REPRESENTATION FOR THE REPRODUCING KERNEL HILBERT SPACE METHOD FOR A NONLINEAR SYSTEM

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Abstract. We apply the reproducing kernel Hilbert space method to a nonlinear system in this work. We utilize this technique to overcome the nonlinearity of the problem. We have obtained accurate results. We have demonstrated our results by tables and figures. We have proved the efficiency of the method.

1. Introduction

Implementations of the kernel methods have been investigated by many authors [?]. Approximation of stochastic partial differential equations [?], numerical solution of integral equations [?], multiple solutions of nonlinear boundary value problems [?] and applications to machine learning algorithms [?]. The reproducing kernel Hilbert space methods have been applied successfully to several nonlinear problems such as, nonlinear singular Lane–Emden type equations and singular nonlinear two-point periodic boundary value problem [?]. For more details see [?, ?].

The governing equations for mass, momentum and energy in unsteady two-dimensional flow of a nano-fluid [?, ?, ?] are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1.1}$$

$$\rho_{nf}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu_{nf}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{1.2}$$

$$\rho_{nf}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu_{nf}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{1.3}$$

2010 *Mathematics Subject Classification.* 47B32, 46E22, 35M32 and 74S30.

Key words and phrases. Reproducing kernel functions, a nonlinear system, bounded linear operator.

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} &= \frac{k_{nf}}{(\rho C_p)_{nf}}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)\\ &+ \frac{\mu_{nf}}{(\rho C_p)_{nf}}\left(4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2\right), \quad (1.4)\end{aligned}$$

$$v=v_w=\frac{dh}{dt},\quad T=T_H \qquad \text{at} \qquad y=h(t),$$
$$v=\frac{\partial u}{\partial y}=\frac{\partial T}{\partial y}=0 \qquad \text{at} \qquad y=0. \quad (1.5)$$

Let us define the similarity transform as below:

$$\begin{aligned}\eta &= \frac{y}{[l(1-\alpha t)^{1/2}]},\quad u=\frac{\alpha x}{[2(1-\alpha t)]}f'(\eta),\\ v &= -\frac{\alpha l}{[2(1-\alpha t)]}f(\eta),\quad \theta=\frac{T}{T_H}\\ A_1 &= (1-\alpha)+\phi\frac{\rho_s}{\rho_f}. \quad (1.6)\end{aligned}$$

Eq. (**??**) easily satisfies Eq. (**??**). The similarity transformation (**??**) reduces the momentum and energy equations, and the boundary conditions (**??**) respectively to

$$f^{iv}-SA_1(1-\phi)^{2.5}(\eta f'''+3f''+f'f''-ff''')=0,$$

$$\theta''+PrS\left(\frac{A_2}{A_3}\right)(f\theta'-\eta\theta')+\frac{PrEc}{A_3(1-\phi)^{2.5}}(f''^2+4\delta^2 f'^2)=0$$

$$f(0)=0,\quad f''(0)=0,\quad f(1)=1,\quad f'(1)=0,$$

$$\theta'(0)=0,\quad \theta(1)=1,$$

where

$$S=\frac{\alpha l}{2v_f},\quad Pr=\frac{\mu_f(\rho C_p)_f}{\rho_f K_f}$$

$$Ec=\frac{\rho f}{(\rho C_p)_f}\left(\frac{\alpha x}{2(1-\alpha t)}\right)^2,\quad \delta=\frac{1}{x}.$$

We investigate the following problem in this paper:

$$\begin{cases} f^{(iv)} - SA_1(1-\phi)^{2.5}(\eta f''' + 3f'' + f'f'' - ff''') = 0, \\ \\ \theta'' + PrS\left(\frac{A_2}{A_3}\right)(f-\eta)\theta' + \frac{PrEc}{A_3(1-\phi)^{2.5}}(f''^2 + 4\delta^2 f'^2) = 0, \\ \\ f(0) = 0, \quad f''(0) = 0, \quad f(1) = 1, \quad f'(1) = 0, \\ \theta'(0) = 0, \quad \theta(1) = 1. \end{cases} \tag{1.7}$$

Ordinary differential systems are important for actual–physical problems. These systems were used for a lot of problems [**?**, **?**, **?**, **?**]. Biswas et al. investigated systems by some varied techniques [**?**].

