\varphi_{m\beta\xi}(\xi,\zeta,\tau^*)\phi_{m\gamma\zeta}(\xi,\zeta,\tau^*)\right\}\right\}$$
$$\varphi_{(m+1)\beta\xi}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{\phi_{m\gamma\xi}(\xi,\zeta,\tau^*)\psi_{m\alpha\zeta}(\xi,\zeta,\tau^*)\right\}\right\}$$
$$\phi_{(m+1)\gamma\xi}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{\psi_{m\alpha\xi}(\xi,\zeta,\tau^*)\varphi_{m\beta\zeta}(\xi,\zeta,\tau^*)\right\}\right\} \tag{74}$$

The non-linear term can calculated by using equation (13), we get

$$\psi_{(m+1)\alpha\zeta}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{\sum_{m=0}^{\infty}A_m\right\}\right\}$$
$$\varphi_{(m+1)\beta\xi}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{w\sum_{m=0}^{\infty}B_m\right\}\right\}$$
$$\phi_{(m+1)\gamma\xi}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{\sum_{m=0}^{\infty}C_m\right\}\right\} \tag{75}$$

From the recursive formula (75),
for $m = 0$

$$\psi_{1\alpha\zeta}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\{A_0\}\right\}$$
$$\varphi_{1\beta\xi}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\{B_0\}\right\}$$
$$\phi_{1\gamma\xi}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\{C_0\}\right\} \tag{76}$$

we get

$$\psi_{1\alpha\zeta}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{\varphi_{0\beta\xi}(\xi,\zeta,\tau^*)\phi_{0\gamma\zeta}(\xi,\zeta,\tau^*)\right\}\right\}$$

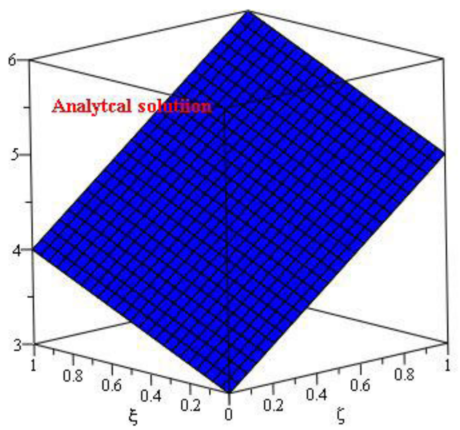

FIGURE 16. $\psi_{\alpha}(\xi,\zeta,\tau^*)$ Represents the analytical solution of example 4.4.

$$\varphi_{1\beta\xi}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{\phi_{0\gamma\xi}(\xi,\zeta,\tau^*)\psi_{0\alpha\zeta}(\xi,\zeta,\tau^*)\right\}\right\}$$
$$\phi_{1\gamma\xi}(\xi,\zeta,\tau^*) = M^{-1}\left\{\frac{1}{\vartheta}M\left\{\psi_{0\alpha\xi}(\xi,\zeta,\tau^*)\varphi_{0\beta\zeta}(\xi,\zeta,\tau^*)\right\}\right\} \tag{77}$$

$$\psi_{1\alpha\zeta}(\xi,\zeta,\tau^*) = 2\frac{\tau^{*\vartheta}}{\vartheta!}$$
$$\varphi_{1\beta\xi}(\xi,\zeta,\tau^*) = -2\frac{\tau^{*\vartheta}}{\vartheta!}$$
$$\phi_{1\gamma\xi}(\xi,\zeta,\tau^*) = -2\frac{\tau^{*\vartheta}}{\vartheta!} \tag{78}$$

for $m = 1$

$$\psi_{2\alpha\zeta}(\xi,\zeta,\tau^*) = 0$$
$$\varphi_{2\beta\xi}(\xi,\zeta,\tau^*) = 0$$
$$\phi_{2\gamma\xi}(\xi,\zeta,\tau^*) = 0 \tag{79}$$

for $m = 2$

$$\psi_{3\alpha\zeta}(\xi,\zeta,\tau^*) = 0$$
$$\varphi_{3\beta\xi}(\xi,\zeta,\tau^*) = 0$$
$$\phi_{3\gamma\xi}(\xi,\zeta,\tau^*) = 0$$
$$\vdots \tag{80}$$

