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Fixed Point Theorems for Multi-Valued Contractions in *b*-Metric Spaces With Applications to Fractional Differential and Integral Equations

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ABSTRACT The aim of this manuscript is to establish common fixed points results for multi-valued mappings via generalized rational type contractions in complete b-metric spaces. Using the derived results, existence of solutions to certain integral equations and fractional differential equations in the frame of Caputo fractional derivative are studied. Examples are provided for the authenticity of the presented work.

INDEX TERMS Common fixed points, *b*-metric space, set valued mappings, generalized rational type contraction, system of integral equation.

I. INTRODUCTION AND PRELIMINARIES

Fractional calculus has been given proper attention in the last few decades by researchers. In fact, it is the generalization of classical calculus with the rapid development and advancement in nanotechnology. It has become a powerful tool (fractional order derivatives and integrals) with succesful and accurate results in modeling of various complex real world problems of science and engineering ([1], [2], [23], [35]). Fractional calculus is not only emerging and a productive field, it also represents a new philosophy how to construct and apply a certain type of nonlocal operators to real world problems. For detail see ([8], [29], [31], [32], [37]–[40]).

The problem of the convergence with respect to a measure of the measurable functions leads to a generalization of the concept of a metric. Using this notion, Czerwik [12] and Bakhtin [9] generalized the idea of metric space and presented metric spaces called b-metric spaces. Many researchers took the clue of Czerwik [12] and obtained interesting results. For detail see ([5], [7], [19], [26]–[28]). Recently, some authors generalized the $b-$metric space to more general type of metric type spaces by using control functions in the triangle inequality. Of special interest, the so-called controlled and double controlled metric type spaces was introduced in [24] and [6], where the contraction principle was proved under new limit conditions for the controlled functions.

A metric space (usual) is obviously a b-metric space. However, in Czerwik [12] and [13] showed that a b-metric on X need not be a metric(usual) on X (see [10], [14].

In Hilbert space, Alber and Gurre [3] presented weak contraction by generalizing contraction and revealed the presence of fixed points for a self-map. Rhoades [33] demonstrated this results in metric space under ϕ-weak contraction. Dutta and Choudhury [15] generalized ϕ-weak contraction for (ψ, ϕ) weak contraction and examined results for fixed point. Zhang and Song [41] described weak contraction under two self-map. The result proved by Zhang and Song [41] was further generalized by Doric [16], Radenović and Kadelburg [30] for the presence of common fixed point under (ψ, ϕ) weak contraction. Gordji et. al. [36] studied common fixed point theorems for (ψ, ϕ) nonlinear weak contraction.

Sehgal [34] ascertained result on periodic points and fixed points for a class of mappings. Murthy *et al.* [22] demonstrated some results for rational contraction in a complex valued metric space. Chen and Sun [11] established fixed points for the (ψ, ϕ) weaker contractive mappings in generalized complete metric spaces.

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In the current work, we discuss unique common fixed point results for set-valued generalized almost and (ψ, ϕ) weak contraction in b-metric spaces with applications. Throughout the paper, $\mathbf{R}^+$, $\mathbf{N}$ and $\mathbf{N}_0$ stand for the set of non-negative real numbers, the set of positive integers and the set of non-negative integers, respectively.

Now, we give some definitions and results for multi-valued mappings defined in a b-metric space (Λ, d, s). Define the function $H : CB(\Lambda) \times CB(\Lambda) \to \mathbf{R}^+$ for $\Omega_1, \Omega_2 \in CB(\Lambda)$ by

$$H(\Omega_1, \Omega_2) = \max\{ \sup_{\varsigma \in \Omega_1} d(\varsigma, \Omega_2), \sup_{\zeta \in \Omega_2} d(\zeta, \Omega_1)\},$$

where

$$d(\xi, \Omega_1) = \inf\{d(\xi, \zeta) : \zeta \in \Omega_1\}.$$

Then, H is called Hausdorff-Pompeiu metric. Also

$$\delta(\Omega_1, \Omega_2) = \sup\{d(\varsigma, \zeta) : \varsigma \in \Omega_1, \zeta \in \Omega_2\}$$

and

$$D(\Omega_1, \Omega_2) = \inf\{d(\varsigma, \zeta) : \varsigma \in \Omega_1, \zeta \in \Omega_2\}.$$

The following can be deduced from the definition of δ

$$\begin{aligned}
&\delta(\Omega_1, \Omega_2) = \delta(\Omega_2, \Omega_1),\\
&\delta(\Omega_1, \Omega_3) \leq \delta(\Omega_1, \Omega_2) + \delta(\Omega_2, \Omega_3),\\
&\delta(\Omega_1, \Omega_2) = 0 \Leftrightarrow \Omega_1 = \Omega_2 = \{\varsigma\},\\
&\delta(\Omega_1, \Omega_1) = diam\Omega_1.
\end{aligned}$$

Definition 1 [25]: For $\Lambda \neq \emptyset$. A fixed point of a multi-valued map $f : \Lambda \to CB(\Lambda)$ is a point $p \in \Lambda$ such that $p \in f(p)$.

Theorem 2 [25]: Let $f : \Lambda \to CB(\Lambda)$ defined on complete (Λ, d) metric space which hold the condition

$$H(f\xi, f\zeta) \leq k(d(\xi, \zeta) \quad \forall\, \xi,\ \zeta \in \Lambda. \tag{I.1}$$

Here, $k \in (0, 1)$, $CB(\Lambda)$ is the collection of bounded closed and nonempty subset of Λ. Then f has a fixed point.

Lemma 3 [12]: Let (Λ, d, s) be a b-metric space with $s > 1$. For any $\Omega_1, \Omega_2, \Omega_3 \in CB(\Lambda)$ and $\xi, \zeta \in \Lambda$. We have the following

1) $d(\xi, \Omega_2) \leq d(\xi, \zeta), \forall\, \zeta \in \Omega_2$;
2) $\delta(\Omega_1, \Omega_2) \leq H(\Omega_1, \Omega_2)$;
3) $d(\xi, \Omega_2) \leq H(\Omega_1, \Omega_2), \forall\, \xi \in \Omega_1$;
4) $H(\Omega_1, \Omega_1) = 0$;
5) $H(\Omega_1, \Omega_2) = H(\Omega_2, \Omega_1)$;
6) $H(\Omega_1, \Omega_3) \leq s[H(\Omega_1, \Omega_2) + H(\Omega_2, \Omega_3)]$;
7) $H(\xi, \Omega_1) \leq s[d(\xi, \zeta) + d(\zeta, \Omega_1)]$."

Lemma 4 [21]: "Assume (Λ, d, s) is a b-metric space and $\Theta_1, \Theta_2 \in CB(\Lambda)$. Then for $h \geq 1$, $a \in \Theta_1$ there exist $b(a) \in \Theta_2$ such that $d(a, b) \leq hH(\Theta_1, \Theta_2)$".

The following recent result of Miculescu and Mihail is useful in the context of b-metric spaces.

Lemma 5 [20]: "Every sequence $(x_n)_{n\in\mathbf{N}}$ of elements from a b-metric space (Λ, d, s), having the property that there exists $\gamma \in [0, 1)$ such that

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}),$$

for every $n \in \mathbf{N}$, is Cauchy."

Lemma 6 [4]: "Let (Λ, d, s) be a b-metric with $s \geq 1$, and suppose that $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ are b-convergent to x, y respectively then we have

$$\begin{aligned}
\frac{1}{s^2} d(x, y) &\leq \lim_{n\to\infty} \inf d(x_n, y_n)\\
&\leq \lim_{n\to\infty} \sup d(x_n, y_n) \leq s^2 d(x, y).
\end{aligned}$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{s} d(x, z) \leq \lim_{n\to\infty} \inf d(x_n, z) \leq \lim_{n\to\infty} \sup d(x_n, z) \leq sd(x, z).\text{''}$$

Geraghty [17] generalized Banach contraction principle by using the following definition.

Definition 7: Assume Θ signify the set of functions $\kappa : \mathbf{R}^+ \to (0, 1]$ with

(i) $\mathbf{R}^+ = \{\theta \in \mathbf{R} \mid \theta > 0\}$,
(ii) $\kappa(\theta_n) \to 1$ implies $\theta_n \to 0$.

Example 8 [18]: Let $\varphi : \mathbf{R}^+ \to [0, 1)$ define by,

$$\varphi(\omega) = \begin{cases} 1 - \dfrac{\omega^3}{2}, & \text{if } \omega \leq 1\\ \beta < 1, & \text{if } \omega > 1. \end{cases}$$

Clearly, $\varphi \in \Theta$.