In this work, we obtain the solutions of (**??**) by reproducing kernel Hilbert space method (RKHSM). We assume that (**??**) has one solution. (**??**) can be written as:

$$\begin{cases} Pu = M(f,\theta), \quad 0 \le x \le 1, \\ u(0) = 0 = u(1), \end{cases} \tag{1.8}$$

where $M = (M_1, M_2)^T$, $u \in V_2^5[0,1] \oplus V_2^5[0,1]$, $M \in V_2^1[0,1] \oplus V_2^1[0,1]$. The space $V_2^5[0,1] \oplus V_2^5[0,1]$ is defined as

$$V_2^5[0,1] \oplus V_2^5[0,1] = \{u = (f,\theta)^T \mid f,\theta \in V_2^5[0,1]\}.$$

The inner product and norm are presented as:

$$\langle m, n\rangle = \sum_{i=1}^{2} \langle m_i, n_i\rangle_{V_2^5}, \quad \|m\| = \left(\sum_{i=1}^{2} \|m_i\|^2\right)^{\frac{1}{2}}, \quad m,n \in V_2^5[0,1] \oplus V_2^5[0,1].$$

$V_2^5[0,1] \oplus V_2^5[0,1]$ is a reproducing kernel Hilbert space. $V_2^1[0,1] \oplus V_2^1[0,1]$ can be identified in a similar way.

This work is arranged as: Section 2 gives some reproducing kernel Hilbert spaces. Solutions in $V_2^5[0,1] \oplus V_2^5[0,1]$ and a related linear operator are shown in Section 3. Numerical experiments are demonstrated in Section 4. The last section includes conclusions.

2. Some useful kernels

Definition 1. We define $V_2^5[0,1]$ by:

$$V_2^5[0,1] = \{m \in AC[0,1] : \ m', m'', m^{(3)}, m^{(4)} \in AC[0,1], \ m^{(5)} \in L^2[0,1], \\ m(0) = m''(0) = m(1) = m'(1) = 0\}.$$

$$\langle m, n\rangle_{V_2^5} = \sum_{i=0}^{4} m^{(i)}(0)n^{(i)}(0) + \int_0^1 m^{(5)}(t)n^{(5)}(t)\mathrm{d}t, \quad m, n \in \ V_2^5[0,1]$$

and

$$\|m\|_{V_2^5} = \sqrt{\langle m, m\rangle_{V_2^5}}, \quad m \in V_2^5[0,1].$$

are inner product and norm in $V_2^5[0,1]$.

Theorem 1. *Reproducing kernel* $\widetilde{A_y}$ *of* $V_2^5[0,1]$ *is acquired as:*

$$\widetilde{A_y}(x) = \begin{cases} \sum_{i=1}^{10} c_i(y)x^{i-1}, & x \le y, \\ \\ \sum_{i=1}^{10} d_i(y)x^{i-1}, & x > y. \end{cases} \tag{2.1}$$

Proof. We get

$$\left\langle v, \widetilde{A_y}\right\rangle_{V_2^5} = \sum_{i=0}^{4} v^{(i)}(0)\widetilde{A_y}^{(i)}(0) + \int_0^1 v^{(5)}(x)\widetilde{A_y}^{(5)}(x)\mathrm{d}x, \quad v, \widetilde{A_y} \in \ V_2^5[0,1]$$

by Definition ??.

We obtain

$$\begin{aligned} \left\langle v, \widetilde{A_y}\right\rangle_{V_2^5} =& v(0)\widetilde{A_y}(0) + v'(0)\widetilde{A_y}'(0) + v''(0)\widetilde{A_y}''(0) \\ &+ v^{(3)}(0)\widetilde{A_y}^{(3)}(0) + v^{(4)}(0)\widetilde{A_y}^{(4)}(0) \\ &+ v^{(4)}(1)\widetilde{A_y}^{(5)}(1) - v^{(4)}(0)\widetilde{A_y}^{(5)}(0) - v^{(3)}(1)\widetilde{A_y}^{(6)}(1) \\ &+ v^{(3)}(0)\widetilde{A_y}^{(6)}(0) + v''(1)\widetilde{A_y}^{(7)}(1) - v''(0)\widetilde{A_y}^{(7)}(0) \\ &- v'(1)\widetilde{A_y}^{(8)}(1) + v'(0)\widetilde{A_y}^{(8)}(0) + v(1)\widetilde{A_y}^{(9)}(1) \\ &- v(0)\widetilde{A_y}^{(9)}(0) - \int_0^1 v(x)\widetilde{A_y}^{(10)}(x)\mathrm{d}x. \end{aligned} \tag{2.2}$$