The Mohand transform solution for example 4.4 is

$$\psi_{\alpha\zeta}(\xi,\zeta,\tau^*) = \psi_{0\alpha\zeta}(\xi,\zeta,\tau^*) + \psi_{1\alpha\zeta}(\xi,\zeta,\tau^*) + \psi_{2\alpha\zeta}(\xi,\zeta,\tau^*) + \psi_{3\alpha\zeta}(\xi,\zeta,\tau^*) + \cdots$$
$$\varphi_{\beta\xi}(\xi,\zeta,\tau^*) = \varphi_{0\beta\xi}(\xi,\zeta,\tau^*) + \varphi_{1\beta\xi}(\xi,\zeta,\tau^*) + \varphi_{2\beta\xi}(\xi,\zeta,\tau^*) + \varphi_{3\beta\xi}(\xi,\zeta,\tau^*) + \cdots$$
$$\phi_{\gamma\xi}(\xi,\zeta,\tau^*) = \phi_{0\gamma\xi}(\xi,\zeta,\tau^*) + \phi_{1\gamma\xi}(\xi,\zeta,\tau^*) + \phi_{2\gamma\xi}(\xi,\zeta,\tau^*) + \phi_{3\gamma\xi}(\xi,\zeta,\tau^*)) + \cdots \tag{81}$$

$$\psi_{\alpha\zeta}(\xi,\zeta,\tau^*) = \xi + 2\zeta + \frac{\tau^{*\vartheta}}{\vartheta!} + 2\frac{\tau^{*\vartheta}}{\vartheta!} + 0 + 0 + \cdots$$
$$\varphi_{\beta\xi}(\xi,\zeta,\tau^*) = \xi - 2\zeta + \frac{5\tau^{*\vartheta}}{\vartheta!} - 2\frac{\tau^{*\vartheta}}{\vartheta!} + 0 + 0 + \cdots$$
$$\phi_{\gamma\xi}(\xi,\zeta,\tau^*) = -\xi + 2\zeta + \frac{5\tau^{*\vartheta}}{\vartheta!} - 2\frac{\tau^{*\vartheta}}{\vartheta!} + 0 + 0 + \cdots \tag{82}$$

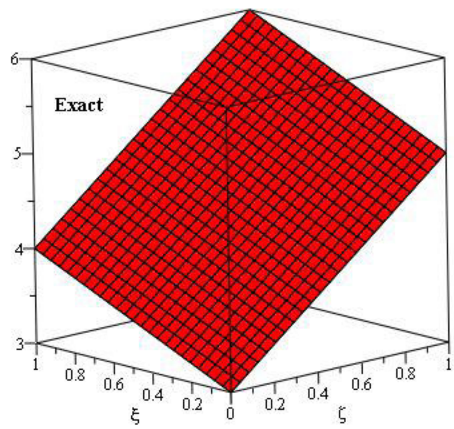

FIGURE 17. $\psi_{\alpha}(\xi,\zeta,\tau^*)$ Represents the exact solution of example 4.4.

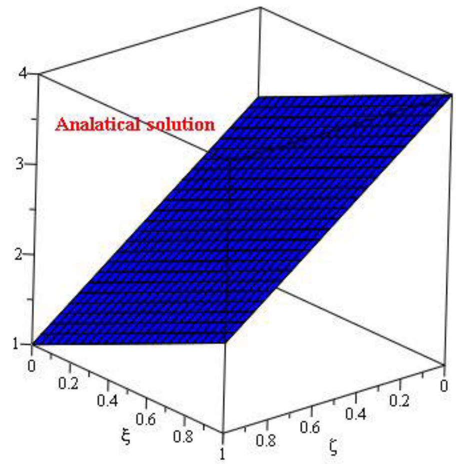

FIGURE 18. $\psi_{\beta}(\xi,\zeta,\tau^*)$ Represents the analytical solution of example 4.4.