II. MAIN RESULTS

In this section, we prove our fixed point theorems for multi-valued mapping on $b-$metric spaces.

Theorem 9: Let Q and R be two multi-valued mapping from Λ to $CB(\Lambda)$ in a complete b-metric space (Λ, d, s) with $s > 1$. Suppose for each $\varsigma, \vartheta \in \Lambda$, the following condition holds:

$$\begin{aligned}
\psi(sH(Q\varsigma, R\vartheta)) \leq \psi(M(\varsigma, \vartheta))\\
- \varphi(\psi(M(\varsigma, \vartheta))) + \theta(N(\varsigma, \vartheta)),
\end{aligned} \tag{II.1}$$

where

$$M(\varsigma, \vartheta) = \max\left\{\frac{d(\varsigma, Q\varsigma)d(\vartheta, R\vartheta)}{1 + d(\varsigma, \vartheta)}, d(\varsigma, \vartheta)\right\}, \tag{II.2}$$

$$\begin{aligned}
N(\varsigma, \vartheta) = \min\Big\{&d(\varsigma, \vartheta), d(\varsigma, Q\varsigma), d(\vartheta, R\vartheta), d(\varsigma, R\vartheta),\\
&d(\vartheta, Q\varsigma)\Big\},
\end{aligned} \tag{II.3}$$

(i) $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ is non-decreasing and continuous function such that $\psi(t) = 0$ if and only if $t = 0$;
(ii) $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is non-decreasing function and a lower semi continuous function such that $\varphi(t) = 0$ if and only if $t = 0$;
(iii) $\theta : \mathbf{R}^+ \to \mathbf{R}^+$ is a continuous function such that $\theta(t) = 0$ if and only if $t = 0$;

(iv) $s > 1$.

Then Q and R has a unique common fixed point.

Proof: Fix any $\varsigma \in \Lambda$. Define $\varsigma_0 = \varsigma$ and let $\varsigma_1 \in Q\varsigma_0$ by Lemma 4 there exist $\varsigma_2 \in R\varsigma_1$ such that

$$d(\varsigma_1, \varsigma_2) \leq \sqrt{s}H(Q\varsigma_0, R\varsigma_1),$$

for $\varsigma_2 \in R\varsigma_1$ there exist $\varsigma_3 \in Q\varsigma_2$ such that

$$d(\varsigma_2, \varsigma_3) \leq \sqrt{s}H(Q\varsigma_1, R\varsigma_2).$$

In this way, we get $\varsigma_{2n+1} \in Q\varsigma_{2n}$ and $\varsigma_{2n+2} \in R\varsigma_{2n+1}$.

If $\varsigma_{2n+1} = \varsigma_{2n+2}$, then ς_n is a Cauchy sequence. Suppose $\varsigma_{2n+1} \neq \varsigma_{2n+2}$. Then by (II.16) we have

$$\begin{aligned}&\psi(\sqrt{s}d(\varsigma_{2n+1}, \varsigma_{2n+2}))\\ &\quad\leq \psi(sH(Q\varsigma_{2n}, R\varsigma_{2n+1})) \leq \psi(M(\varsigma_{2n}, \varsigma_{2n+1}))\\ &\qquad - \varphi(\psi(M(\varsigma_{2n}, \varsigma_{2n+1}))) + \theta(N(\varsigma_{2n}, \varsigma_{2n+1})),\end{aligned} \tag{II.4}$$

where

$$M(\varsigma_{2n}, \varsigma_{2n+1}) = \max\left\{\frac{d(\varsigma_{2n}, Q\varsigma_{2n})d(\varsigma_{2n+1}, R\varsigma_{2n+1})}{1 + d(\varsigma_{2n}, \varsigma_{2n+1})},\right. \tag{II.5}$$

$$\left. d(\varsigma_{2n}, \varsigma_{2n+1})\right\}. \tag{II.6}$$

By using Lemma 3, we have

$$\begin{aligned}M(\varsigma_{2n}, \varsigma_{2n+1}) &\leq \max\left\{\frac{d(\varsigma_{2n}, \varsigma_{2n+1})d(\varsigma_{2n+1}, \varsigma_{2n+2})}{1 + d(\varsigma_{2n}, \varsigma_{2n+1})},\right.\\ &\qquad\left. d(\varsigma_{2n}, \varsigma_{2n+1})\right\}\\ &\leq \max\left\{d(\varsigma_{2n+1}, \varsigma_{2n+2}), d(\varsigma_{2n}, \varsigma_{2n+1})\right\}.\end{aligned}$$

Suppose now

$$\max\left\{d(\varsigma_{2n+1}, \varsigma_{2n+2}), d(\varsigma_{2n}, \varsigma_{2n+1})\right\} = d(\varsigma_{2n+1}, \varsigma_{2n+2}).$$

Then (II.4) becomes

$$\begin{aligned}&\psi(\sqrt{s}d(\varsigma_{2n+1}, \varsigma_{2n+2}))\\ &\quad\leq \psi(d(\varsigma_{2n+1}, \varsigma_{2n+2}))\\ &\qquad - \varphi(\psi(d(\varsigma_{2n+1}, \varsigma_{2n+2}))) + \theta(N(\varsigma_{2n}, \varsigma_{2n+1})).\end{aligned} \tag{II.7}$$

But

$$\begin{aligned}N(\varsigma_{2n}, \varsigma_{2n+1}) &= \min\left\{d(\varsigma_{2n}, \varsigma_{2n+1}), d(\varsigma_{2n}, Q\varsigma_{2n}),\right.\\ &\qquad d(\varsigma_{2n+1}, R\varsigma_{2n+1}),\\ &\qquad \left.d(\varsigma_{2n}, R\varsigma_{2n+1}), d(\varsigma_{2n+1}, Q\varsigma_{2n})\right\}\\ &\leq \min\left\{d(\varsigma_{2n}, \varsigma_{2n+1}),\right.\\ &\qquad d(\varsigma_{2n}, \varsigma_{2n+1})),\\ &\qquad d(\varsigma_{2n+1}, \varsigma_{2n+2}),\\ &\qquad \left.d(\varsigma_{2n}, \varsigma_{2n+2}), d(\varsigma_{2n+1}, \varsigma_{2n+1})\right\}\end{aligned}$$

which implies that

$$N(\varsigma_{2n}, \varsigma_{2n+1}) = 0. \tag{II.8}$$

Then, from equation (II.7) contradiction arises. Therefore

$$M(\varsigma_{2n}, \varsigma_{2n+1}) \leq d(\varsigma_{2n}, \varsigma_{2n+1}). \tag{II.9}$$

Using equations (II.8) and (II.9), from equation (II.4) we have

$$\begin{aligned}\psi(\sqrt{s}d(\varsigma_{2n+1}, \varsigma_{2n+2})) &\leq \psi(sH(Q\varsigma_{2n}, R\varsigma_{2n+1}))\\ &\leq \psi(d(\varsigma_{2n}, \varsigma_{2n+1}))\\ &\quad - \varphi(\psi(d(\varsigma_{2n}, \varsigma_{2n+1})))\\ &\leq \psi(d(\varsigma_{2n}, \varsigma_{2n+1})).\end{aligned}$$

Hence

$$\sqrt{s}d(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq d(\varsigma_{2n}, \varsigma_{2n+1}) \quad \text{for all } n \in \mathbf{N}. \tag{II.10}$$

Similarly, replacing ς by ς_{2n+2} and ϑ by ς_{2n+3}, we have

$$\sqrt{s}d(\varsigma_{2n+2}, \varsigma_{2n+3}) \leq d(\varsigma_{2n+1}, \varsigma_{2n+2}) \quad \text{for all } n \in \mathbf{N}. \tag{II.11}$$

From (II.10) and (II.11), we have

$$d(\varsigma_n, \varsigma_{n+1}) \leq \frac{1}{\sqrt{s}}d(\varsigma_{n-1}, \varsigma_n) \quad \text{for all } n \in \mathbf{N}.$$