by integrating by parts.By reproducing property, we have

$$\langle v, \widetilde{A_y}\rangle_{V_2^5} = v(y). \tag{2.3}$$

Since $\widetilde{A_y} \in V_2^5[0,1]$, we get

$$\widetilde{A_y}(0) = \widetilde{A''_y}(0) = \widetilde{A_y}(1) = \widetilde{A'_y}(1) = 0. \tag{2.4}$$

If

$$\begin{cases} \widetilde{A_y}'(0) + \widetilde{A_y}^{(8)}(0) = 0, \\ \widetilde{A_y}^{(3)}(0) + \widetilde{A_y}^{(6)}(0) = 0, \\ \widetilde{R_y}^{(4)}(0) - \widetilde{A_y}^{(5)}(0) = 0, \\ \widetilde{A_y}^{(5)}(1) = 0, \\ \widetilde{A_y}^{(6)}(1) = 0, \\ \widetilde{A_y}^{(7)}(1) = 0. \end{cases} \tag{2.5}$$

then (**??**) gives

$$\widetilde{A_y}^{(10)}(x) = -\delta(x-y).$$

When $x \neq y$,

$$\widetilde{A_y}^{(10)}(x) = 0,$$

therefore

$$\widetilde{A_y}(x) = \begin{cases} \sum_{i=1}^{10} c_i(y)x^{i-1}, & x \leq y, \\ \\ \sum_{i=1}^{10} d_i(y)x^{i-1}, & x > y. \end{cases} \tag{2.6}$$

Since

$$\widetilde{A_y}^{(10)}(x) = \delta(x-y),$$

we get

$$\partial^k \widetilde{A}_{y^+}(y) = \partial^k \widetilde{A}_{y^-}(y), \quad k = 0, 1, \ldots, 8 \tag{2.7}$$

and

$$\partial^9 \widetilde{A}_{y^+}(y) - \partial^9 \widetilde{A}_{y^-}(y) = -1. \tag{2.8}$$

$c_i(y)$ and $d_i(y)$ $(i = 1, 2, \ldots, 10)$ can be obtained by (**??**)–(**??**). So the proof is completed. □

Definition 2. $V_2^1[0,1]$ is described by

$$V_2^1[0,1] = \{m \in AC[0,1] : \ m' \in L^2[0,1]\}.$$

$$\langle m, n \rangle_{V_2^1} = m(0)n(0) + \int_0^1 m'(t)n'(t)\mathrm{d}t, \quad m, n \in V_2^1[0,1]$$

and

$$\|m\|_{V_2^1} = \sqrt{\langle m, m \rangle_{V_2^1}}, \quad m \in V_2^1[0,1].$$

are inner product and the norm in $V_2^1[0,1]$.

Lemma 1. *Kernel function $\widetilde{Q_z}$ of $V_2^1[0,1]$ is gotten as [?]:*

$$\widetilde{Q_z}(t) = \begin{cases} 1+t, & 0 \le t \le z \le 1, \\ 1+z, & 0 \le z < t \le 1. \end{cases}$$

3. Solutions in $V_2^5[0,1] \oplus V_2^5[0,1]$

Lemma 2. *$P : V_2^5[0,1] \oplus V_2^5[0,1] \to V_2^1[0,1] \oplus V_2^1[0,1]$ is a bounded linear operator.*

Proof. We get

$$\begin{aligned} \|Pu\| &= \left(\sum_{i=1}^{2} \| \sum_{j=1}^{2} P_{ij} v_j \|^2 \right)^{\frac{1}{2}} \\ &\le \left[\sum_{i=1}^{2} \left(\sum_{j=1}^{2} \|P_{ij}\| \|v_j\| \right)^2 \right]^{\frac{1}{2}} \\ &\le \left[\sum_{i=1}^{2} \left(\sum_{j=1}^{2} \|P_{ij}\|^2 \right) \left(\sum_{j=1}^{2} \|v_j\|^2 \right) \right]^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{2} \sum_{j=1}^{2} \|P_{ij}\|^2 \right)^{\frac{1}{2}} \|v\|. \end{aligned}$$