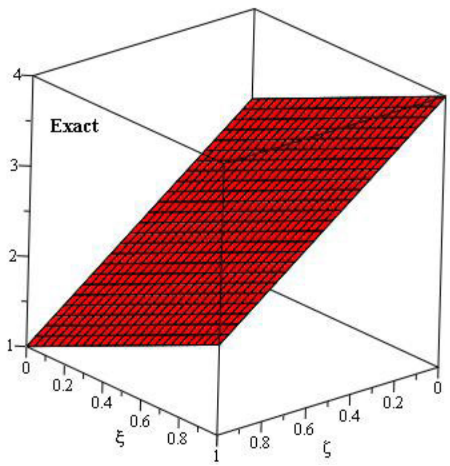

FIGURE 19. $\psi_{\beta}(\xi,\zeta,\tau^*)$ Represents the exact solution of example 4.4.

By simplify, we get

$$\psi_{\alpha\zeta}(\xi,\zeta,\tau^*) = \xi + 2\zeta + \frac{3\tau^{*\vartheta}}{\vartheta!}$$
$$\varphi_{\beta\xi}(\xi,\zeta,\tau^*) = \xi - 2\zeta + \frac{3\tau^{*\vartheta}}{\vartheta!}$$
$$\phi_{\gamma\xi}(\xi,\zeta,\tau^*) = -\xi + 2\zeta + \frac{3\tau^{*\vartheta}}{\vartheta!} \tag{83}$$

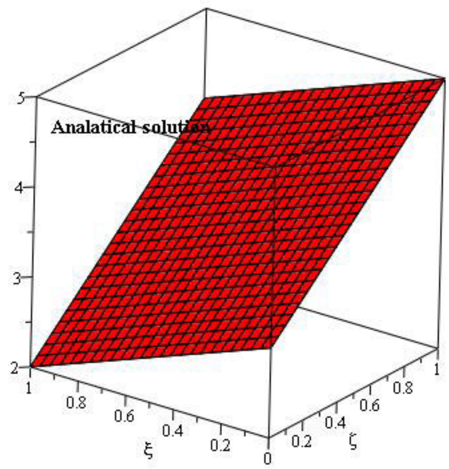

FIGURE 20. $\psi_\gamma(\xi, \zeta, \tau^*)$ Represents the analytical solution of example 4.4.

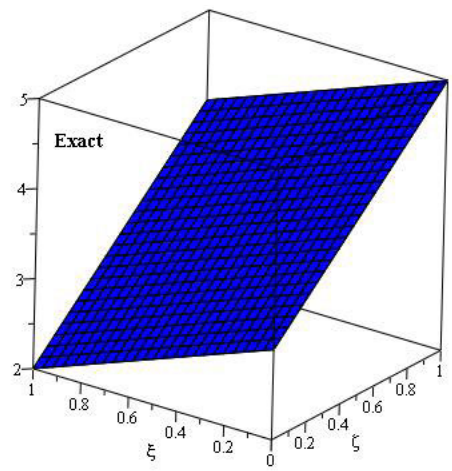

FIGURE 21. $\psi_\gamma(\xi, \zeta, \tau^*)$ Represents the exact solution of example 4.4.

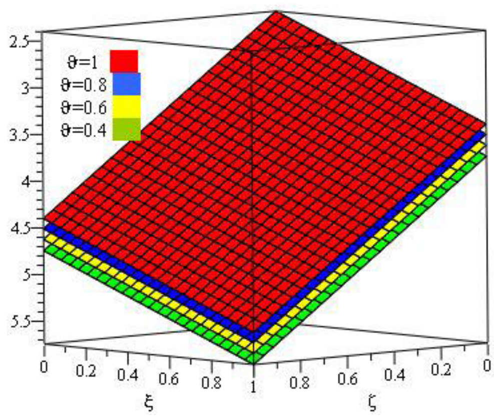

FIGURE 22. $\psi_\alpha(\xi, \zeta, \tau^*)$ Represents the solution at different fractional order of example 4.3.