Now, from Lemma 5, we obtain that the sequence $\{\varsigma_n\}$ is a Cauchy sequence. Since Λ is complete, every Cauchy sequence in Λ is convergent and converges to some point u in Λ. Suppose $u \notin R(u)$. Then,

$$\begin{aligned}d(\varsigma_{2n+1}, Ru) \leq d(Q\varsigma_{2n}, Ru) &\leq \psi(M(\varsigma_{2n}, u))\\ &\quad - \varphi(\psi(M(\varsigma_{2n}, u))) + \theta(N(\varsigma_{2n}, u)),\end{aligned} \tag{II.12}$$

where

$$\begin{aligned}M(\varsigma_{2n}, u) &= \max\left\{\frac{d(\varsigma_{2n}, Q\varsigma_{2n})d(u, Ru)}{1 + d(\varsigma_{2n}, u)}, d(\varsigma_{2n}, u)\right\}\\ &\leq \max\left\{\frac{d(\varsigma_{2n}, \varsigma_{2n+1})d(u, Ru)}{1 + d(\varsigma_{2n}, u)}, d(\varsigma_{2n}, u)\right\}\\ &\leq \max\left\{d(\varsigma_{2n}, \varsigma_{2n+1})d(u, Ru), d(\varsigma_{2n}, u)\right\},\end{aligned}$$

Taking the upper limit $n \to \infty$ and using Lemma6, we have

$$\limsup_{n\to\infty} M(\varsigma_{2n}, u) \leq \max\left\{s^2 d(u, u)\limsup_{n\to\infty} d(u, Ru), sd(u, u)\right\}$$

which implies that Taking the upper limit $n \to \infty$ and using Lemma 6, we have

$$\begin{aligned}\limsup_{n\to\infty} N(\varsigma_{2n}, u) \leq \min\Big\{&sd(u, u), s^2 d(u, u), \limsup_{n\to\infty} d(u, Ru),\\ &\limsup_{n\to\infty} d(\varsigma_{2n}, Ru), sd(u, u)\Big\}.\end{aligned}$$

Which gives that

$$\limsup_{n\to\infty} N(\varsigma_{2n}, u) = 0.$$

Taking the upper limit $n \to \infty$ in $(II.12)$ and using

$$\limsup_{n\to\infty} N(\varsigma_{2n}, u) = 0$$

and

$$\limsup_{n\to\infty} M(\varsigma_{2n}, u) = 0,$$

we have

$$\limsup_{n\to\infty} d(\varsigma_{2n+1}, Ru) = 0.$$

But from Lemma 3

$$d(u, Ru) \leq d(u, \varsigma_{n+1}) + d(\varsigma_{n+1}, Ru).$$

By passing limit we have $d(u, Ru) \to 0$. Therefore $u \in Ru$. Correspondingly, we can show that $u \in Qu$ and so u is a common fixed point of Q and R. Now, we shall show that this point is unique. If possible, let v be a different common fixed point of Q and R. i.e. $v \in Qv$ and $v \in Rv$. Then

$$\begin{aligned}\psi(d(u, v)) &\leq \psi(H(Qu, Rv)) \\ &\leq \psi(M(u, v)) \quad \text{(II.13)} \\ &\quad - \varphi(\psi(M(u, v))) + \theta(N(u, v)), \quad \text{(II.14)}\end{aligned}$$

where

$$\begin{aligned}M(u, v) &= \max\left\{\frac{d(u, Qu)d(v, Rv)}{1 + d(u, v)}, d(u, v)\right\} \\ &= d(u, v), \\ N(u, v) &= \min\{d(u, v), d(u, Qv), \\ &\quad d(v, Rv), d(v, Rv), d(v, Qu)\} = 0. \quad \text{(II.15)}\end{aligned}$$

So, (II.13) implies that $u = v$. Therefore u is a unique common fixed point of Q and R.

Remark1: The author in [36] proved common fixed points for single valued mapping in metric spaces while in Theorem 2.1 we proved common fixed points for multi-valued mapping in b-metric space.

$$\begin{aligned}\psi(sH(Q\varsigma, R\vartheta)) \leq \psi(M(\varsigma, \vartheta)) - \varphi(\psi(M(\varsigma, \vartheta))) \\ + \theta(N(\varsigma, \vartheta)), \quad \text{(II.16)}\end{aligned}$$

Theorem 10: Let η and θ be two multi-valued mapping from Λ to $CB(\Lambda)$ in a complete b-metric space (Λ, d, s) with $s > 1$. Suppose for all $\varsigma, \vartheta \in \Lambda$,

$$sH(\eta\varsigma, \theta\vartheta) \leq \alpha(d(\varsigma, \vartheta))\Psi(\varsigma, \vartheta) + \beta(d(\varsigma, \vartheta))\Phi(\varsigma, \vartheta), \quad \text{(II.17)}$$

where

$$\Psi(\varsigma, \vartheta) = \max\left\{\frac{d(\varsigma, \eta\varsigma)d(\vartheta, \theta\vartheta)}{1 + d(\varsigma, \vartheta)}, d(\varsigma, \vartheta)\right\}, \quad \text{(II.18)}$$

$$\begin{aligned}\Phi(\varsigma, \vartheta) = \min\{&d(\varsigma, \vartheta), d(\varsigma, \eta\varsigma), d(\vartheta, \theta\vartheta), \\ &d(\varsigma, \theta\vartheta), d(\vartheta, \eta\varsigma)\} \quad \text{(II.19)}\end{aligned}$$

and $\alpha(\varsigma), \beta(\varsigma) \in \Theta$. Then, η and θ has a unique common fixed point.

Proof: Fix any $\varsigma \in \Lambda$, define $\varsigma_0 = \varsigma$ and let $\varsigma_1 \in \eta\varsigma_0$. By Lemma 4, there exists $\varsigma_2 \in \theta\varsigma_1$ such that

$$d(\varsigma_1, \varsigma_2) \leq H(\eta\varsigma_0, \theta\varsigma_1).$$

Now, for $\varsigma_2 \in \theta\varsigma_1$ there exists $\varsigma_3 \in \eta\varsigma_2$ such that

$$d(\varsigma_2, \varsigma_3) \leq H(\eta\varsigma_1, \theta\varsigma_2).$$

Similarly, in this way we can obtain $\varsigma_{2n+1} \in \eta\varsigma_{2n}$ and $\varsigma_{2n+2} \in \theta\varsigma_{2n+1}$.

If $\varsigma_{2n+1} = \varsigma_{2n+2}$, then ς_n is a Cauchy sequence.

Suppose $\varsigma_{2n+1} \neq \varsigma_{2n+2}$. Then

$$sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq sH(\eta\varsigma_{2n}, \theta\varsigma_{2n+1}).$$

Using (II.17), one has

$$\begin{aligned}sH(\eta\varsigma_{2n}, \theta\varsigma_{2n+1}) \leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))\Psi(\varsigma_{2n}, \varsigma_{2n+1}) \\ + \beta(d(\varsigma_{2n}, \varsigma_{2n+1}))\Phi(\varsigma_{2n}, \varsigma_{2n+1}).\end{aligned}$$

Thus,

$$\begin{aligned}&sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \\ &\quad\leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))\Psi(\varsigma_{2n}, \varsigma_{2n+1}) \\ &\quad\quad + \beta(d(\varsigma_{2n}, \varsigma_{2n+1}))\Phi(\varsigma_{2n}, \varsigma_{2n+1}), \quad \text{(II.20)}\end{aligned}$$

where

$$\begin{aligned}&\Psi(\varsigma_{2n}, \varsigma_{2n+1}) \\ &\quad= \max\left\{\frac{d(\varsigma_{2n}, \eta\varsigma_{2n})d(\varsigma_{2n+1}, \theta\varsigma_{2n+1})}{1 + d(\varsigma_{2n}, \varsigma_{2n+1})}, d(\varsigma_{2n}, \varsigma_{2n+1})\right\}.\end{aligned}$$

By using Lemma 3, we have

$$\begin{aligned}&\Psi(\varsigma_{2n}, \varsigma_{2n+1}) \\ &\quad\leq \max\left\{\frac{d(\varsigma_{2n}, \varsigma_{2n+1})d(\varsigma_{2n+1}, \varsigma_{2n+2})}{1 + d(\varsigma_{2n}, \varsigma_{2n+1})}, d(\varsigma_{2n}, \varsigma_{2n+1})\right\}.\end{aligned}$$

Hence,

$$\Psi(\varsigma_{2n}, \varsigma_{2n+1}) \leq \max\left\{d(\varsigma_{2n+1}, \varsigma_{2n+2}), d(\varsigma_{2n}, \varsigma_{2n+1})\right\}. \quad \text{(II.21)}$$

Now, suppose that

$$\max\left\{d(\varsigma_{2n+1}, \varsigma_{2n+2}), d(\varsigma_{2n}, \varsigma_{2n+1})\right\} = d(\varsigma_{2n+1}, \varsigma_{2n+2}).$$