P is bounded by the boundedness of P_{ij}. □

Now, put

$$\varphi_{ij}(x) = \widetilde{Q}_{x_i}(x)\overrightarrow{e_j} = \begin{cases} (\widetilde{Q}_{x_i}(x), 0)^T, & j = 1, \\ (0, \widetilde{Q}_{x_i}(x))^T, & j = 2, \end{cases}$$

and $\psi_{ij}(x) = P^* \varphi_{ij}(x)$, $i = 1, 2, \ldots,$ $j = 1, 2$. The orthonormal system of $\{\widehat{\psi}_{ij}(x)\}_{(1,1)}^{(\infty,2)}$ of $V_2^5[0,1] \oplus V_2^5[0,1]$ is acquired as:

$$\widehat{\psi}_{ij}(x) = \sum_{z=1}^{i} \sum_{q=1}^{j} \beta_{zq}^{ij} \psi_{zq}(x), \quad i = 1, 2, \ldots, j = 1, 2.$$

Theorem 2. *Suppose that $\{p_\eta\}_{\eta=1}^{\infty}$ is dense in $[0,1]$. Thus, $\{\psi_{\eta\tau}(p)\}_{(1,1)}^{(\infty,2)}$ is a complete system in $V_2^5[0,1] \oplus V_2^5[0,1]$.*

Proof. Let $\langle v(p), \psi_{\eta\tau}(p)\rangle = 0$ $(\eta = 1, 2, \ldots)$. We obtain

$$\langle Pv(p), \varphi_{\eta\tau}(p)\rangle = 0. \tag{3.1}$$

We have

$$v(p) = \sum_{\tau=1}^{2} v_\tau(p)\overrightarrow{e_\tau} = \sum_{\tau=1}^{2}\langle v(.), P_p(.)\overrightarrow{e_\tau}\rangle\overrightarrow{e_\tau}.$$

Thus, we get

$$Av(p_\eta) = \sum_{\tau=1}^{2}\langle Pv(y), \varphi_{\eta\tau}(y)\rangle\overrightarrow{e_\tau} = 0 \ (\eta = 1, 2, \ldots).$$

We take $(Pv)(p) = 0$. In conclusion, $\{\psi_{\eta\tau}(p)\}_{(1,1)}^{(\infty,2)}$ is a complete system in $V_2^5[0,1] \oplus V_2^5[0,1]$. □

Theorem 3. *If* $\{p_\eta\}_{\eta=1}^{\infty}$ *is dense in* $[0,1]$*, the solution of* (**??**) *fulfills*

$$v = \sum_{\eta=1}^{\infty}\sum_{\tau=1}^{2}\sum_{z=1}^{i}\sum_{q=1}^{\tau}\beta_{zq}^{\eta\tau} M(p_z, f(p_z), \theta(p_z)). \tag{3.2}$$

Proof. We get

$$\begin{aligned}
v &= \sum_{\eta=1}^{\infty}\sum_{\tau=1}^{2}\langle v(p), \widehat{\psi}_{\eta\tau}(p)\rangle\widehat{\psi}_{ij}(p)\\
&= \sum_{\eta=1}^{\infty}\sum_{\tau=1}^{2}\langle v(p), \sum_{z=1}^{\eta}\sum_{q=1}^{\tau}\beta_{zq}^{\eta\tau}\widehat{\psi}_{zq}(p)\rangle\widehat{\psi}_{ij}(p)\\
&= \sum_{\eta=1}^{\infty}\sum_{\tau=1}^{2}\sum_{z=1}^{\eta}\sum_{q=1}^{\tau}\beta_{zq}^{\eta\tau}\langle v(p), P^*\varphi_{zq}(p)\rangle\widehat{\psi}_{\eta\tau}(p)\\
&= \sum_{\eta=1}^{\infty}\sum_{\tau=1}^{2}\sum_{z=1}^{\eta}\sum_{q=1}^{\tau}\beta_{zq}^{\eta\tau}\langle Pv(p), \varphi_{zq}(p)\rangle\widehat{\psi}_{ij}(p)\\
&= \sum_{\eta=1}^{\infty}\sum_{\tau=1}^{2}\sum_{z=1}^{\eta}\sum_{q=1}^{\tau}\beta_{zq}^{\eta\tau} M(p_z, f(p_z), \theta(p_z))\widehat{\psi}_{\eta\tau}(p).
\end{aligned}$$