For particular case $\vartheta = 1$, the Mohand transform solution become as

$$
\begin{aligned}
\psi_{\alpha\zeta}(\xi, \zeta, \tau^*) &= \xi + 2\zeta + 3\tau^* \\
\varphi_{\beta\xi}(\xi, \zeta, \tau^*) &= \xi - 2\zeta + 3\tau^* \\
\phi_{\gamma\xi}(\xi, \zeta, \tau^*) &= -\xi + 2\zeta + 3\tau^* \qquad (84)
\end{aligned}
$$

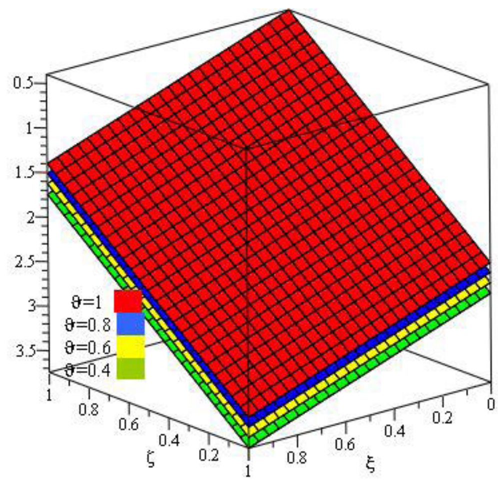

FIGURE 23. $\varphi_\beta(\xi, \zeta, \tau^*)$ Represents the solution at different fractional order of example 4.3.

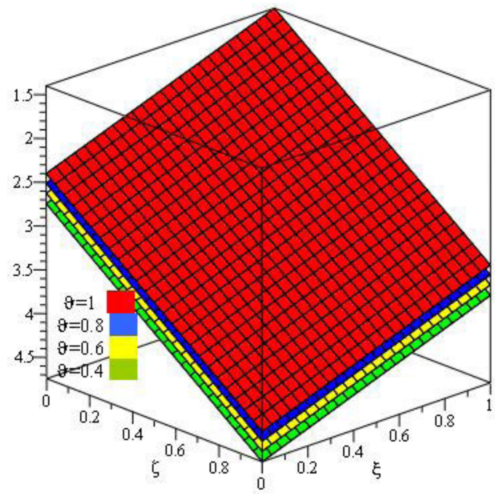

FIGURE 24. $\phi_\gamma(\xi, \zeta, \tau^*)$ Represents the solution at different fractional order of example 4.3.

The calculated result provide the exact solution in the close form

$$
\begin{aligned}
\psi_{\alpha\zeta}(\xi, \zeta, \tau^*) &= \xi + 2\zeta + 3\tau^* \\
\varphi_{\beta\xi}(\xi, \zeta, \tau^*) &= \xi - 2\zeta + 3\tau^* \\
\phi_{\gamma\xi}(\xi, \zeta, \tau^*) &= -\xi + 2\zeta + 3\tau^* \qquad (85)
\end{aligned}
$$

V. CONCLUSION

Mohand transformation along with Decomposition method is a hybrid analytical technique that can be used effectively to solve system of fractional-order nonlinear partial differential equations. The fractional-order derivatives are defined the term of Caputo operator. The applicability and credibility of the suggested method is investigated through illustration examples. Higher rate of convergence is achieved of the proposed method. Moreover, the closed form analytical solution is obtained, which has classified the current method among the best analytical techniques. The easy and straightforward implementations of the method have extended the idea to solve other nonlinear systems of fractional-Order differential equations.

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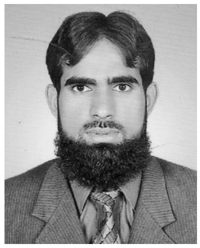

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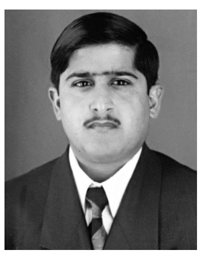

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