Then, (II.20) becomes

$$\begin{aligned}sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))d(\varsigma_{2n+1}, \varsigma_{2n+2}) \\ + \beta(d(\varsigma_{2n}, \varsigma_{2n+1}))\Phi(\varsigma_{2n}, \varsigma_{2n+1}). \quad \text{(II.22)}\end{aligned}$$

But

$$\begin{aligned}\Phi(\varsigma_{2n}, \varsigma_{2n+1}) = \min\Big\{&d(\varsigma_{2n}, \varsigma_{2n+1}), d(\varsigma_{2n+1}, \eta\varsigma_{2n}), \\ &d(\varsigma_{2n}, \eta\varsigma_{2n}), \\ &d(\varsigma_{2n+1}, \theta\varsigma_{2n+1}), d(\varsigma_{2n}, \theta\varsigma_{2n+1})\Big\} \\ \leq \min\Big\{&d(\varsigma_{2n}, \varsigma_{2n+1}), d(\varsigma_{2n+1}, \\ &\varsigma_{2n+1}), d(\varsigma_{2n}, \varsigma_{2n+1}), \\ &d(\varsigma_{2n+1}, \varsigma_{2n+2}), d(\varsigma_{2n}, \varsigma_{2n+2})\Big\} = 0.\end{aligned}$$

This implies that

$$\Phi(\varsigma_{2n}, \varsigma_{2n+1}) = 0. \quad \text{(II.23)}$$

Then, by using (II.23) in (II.22), we get contradiction. Therefore,

$$\Psi(\varsigma_{2n}, \varsigma_{2n+1}) \leq d(\varsigma_{2n}, \varsigma_{2n+1}).$$

From equation (II.20), we have

$$\begin{aligned} sd(\varsigma_{2n+1}, \varsigma_{2n+2}) &\leq sH(\eta\varsigma_{2n}, \theta\varsigma_{2n+1}) \\ &\leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))d(\varsigma_{2n}, \varsigma_{2n+1}). \end{aligned}$$

Which implies that

$$sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))d(\varsigma_{2n}, \varsigma_{2n+1}). \tag{II.24}$$

Similarly, replacing ς by ς_{2n+2} and ϑ by ς_{2n+3}, we have

$$sd(\varsigma_{2n+2}, \varsigma_{2n+3}) \leq \alpha(d(\varsigma_{2n+1}, \varsigma_{2n+2}))d(\varsigma_{2n+1}, \varsigma_{2n+2}). \tag{II.25}$$

From (II.24) and (II.25), we have

$$sd(\varsigma_n, \varsigma_{n+1}) \leq \alpha(d(\varsigma_{n-1}, \varsigma_n))d(\varsigma_{n-1}, \varsigma_n)$$

for all $n \in \mathbf{N}$, which implies that

$$d(\varsigma_n, \varsigma_{n+1}) \leq \frac{1}{s}d(\varsigma_{n-1}, \varsigma_n).$$

Now, from Lemma 5, we obtain that $\{\varsigma_n\}$ is a Cauchy sequence. Since Λ is complete, therefore every Cauchy sequence in Λ is convergent and converges to a point u (say) in Λ. Suppose $u \notin \theta(u)$

$$d(\varsigma_{2n+1}, \theta u) \tag{II.26}$$

$$\begin{aligned} &\leq d(\eta\varsigma_{2n}, \theta u) \leq \alpha(d(\varsigma_{2n}, u))\Psi(\varsigma_{2n}, u) \\ &\quad + \beta(d(\varsigma_{2n}, u))\Phi(\varsigma_{2n}, u), \end{aligned} \tag{II.27}$$

where

$$\begin{aligned} \Psi(\varsigma_{2n}, u) &= \max\left\{\frac{d(\varsigma_{2n}, \eta\varsigma_{2n})d(u, \theta u)}{1 + d(\varsigma_{2n}, u)}, d(\varsigma_{2n}, u)\right\} \\ &\leq max\left\{\frac{d(\varsigma_{2n}, \varsigma_{2n+1})d(u, \theta u)}{1 + d(\varsigma_{2n}, u)}, d(\varsigma_{2n}, u)\right\} \\ &\leq max\left\{d(\varsigma_{2n}, \varsigma_{2n+1})d(u, \theta u), d(\varsigma_{2n}, u)\right\}. \end{aligned}$$

Taking the upper limit $n \to \infty$ and using Lemma6, we have

$$\limsup_{n\to\infty}\Psi(\varsigma_{2n}, u) \leq max\left\{s^2d(u, u)\limsup_{n\to\infty}d(u, \theta u), sd(u, u)\right\}$$

which gives

$$\limsup_{n\to\infty}\Psi(\varsigma_{2n}, u) = 0.$$

$$\begin{aligned} \Phi(\varsigma_{2n}, u) &= \min\{d(\varsigma_{2n}, u), d(\varsigma_{2n}, \eta\varsigma_{2n}), d(u, \theta u), \\ &\qquad d(\varsigma_{2n}, \theta u), d(u, \eta\varsigma_{2n})\} \\ &\leq \min\{d(\varsigma_{2n}, u), d(\varsigma_{2n}, \varsigma_{2n+1}), \\ &\qquad d(u, \theta u), d(\varsigma_{2n}, \theta u), d(u, \varsigma_{2n+1})\}. \end{aligned}$$

Taking the upper limit $n \to \infty$ and using Lemma6, we have

$$\begin{aligned} &\limsup_{n\to\infty}\Phi(\varsigma_{2n}, u) \\ &\leq \min\{sd(u, u), s^2d(u, u), \limsup_{n\to\infty}d(u, \theta u), \\ &\quad \limsup_{n\to\infty}d(\varsigma_{2n}, \theta u), sd(u, u)\}. \\ &\limsup_{n\to\infty}\Phi(\varsigma_{2n}, u) = 0. \end{aligned}$$

Put on upper limit $n \to \infty$ in (II.26), and using

$$\limsup_{n\to\infty}\Psi(\varsigma_{2n}, u) = 0, \ \limsup_{n\to\infty}\Phi(\varsigma_{2n}, u) = 0,$$

we have

$$\limsup_{n\to\infty}d(\varsigma_{2n+1}, \theta u) = 0.$$

But, from Lemma3

$$d(u, \theta u) \leq d(u, \varsigma_{n+1}) + d(\varsigma_{n+1}, \theta u).$$

By taking limit we get $d(u, \theta u) = 0$. Thus $u \in \theta u$. We can show similarly that $u \in \eta u$, so u is a common fixed point of η and θ. Now to show that this point is unique. Let v be another common fixed point of η and θ i.e $u \in \eta u$ and $v \in \theta v$. Then,

$$\begin{aligned} d(u, v) &\leq sH(\eta u, \theta v) \\ &\leq \alpha(d(u, v))\Psi(u, v) + \beta(d(u, v))\Phi(u, v), \end{aligned}$$

where

$$\Psi(u, v) = \max\left\{\frac{d(u, \eta u)d(v, \theta v)}{1 + d(u, v)}, d(u, v)\right\} = d(u, v), \tag{II.28}$$

$$\begin{aligned} \Phi(u, v) &= \min\{d(u, v), d(u, \eta v), d(v, \theta v), d(v, \theta v), d(v, \eta u)\} \\ &= 0. \end{aligned} \tag{II.29}$$

By using (II.28) and (II.29) from (II.28) since $0 < \alpha(d(u, v)) \leq 1$, we have $u = v$. Hence, u is a unique common fixed point of η and θ.

Remark2: The author in [22] proved periodic points for single valued mapping while in Theorem 2.3 we proved common fixed points for multi-valued mapping in b-metric space.