□

The approximate solution v_n can be found as:

$$v_n = \sum_{\eta=1}^{n}\sum_{\tau=1}^{2}\sum_{z=1}^{\eta}\sum_{q=1}^{\tau}\beta_{zq}^{\eta\tau} M(p_z, f(p_z), \theta(p_z)). \tag{3.3}$$

4. Numerical results

We consider (**??**) in the reproducing kernel Hilbert space in this paper. After homogenizing the conditions we obtained the numerical results for different values of S, Pr, Ec and ϕ. We showed our results by Figures **??**–**??** and Table **??**.

Heat transfer in the unsteady nano-fluid flow (**??**) is studied using reproducing kernel method. After homogenizing the conditions we obtained the numerical results for different values of S, Pr, Ec and ϕ. We showed our results by Figures 1-5 and Table 1. To show the influence of inserting physical parameters on the temperature, Figs. 1-5 have been plotted.
From Fig. 1, we can observe that temperature distribution is decreasing for increasing values of ϕ. Fig. 2 shows the effect of positive and negative squeeze number on the temperature distribution. The aim of squeeze number (S) describes the movement of the Plates. The effect of increasing the squeeze number can be described in following ways:

I . decrease in the kinematic viscosity
II . an increase in the distance between the plates
III . an increase in the speed at which the plates move

Fig. 3 demonstrates the effect of Eckert number, squeeze number and volume fraction on. temperature
The influence of Eckert number and Prandtl number on the temperature θ are illustrated in Figs. 4 and 5. The small values of Pr (< 1) typify liquid materials, which have high thermal diffusivity but low viscosity.

FIGURE 1. Approximate solutions of $\theta(x)$ for $Pr = 6.2, \delta = 0.1, Ec = 0.5,$ and $S = 1.0$.

FIGURE 2. Approximate solutions of $\theta(x)$ for $Pr = 6.2, \delta = 0.1, Ec = 0.5,$ and $\phi = 0.06$.

FIGURE 3. Approximate solutions of $\theta(x)$ for $Pr = 6.2$ and $\delta = 0.1$.

x	$f(x)$	$\theta(x)$
0.0	0.000000000	1.033181642
0.1	0.141886354	1.033171580
0.2	0.281700570	1.033086792
0.3	0.417232198	1.032745865
0.4	0.546087818	1.031853184
0.5	0.665616612	1.030014629
0.6	0.772815346	1.026840287
0.7	0.864226829	1.022250044
0.8	0.935830527	1.016245974
0.9	0.982924803	1.008829154
1.0	1.000000000	1.000000000

Table 1. Approximate solutions of $f(x)$ and $\theta(x)$ for $S = 1,\ Pr = 6.2,\ Ec = 0.01,\ \phi = 0.02$ and $\delta = 0.01$.

Figure 4. Approximate solutions of $\theta(x)$ for $S = Pr = 1.0$ and $\delta = 0.1$.

Figure 5. Approximate solutions of $\theta(x)$ for $S = Ec = 1.0$ and $\delta = 0.1$.

5. Conclusion

We obtained solutions of nonlinear system in this paper. We supplied evidence that the reproducing kernel Hilbert space method is a very powerful method. Moreover, this method is practical and proper to solve many problems.

6. Nomenclature

- ρ_{nf} Effective density of fluid
- μ_{nf} effective dynamic viscosity
- $(\rho C_p)_{nf}$ effective heat capacity
- k_{nf} effective thermal conductivity

- f dimensionless velocity profile
- θ dimensionless temperature
- p pressure
- T Fluid Temperature
- A_2 and A_3 Dimensionless constants
- u velocity component in x direction
- v velocity component in y direction
- η Independent dimensionless parameter
- S Squeeze number
- Pr Prandtl number
- Ec Eckert number
- ϕ nanoparticle volume fraction

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