Theorem 11: Let E and F be two multi-valued mapping from Λ to $CB(\Lambda)$ in a complete b-metric space (Λ, d, s) with $s > 1$. Suppose for all $\varsigma, \vartheta \in \Lambda$,

$$sH(E\varsigma, F\vartheta) \leq \alpha(d(\varsigma, \vartheta))\Delta_3(\varsigma, \vartheta) + \beta(d(\varsigma, \vartheta))\Delta_4(\varsigma, \vartheta),$$

where

$$\begin{aligned} \Delta_3(\varsigma, \vartheta) = \max\Bigg\{&\frac{d(\varsigma, E\varsigma)d(\vartheta, F\vartheta)}{1 + d(\varsigma, \vartheta)}, \\ &\frac{d(\varsigma, E\varsigma)d(\varsigma, F\vartheta) + d(\vartheta, F\vartheta)d(\vartheta, E\varsigma)}{1 + d(\varsigma, F\vartheta)}, \\ &d(\varsigma, \vartheta)\Bigg\}, \end{aligned}$$

$$\Delta_4(\varsigma, \vartheta) = \max\left\{d(\varsigma, \vartheta), \frac{d(\varsigma, E\varsigma) + d(\vartheta, F\vartheta)}{2s}, \frac{d(\varsigma, F\vartheta) + d(\vartheta, E\varsigma)}{2s}\right\}.$$

and $\alpha(\varsigma), \beta(\varsigma) \in \Theta$ such that, $\alpha(\varsigma)+\beta(\varsigma) < 1$. Then, E and F has a unique common fixed point.

Proof: Fix any $\varsigma \in \Lambda$. Define $\varsigma_0 = \varsigma$ and let $\varsigma_1 \in E\varsigma_0$. By Lemma 4, there exists $\varsigma_2 \in F\varsigma_1$ such that

$$d(\varsigma_1, \varsigma_2) \leq H(E\varsigma_0, F\varsigma_1).$$

Now, for $\varsigma_2 \in F\varsigma_1$, there exists $\varsigma_3 \in E\varsigma_2$ such that

$$d(\varsigma_2, \varsigma_3) \leq H(E\varsigma_1, F\varsigma_2).$$

Similarly, in this way we can obtain $\varsigma_{2n+1} \in E\varsigma_{2n}$ and $\varsigma_{2n+2} \in F\varsigma_{2n+1}$.

If $\varsigma_{2n+1} = \varsigma_{2n+2}$, then ς_n is a Cauchy sequence.

Suppose $\varsigma_{2n+1} \neq \varsigma_{2n+2}$. Then

$$sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq sH(E\varsigma_{2n}, F\varsigma_{2n+1})$$

and

$$sH(E\varsigma_{2n}, F\varsigma_{2n+1}) \leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))\Delta_3(\varsigma_{2n}, \varsigma_{2n+1}) + \beta(d(\varsigma_{2n}, \varsigma_{2n+1}))\Delta_3(\varsigma_{2n}, \varsigma_{2n+1}),$$

Thus,

$$sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))\Delta_3(\varsigma_{2n}, \varsigma_{2n+1}) + \beta(d(\varsigma_{2n}, \varsigma_{2n+1}))\Delta_4(\varsigma_{2n}, \varsigma_{2n+1}), \tag{II.30}$$

where $\Delta_3(\varsigma_{2n}, \varsigma_{2n+1})$, as shown at the top of the next page.

By using Lemma 3, we have $\Delta_3(\varsigma_{2n}, \varsigma_{2n+1})$, as shown at the top of the next page.

so,

$$\Delta_3(\varsigma_{2n}, \varsigma_{2n+1}) \leq \max\left\{d(\varsigma_{2n+1}, \varsigma_{2n+2}), d(\varsigma_{2n}, \varsigma_{2n+1})\right\}. \tag{II.31}$$

Now, suppose that

$$\max\left\{d(\varsigma_{2n+1}, \varsigma_{2n+2}), d(\varsigma_{2n}, \varsigma_{2n+1})\right\} = d(\varsigma_{2n+1}, \varsigma_{2n+2}).$$

Then, (II.30) becomes

$$sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))d(\varsigma_{2n+1}, \varsigma_{2n+2}) + \beta(d(\varsigma_{2n}, \varsigma_{2n+1}))\Delta_4(\varsigma_{2n}, \varsigma_{2n+1}). \tag{II.32}$$

But

$$\Delta_4(\varsigma_{2n}, \varsigma_{2n+1}) = \max\left\{d(\varsigma_{2n}, \varsigma_{2n+1}), \frac{d(\varsigma_{2n}, E\varsigma_{2n}) + d(\varsigma_{2n+1}, F\varsigma_{2n+1})}{2s}, \frac{d(\varsigma_{2n}, F\varsigma_{2n+1}) + d(\varsigma_{2n+1}, E\varsigma_{2n})}{2s}\right\}. \tag{II.33}$$

$$\Delta_4(\varsigma_{2n}, \varsigma_{2n+1}) = \max\left\{d(\varsigma_{2n}, \varsigma_{2n+1}), \frac{d(\varsigma_{2n}, \varsigma_{2n+1}) + d(\varsigma_{2n+1}, \varsigma_{2n+2})}{2s}, \frac{d(\varsigma_{2n}, \varsigma_{2n+2}) + d(\varsigma_{2n+1}, \varsigma_{2n+1})}{2s}\right\}. \tag{II.34}$$

Which implies

$$\Delta_4(\varsigma_{2n}, \varsigma_{2n+1}) \leq d(\varsigma_{2n+1}, \varsigma_{2n+2}).$$

By (II.32), we get contradiction. Therefore

$$\Delta_3(\varsigma_{2n}, \varsigma_{2n+1}) \leq d(\varsigma_{2n}, \varsigma_{2n+1}).$$

and

$$\Delta_4(\varsigma_{2n}, \varsigma_{2n+1}) \leq d(\varsigma_{2n}, \varsigma_{2n+1}).$$

From equation (II.30), we have

$$sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq sH(E\varsigma_{2n}, F\varsigma_{2n+1}) \leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))d(\varsigma_{2n}, \varsigma_{2n+1}) + \beta(d(\varsigma_{2n}, \varsigma_{2n+1}))d(\varsigma_{2n}, \varsigma_{2n+1}).$$

Which implies that

$$sd(\varsigma_{2n+1}, \varsigma_{2n+2}) \leq \alpha(d(\varsigma_{2n}, \varsigma_{2n+1}))d(\varsigma_{2n}, \varsigma_{2n+1}) + \beta(d(\varsigma_{2n}, \varsigma_{2n+1}))d(\varsigma_{2n}, \varsigma_{2n+1}). \tag{II.35}$$

Similarly, replacing ς by ς_{2n+2} and ϑ by ς_{2n+3}, we have

$$sd(\varsigma_{2n+2}, \varsigma_{2n+3}) \leq \alpha(d(\varsigma_{2n+1}, \varsigma_{2n+2}))d(\varsigma_{2n+1}, \varsigma_{2n+2}) + \beta(d(\varsigma_{2n+1}, \varsigma_{2n+2}))d(\varsigma_{2n+1}, \varsigma_{2n+2}). \tag{II.36}$$

From (II.35) and (II.36), we have

$$sd(\varsigma_n, \varsigma_{n+1}) \leq \alpha(d(\varsigma_{n-1}, \varsigma_n))d(\varsigma_{n-1}, \varsigma_n) + \beta(d(\varsigma_{n-1}, \varsigma_n))d(\varsigma_{n-1}, \varsigma_n)$$

for all $n \in \mathbf{N}$, which implies that

$$d(\varsigma_n, \varsigma_{n+1}) \leq \frac{1}{s}d(\varsigma_{n-1}, \varsigma_n).$$

Now, from Lemma 5, we obtain that $\{\varsigma_n\}$ is a Cauchy sequence. Since Λ is complete, it converges to a point u (say) in Λ. Suppose $u \notin F(u)$. We have

$$d(\varsigma_{2n+1}, Fu) \leq d(E\varsigma_{2n}, Fu) \leq \alpha(d(\varsigma_{2n}, u))\Delta_3(\varsigma_{2n}, u) + \beta(d(\varsigma_{2n}, u))\Delta_4(\varsigma_{2n}, u), \tag{II.37}$$

where

$$\Delta_3(\varsigma_{2n}, u) = \max\left\{\frac{d(\varsigma_{2n}, E\varsigma_{2n})d(u, Fu)}{1 + d(\varsigma_{2n}, u)}, \frac{d(\varsigma_{2n}, E\varsigma_{2n})d(\varsigma_{2n}, Fu) + d(u, Fu)d(u, E\varsigma_{2n})}{1 + d(\varsigma_{2n}, Fu)}, d(\varsigma_{2n}, \varsigma_{2n+1})\right\},$$

$$\Delta_3(\varsigma_{2n},\varsigma_{2n+1}) = \max\left\{\frac{d(\varsigma_{2n},E\varsigma_{2n})d(\varsigma_{2n+1},F\varsigma_{2n+1})}{1+d(\varsigma_{2n},\varsigma_{2n+1})}, \frac{d(\varsigma_{2n},E\varsigma_{2n})d(\varsigma_{2n},F\varsigma_{2n+1}+d(\varsigma_{2n+1},F\varsigma_{2n+1})d(\varsigma_{2n+1},E\varsigma_{2n})}{1+d(\varsigma_{2n},F\varsigma_{2n+1})}, d(\varsigma_{2n},\varsigma_{2n+1})\right\},$$

$$\Delta_3(\varsigma_{2n},\varsigma_{2n+1}) \le \max\left\{\frac{d(\varsigma_{2n},\varsigma_{2n+1})d(\varsigma_{2n+1},\varsigma_{2n+2})}{1+d(\varsigma_{2n},\varsigma_{2n+1})}, \frac{d(\varsigma_{2n},\varsigma_{2n+1})d(\varsigma_{2n},F\varsigma_{2n+1})+d(\varsigma_{2n+1},\varsigma_{2n+2})d(\varsigma_{2n+1},\varsigma_{2n+1})}{1+d(\varsigma_{2n},F\varsigma_{2n+1})}, d(\varsigma_{2n},\varsigma_{2n+1})\right\},$$

$$\Delta_3(\varsigma_{2n},u) \le \max\left\{\frac{d(\varsigma_{2n},\varsigma_{2n+1})d(u,Fu)}{1+d(\varsigma_{2n},u)}, \frac{d(\varsigma_{2n},\varsigma_{2n+1})d(\varsigma_{2n},Fu)+d(u,Fu)d(u,\varsigma_{2n+1})}{1+d(\varsigma_{2n},Fu)}, d(\varsigma_{2n},\varsigma_{2n+1})\right\},$$

$$\Delta_3(\varsigma_{2n},u) \le \max\Big\{d(\varsigma_{2n},\varsigma_{2n+1})d(u,Fu), d(\varsigma_{2n},\varsigma_{2n+1})d(\varsigma_{2n},Fu) + d(u,Fu)d(u,\varsigma_{2n+1}), d(\varsigma_{2n},\varsigma_{2n+1})\Big\}.$$

Taking the upper limit $n\to\infty$ and using Lemma 6, we have

$$\limsup_{n\to\infty}\Delta_3(\varsigma_{2n},u) \le \max\left\{s^2d(u,u)\limsup_{n\to\infty}d(u,Fu), s^2d(u,u)\limsup_{n\to\infty}d(\varsigma_{2n},Fu), sd(u,u)\right\} \tag{II.38}$$

which gives

$$\limsup_{n\to\infty}\Delta_3(\varsigma_{2n},u) = 0.$$

$$\Delta_4(\varsigma_{2n},u) = \max\left\{d(\varsigma_{2n},u), \frac{d(\varsigma_{2n},E\varsigma_{2n})+d(u,Fu)}{2s}, \frac{d(\varsigma_{2n},Fu)+d(u,E\varsigma_{2n})}{2s}\right\}.$$

$$\Delta_4(\varsigma_{2n},u) \le \max\left\{d(\varsigma_{2n},u), \frac{d(\varsigma_{2n},\varsigma_{2n+1})+d(u,Fu)}{2s}, \frac{d(\varsigma_{2n},Fu)+d(u,\varsigma_{2n+1})}{2s}\right\}.$$

Taking the upper limit $n\to\infty$ and using Lemma 6, we have

$$\limsup_{n\to\infty}\Delta_4(\varsigma_{2n},u) \le \frac{d(u,Fu)}{2s}.$$

Put on upper limit $n\to\infty$ in (II.37), and using

$$\limsup_{n\to\infty}\Delta_3(\varsigma_{2n},u) = 0 \;\; \textit{and} \;\; \limsup_{n\to\infty}\Delta_4(\varsigma_{2n},u) \le \frac{d(u,Fu)}{2s},$$

we have

$$\limsup_{n\to\infty}d(\varsigma_{2n+1},Fu) < \frac{d(u,Fu)}{2s}.$$

But from Lemma 3

$$d(u,Fu) \le d(u,\varsigma_{n+1}) + d(\varsigma_{n+1},Fu).$$

By taking limit sup we get $d(u,Fu) = 0$. Thus $u \in Fu$. We can show similarly that $u \in Eu$, so u is a common fixed point of E and F. Now to show that this point is unique. Let v be another common fixed point of E and F i.e $u \in Eu$ and $v \in Fv$. Then

$$d(u,v) \le sH(Eu,Fv) \le \alpha(d(u,v))\Delta_3(u,v) + \beta(d(u,v))\Delta_4(u,v), \tag{II.39}$$

where

$$\Delta_3(u,v) = \max\left\{\frac{d(u,Eu)d(v,Fv)}{1+d(u,v)}, \frac{d(u,Eu)d(u,Fv)+d(v,Fv)d(v,Eu)}{1+d(u,Fv)}, d(u,v)\right\},$$

which implies

$$\Delta_3(u,v) = d(u,v), \tag{II.40}$$

Similarly, we have

$$\Delta_4(u,v) = d(u,v). \tag{II.41}$$

By using (II.40) and (II.41) from (II.39) since $0 < \alpha(d(u,v)) + \beta(d(u,v)) \le 1$, but $\alpha(\varsigma), \beta(\varsigma) \in \Theta$, we have $u = v$. Hence u is a unique common fixed point of E and F.

Remark3: The author in [22] proved periodic points for single valued mapping while in Theorem 2.5 we proved common fixed points for set-valued mapping in b-metric space by using generalized type contraction.

Example 12: Suppose $\Lambda = \mathbf{R}$. A metric $d : \Lambda\times\Lambda \to \mathbf{R}^+$ is defined by,

$$d(\varsigma,\vartheta) = |\varsigma-\vartheta|^l,$$

for all $\varsigma,\vartheta \in \Lambda$ with $l > 1$. Then (Λ, d) is b-metric space. We define $Q, R : \Lambda \to CB(\Lambda)$ and $\psi,\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ defined by

$$Q\vartheta = \left[0, \frac{\vartheta^{2m}}{s(k+2)}\right]$$

and

$$R\varsigma = \left[0, \frac{\varsigma^{2n}}{s(k+2)}\right] \quad \forall k \geq s$$
$$\psi(t) = t, \varphi(t) = \frac{s}{k+1}t$$

Now,

$$\begin{aligned} sH(R\varsigma, Q\vartheta) &= sH\left(\left[0, \frac{\varsigma^{2m}}{s(k+2)}\right], \left[0, \frac{y^{2n}}{s(k+2)}\right]\right) \\ &= s\frac{1}{s(k+2)}|\varsigma^{2m} - \vartheta^{2n}|^l \\ &\leq \frac{1}{k+1}d(\varsigma, \vartheta) \\ &\leq \frac{1}{k+1}M(\varsigma, \vartheta) \\ &\leq \psi(M(\varsigma, \vartheta)) - \varphi(\psi(M(\varsigma, \vartheta))) + \theta(N(\varsigma, \vartheta)). \end{aligned}$$

Thus, for different value of m, n and k we have Q and R satisfying all conditions of Theorem 9 so Q and R has a unique common fixed point.

Example 13: Let $\Lambda = [0, \infty)$. Define $d : \Lambda \times \Lambda \to \mathbf{R}^+$ by,

$$d(\varsigma, \vartheta) = |\varsigma - \vartheta|^2.$$

Then (Λ, d, s) is b-metric space. Let $\eta, \theta : \Lambda \to CB(\Lambda)$, and $\alpha, \beta : \mathbf{R}^+ \to [0, 1)$ define by,

$$\begin{aligned} \eta(\varsigma) &= \left[\frac{\varsigma}{2}, \frac{\varsigma}{4}\right] \\ \theta(\mu) &= \left[\frac{\vartheta}{2}, \frac{\vartheta}{4}\right] \\ \alpha(t) &= \beta(t) = k < 1 \quad \forall\, t \in [0, \infty) \end{aligned}$$

$$H(\eta\varsigma, \theta\vartheta) = \max\left\{\sup_{\varsigma \in \eta\varsigma} d(\varsigma, \theta\vartheta), \sup_{\zeta \in \theta\vartheta} d(\zeta, \eta\varsigma)\right\},$$
$$H(\eta\varsigma, \theta\vartheta) = \max\left\{\sup_{\varsigma \in \eta\varsigma} d(\varsigma, [0, \frac{\mu}{4}]), \sup_{\zeta \in \theta\vartheta} d(\zeta, [\frac{\mu}{2}, 1])\right\},$$
$$H(\eta\varsigma, \theta\vartheta) = \max\left\{|\frac{\vartheta}{2} - \frac{\varsigma}{2}|^2, |\frac{\vartheta}{4} - \frac{\varsigma}{4}|^2\right\}$$
$$\begin{aligned} H(\eta\varsigma, \theta\vartheta) &= \max\left\{\frac{1}{4}|\vartheta - \varsigma|^2, \frac{1}{16}|\vartheta - \varsigma|^2\right\} \\ &\leq \frac{1}{4}\max\left\{|\vartheta - \varsigma|^2, |\vartheta - \varsigma|^2\right\} \\ &\leq \frac{1}{4}\max\left\{|\vartheta - \varsigma|^2, |\vartheta - \varsigma|^2\right\} \\ &\leq \frac{1}{4}\max\left\{\frac{d(\varsigma, \eta\varsigma)d(\vartheta, \theta\vartheta)}{1 + d(\varsigma, \vartheta)}, d(\varsigma, \vartheta)\right\}, \end{aligned}$$

which implies that

$$sH(\eta\varsigma, \theta\vartheta) \leq \alpha(d(\varsigma, \vartheta))\Delta_1(\varsigma, \vartheta) + \beta(d(\varsigma, \vartheta))\Delta_2(\varsigma, \vartheta).$$

Thus, all the conditions of Theorem 10 hold. Therefore η and θ have a unique fixed point.

Example 14: Let $\Lambda = [0, 1]$. Define $d : \Lambda \times \Lambda \to \mathbf{R}^+$ by,

$$d(\varsigma, \vartheta) = |\varsigma - \vartheta|^2.$$

Then (Λ, d, s) is b-metric space.

Let $E, F : \Lambda \to CB(\Lambda)$ defined by $F\vartheta = \{\frac{\vartheta}{49}\}$ and $E\varsigma = [0, \frac{\varsigma}{49}]$,

$$H(E\varsigma, F\vartheta) = \max\left\{\sup_{\varsigma \in E\varsigma} d(\varsigma, F\vartheta), \sup_{\zeta \in F\vartheta} d(\zeta, E\varsigma)\right\},$$
$$H(E\varsigma, F\vartheta) = \max\left\{\sup_{\varsigma \in E\varsigma} d(\varsigma, \{\frac{\vartheta}{49}\}), \sup_{\zeta \in F\vartheta} d(\zeta, [0, \frac{\varsigma}{49}])\right\},$$
$$\begin{aligned} H(E\varsigma, F\vartheta) &= \max\left\{|\frac{\vartheta}{49} - \frac{\varsigma}{49}|^2, |\frac{\vartheta}{49}|^2\right\} \\ &= \frac{1}{7}\max\left\{|\vartheta - \varsigma|^2, |\vartheta - \frac{\vartheta}{7}|^2\right\} \\ &\leq \frac{1}{7}\max\left\{|\vartheta - \varsigma|^2, |\vartheta - \frac{\vartheta}{49}|^2\right\} \\ &\leq \frac{1}{7}\max\left\{|\vartheta - \varsigma|^2, |\vartheta - \frac{\vartheta}{49}|^2\right\} \\ &= \frac{1}{7}\max\left\{d(\varsigma, \vartheta), \frac{1}{4}d(\vartheta, F\vartheta))\right\} \\ &\leq \frac{1}{7}\max\left\{d(\varsigma, \vartheta), \frac{d(\varsigma, E\varsigma) + d(\vartheta, F\vartheta)}{4}, \frac{d(\varsigma, F\vartheta) + d(\vartheta, E\varsigma)}{4}\right\}. \\ &\leq \frac{1}{7}\max\left\{d(\varsigma, \vartheta), \frac{d(\varsigma, E\varsigma) + d(\vartheta, F\vartheta)}{2s}, \frac{d(\varsigma, F\vartheta) + d(\vartheta, E\varsigma)}{2s}\right\}. \end{aligned}$$

This implies that

$$sH(E\varsigma, F\vartheta) \leq \alpha(d(\varsigma, \vartheta))\Delta_3(\varsigma, \vartheta) + \beta(d(\varsigma, \vartheta))\Delta_4(\varsigma, \vartheta).$$

By taking natural log both side and then consider $s = 2$, $\alpha(\varsigma) = \beta(\varsigma) = \frac{1}{7}$. All axiom of Theorem 11 are held. therefore, E and F have a unique common fixed point $\varsigma = 0$.

III. APPLICATIONS

In this section we discuss an existence results for the solution of the system of non-linear integral, fractional differential and surface integral equations.

Consider the following general system of nonlinear Fredholm integral equations of the 2nd kind

$$\begin{cases} \varsigma(\gamma) = \kappa(\gamma) + \int_\theta^\vartheta \lambda_1(\gamma, s, \varsigma(s))ds, & \gamma \in [\theta, \vartheta], \\ \vartheta(\gamma) = \kappa(\gamma) + \int_\theta^\vartheta \lambda_2(\gamma, s, \vartheta(s))ds, & \gamma \in [\theta, \vartheta]. \end{cases} \tag{III.1}$$

Let $\Lambda = C[\theta, \vartheta]$ be the set of all continuous function defined on $[\theta, \vartheta]$. Define $d : \Lambda \times \Lambda \to \mathbf{R}^+$ by $d(\eta, \zeta) = \left(\sup_{\gamma \in I} |\eta(\gamma) - \zeta(\gamma)|\right)^p$ for all $\eta, \zeta \in \Lambda$ Then (Λ, d, s) is a complete b-metric space on Λ with $s = 2^{p-1}$ and $p > 1$. For the derivation of existence results for the solution of the system (III.1) we provide the below theorem.

Theorem 15: Assume that the assumptions below hold

(A_1) $\lambda_j : [\theta, \vartheta] \times [\theta, \vartheta] \times \mathbf{R}^+ \to \mathbf{R}^+$, for $j = 1, 2$ and $\kappa : \mathbf{R}^+ \to \mathbf{R}^+$ is continuous;

(A_2) there exist a continuous function $K : [\theta, \vartheta] \times [\theta, \vartheta] \to [0, \infty)$ such that,

$$|\lambda_1(\gamma, s, u) - \lambda_2(\gamma, s, v)| \leq K(\gamma, s)|u - v|$$

for each $\gamma, s \in [\theta, \vartheta]$,

(A_3) $\sup_{\gamma, s\in[\theta,\vartheta]} \int_0^1 |K(\gamma, s)| \leq q$ for $q < 1$.

Then, the system (III.1) has a unique solution in $C([\theta, \vartheta])$.

Proof: Define $Q, R : C([\theta, \vartheta]) \to C([\theta, \vartheta])$ by,

$$Q\varsigma(\gamma) = \kappa(\gamma) + \int_\theta^\vartheta \lambda_1(\gamma, s, \varsigma(s))ds, \quad \gamma \in [\theta, \vartheta].$$

$$R\vartheta(\gamma) = \kappa(\gamma) + \int_\theta^\vartheta \lambda_2(\gamma, s, \vartheta(s))ds, \quad t \in [\theta, \vartheta].$$

Now we have,

$$\begin{aligned}
&2^{p-1}d(Q\varsigma(\gamma), R\vartheta(\gamma)) \\
&= 2^{p-1}\Big(\sup_{\gamma\in[\theta,\vartheta]} |Q\varsigma(\gamma) - R\vartheta(\gamma)|\Big)^p \\
&\leq 2^{p-1}\Big(\sup_{\gamma\in[\theta,\vartheta]} \int_\theta^\vartheta |\lambda_1(\gamma, s, \varsigma(s)) - \lambda_2(\gamma, s, \vartheta(s))|ds\Big)^p \\
&\leq 2^{p-1}\Big(\sup_{\gamma\in[\theta,\vartheta]} \int_\theta^\vartheta K(\gamma, s)|\varsigma(s) - \vartheta(s)|ds\Big)^p \\
&\leq 2^{p-1}\Big(\sup_{\gamma\in[\theta,\vartheta]} |\varsigma(\gamma) - \vartheta(\gamma)|\Big)^p\Big(\sup_{\gamma\in[\theta,\vartheta]} \int_\theta^\vartheta K(\gamma, s)ds\Big)^p \\
&\leq 2^{p-1}\Big(\sup_{\gamma\in[\theta,\vartheta]} |\varsigma(\gamma) - \vartheta(\gamma)|\Big)^p \\
&= d(\varsigma(\gamma), \vartheta(\gamma)).
\end{aligned}$$

This implies that

$$\begin{aligned}
2^{p-1}d(Q\varsigma(\gamma), R\vartheta(\gamma)) &\leq 2^{p-1}d(\varsigma(\gamma), \vartheta(\gamma)) \\
&\leq 2^{p-1}M(\varsigma(\gamma), \vartheta(\gamma)). \\
2^{p-1}d(Q\varsigma(\gamma), R\vartheta(\gamma)) &\leq 2^{p-1}M(\varsigma(\gamma), \vartheta(\gamma)) \\
&\leq \psi(M(\varsigma, \vartheta)) \\
&\quad - \varphi(\psi(M(\varsigma, \vartheta))) + \theta(N(\varsigma, \vartheta)).
\end{aligned}$$

Define $\psi(\gamma) = 4^{p-1}\gamma$, $\varphi(\gamma) = \frac{\gamma}{16^{p-1}}$, then by Theorem 9 the system (III.1) has a unique common solution in Λ.

Next we notice that one can also show the existence of solution to the following system of nonlinear fractional ordered differential equations

$$\begin{cases}
{}^cD^\beta u(\gamma) + Q(v(\gamma)) = 0, & 1 < \beta \leq 2,\ \gamma \in [0, 1], \\
{}^cD^\beta v(\gamma) + R(w(\gamma)) = 0, & 1 < \beta \leq 2,\ \gamma \in [0, 1] \\
u(0) = v(0) = l, \quad u(1) = v(1) = m, \\
\text{where } l, m \text{ are constants.}
\end{cases} \tag{III.2}$$

Here $Q, R : [0, 1] \times [0, \infty) \to [0, \infty)$, and ${}^cD^\beta$ represent the Caputo derivative of order β. The equivalent system of integral equations corresponding to (III.2) is given by

$$\begin{cases}
u(\gamma) = \kappa(\gamma) + \int_0^1 \mathcal{G}(\gamma, s)Q(v(s)ds, & \gamma \in [0, 1], \\
v(\gamma) = \kappa(\gamma) + \int_0^1 \mathcal{G}(\gamma, s)R(w(s)ds, & \gamma \in [0, 1].
\end{cases} \tag{III.3}$$

Here, the Green's function $\mathcal{G}(\gamma, s)$ is continuous on $[0, 1] \times [0, 1]$ and defined as

$$\mathcal{G}(\gamma, s) = \begin{cases}
\dfrac{(\gamma - s)^{\beta-1} - \gamma(1 - s)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq s \leq \gamma \leq 1, \\
\dfrac{-\gamma(1 - s)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq s \leq \gamma \leq 1,
\end{cases} \tag{III.4}$$

Moreover $\sup_{\gamma\in[0,1]} \int_0^1 |\mathcal{G}(\gamma, s)|ds \leq 1$. By letting $\lambda(\gamma, s, \varsigma(s)) = \mathcal{G}(\gamma, s)Q(v(s)$ etc. Then, the system (III.3) turns into

$$\begin{cases}
\varsigma(\gamma) = \kappa(\gamma) + \int_0^1 \lambda_1(\gamma, s, \varsigma(s))ds, & \gamma \in [0, 1], \\
\vartheta(\gamma) = \kappa(\gamma) + \int_0^1 \lambda_2(\gamma, s, \varsigma(s))ds, & \gamma \in [0, 1].
\end{cases} \tag{III.5}$$

By using Theorem 9, one can say that the system (III.5) has a unique common solution, which is the corresponding unique common solution of the system of nonlinear fractional differential equation (III.2).

Finally, we derive sufficient conditions for the existence of solutions for the following general non-linear surface integral equation

$$R(t, \mu(\varsigma, t)) = \int_0^1\int_0^1 W(t, \zeta, s, \mu(\zeta, s))d\zeta ds, \quad t \in [\varsigma, \sigma], \tag{III.6}$$

where $\mu \in L^p(C([0, 1] \times [0, 1]))$, $1 < p < \infty$ and $t, \zeta, s \in [0, 1]$.

Let $\Lambda = C[\eta, \zeta]$ be the set of all continuous function defined on $[\eta, \zeta]$. Define $d : \Lambda \times \Lambda \to \mathbf{R}^+$ by

$$d(\eta, \zeta) = \Big(\max_{t\in I} |\eta(t) - \zeta(t)|\Big)^p \quad \textit{for all } \eta, \zeta \in \Lambda.$$

Then d is a complete b-metric space with $s = 2^{p-1}$ and $p > 1$. For the existence we have the following result.

Theorem 16: Assume that the assumptions below holds.

(A_1) $R(t, \mu(\varsigma, t)) - R(t, \nu(\varsigma, t)) \leq \frac{\alpha^{1/p}(d(\mu(\varsigma,t),\nu(\varsigma,t)))}{2^{\frac{p-1}{p}}}$ $(\mu(\varsigma, t) - \nu(\varsigma, t))\ \alpha \in S$;

(A_2) $W(t, \zeta, s, \mu(\zeta, s)) \geq R(t, \mu(\varsigma, t))$ for all $\zeta, t, s \in [0, 1]$;

(A_3) $W(t, \zeta, s, \mu(\zeta, s)) \leq \mu(\varsigma, t)$ for all $t \in [0, 1]$.

Then the system (III.6) has a solution in $L^p(C([0, 1]\times[0, 1]))$.

Proof: Define $\eta(\mu(\varsigma,t)) = R(t,\mu(\varsigma,t))$ and $\theta(\mu(\varsigma,t)) = \int_0^1\int_0^1 W(t,\zeta,s,\mu(\zeta,s))d\zeta ds$, we have

$$\begin{aligned}
&2^{p-1}\left|\eta(\mu(\varsigma,t)) - \theta(\nu(\varsigma,t)\right|^p \\
&= 2^{p-1}\Big|R(t,\mu(\varsigma,t)) \\
&\quad - \int_0^1\int_0^1 W(t,\zeta,s,\nu(\zeta,s))d\zeta ds\Big|^p \\
&\leq 2^{p-1}\left|R(t,\mu(\varsigma,t)) - \int_0^1\int_0^1 R(t,\nu(\varsigma,t))d\zeta ds\right|^p \\
&= 2^{p-1}\left|R(t,\mu(\varsigma,t)) - R(t,\nu(\varsigma,t)\right|^p.
\end{aligned}$$

Using assumption (A_1) and taking max on both sides, we have

$$\begin{aligned}
&2^{p-1}\big(\max|\eta(\mu(\varsigma,t)) - \theta(\nu(\varsigma,t)|\big)^p \\
&\leq 2^{p-1}\frac{\alpha(d(\mu(\varsigma,t),\nu(\varsigma,t)))}{(2^{\frac{p-1}{p}})^p}(\max\{\mu(\varsigma,t)-\nu(\varsigma,t)\})^p \\
&= \alpha(d(\mu(\varsigma,t),\nu(\varsigma,t)))(d(\mu(\varsigma,t),\nu(\varsigma,t))) \\
&\leq \alpha(d(\mu,\nu))d(\mu,\nu) + \frac{d(\mu,\nu)}{1+d(\mu,\nu)} \\
&\leq \alpha(d(\mu,\nu))\Psi((\mu,\nu)) + \beta(d(\mu,\nu))\Phi(d(\mu,\nu)).
\end{aligned}$$

Which implies that

$$\begin{aligned}
sd(\eta(\mu(\varsigma,t)),\theta(\nu(\varsigma,t)) \leq{}& \alpha(d(\mu,\nu))\Psi(\mu,\nu) \\
&+ \beta(d(\mu,\nu))\Phi(\mu,\nu).
\end{aligned}$$

By taking $s = 2^{p-1}$ from Theorem 10 the integral equation (III.6) has a solution in Λ.

IV. CONCLUSION

During modeling, real-world problems in more accurate and significant ways related to engineering and scientific field's fractional order differential equation and integrals equation are the best tools. One of the most preferable research areas in this field is the existence theory of solutions. In the current work, we have discussed some new fixed point theorem for multi-valued mapping in b-metric spaces with application to the existence of solutions to certain fractional differential equations and integral equations in the frame of Caputo fractional derivative.

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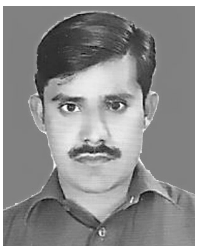

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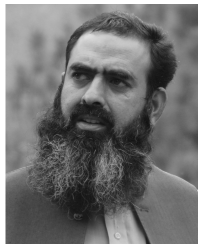

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