Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1518 – 1532
© TÜBİTAK
doi:10.3906/mat-1812-36

Third-order boundary value transmission problems

Ekin UĞURLU*
Department of Mathematics, Faculty of Arts and Sciences, Çankaya University, Ankara, Turkey

Received: 12.12.2018 • **Accepted/Published Online:** 03.04.2019 • **Final Version:** 29.05.2019

Abstract: In this paper, we consider some third-order operators with transmission conditions. In particular, it is shown that such operators are formally symmetric in the corresponding Hilbert spaces and we introduce the resolvent operators associated with the differential operators. After showing that the eigenvalues of the problems are real and discrete we introduce some ordinary and Frechet derivatives of the eigenvalues with respect to some elements of data.

Key words: Third-order differential operator, transmission conditions, Frechet derivative

1. Introduction

Recently, in [8], the nature of the formally symmetric separated, real-coupled, and complex-coupled boundary conditions for the formally symmetric regular third-order differential equation was introduced and some spectral properties were shared. Readers may find the historical background on odd-order formally symmetric boundary value problems in [8].

Although the results filled some gaps on third-order boundary value problems, there is still a huge amount of work that needs to be done on such problems. This includes the imposing separated, real-coupled, complex-coupled transmission conditions to the solutions of these third-order equations and investigating the spectral properties of such problems. For second-order equations with transmission conditions we refer to [1, 2, 5–7].

In this paper, we consider the following third-order equation:

$$\ell(y) = \lambda y, \; x \in [a, c) \cup (c, b], \tag{1.1}$$

where

$$\ell(y) = \frac{1}{w}\left\{-i\left(q_0(q_0y')'\right)' - (p_0y')' + i\left[q_1y' + (q_1y)'\right] + p_1y\right\},$$

and q_0, q_1, p_0, p_1, and w are real-valued and continuous functions on each interval $[a, c)$ and $(c, b]$ such that $w > 0$ and $q_0 \neq 0$ on $[a, c)$ and $(c, b]$.

Let $L^2_w([a, c) \cup (c, b]) = L^2_{w_1}[a, c) \oplus L^2_{w_2}(c, b]$ be the Hilbert space with the inner product

$$(y, z) = \int_a^c y_1\overline{z}_1w_1dx + \int_c^b y_2\overline{z}_2w_2dx,$$

where

*Correspondence: ekinugurlu@cankaya.edu.tr
2000 *AMS Mathematics Subject Classification:* Primary 34B05; Secondary 34B37

$$y = \begin{cases} y_1, & x \in [a,c) \\ y_2, & x \in (c,b] \end{cases}, z = \begin{cases} z_1, & x \in [a,c) \\ z_2, & x \in (c,b] \end{cases} \in L_w^2, \quad w = \begin{cases} w_1, & x \in [a,c) \\ w_2, & x \in (c,b] \end{cases}.$$

Consider the subspace D of L_w^2 consisting of all functions $y \in L_w^2$ such that y', $(q_0 y')'$ exist and $\ell(y) \in L_w^2$. We set the maximal operator L with the rule

$$Ly = \ell(y),\ y \in D,\ x \in [a,c) \cup (c,b].$$

For $y, z \in D$ one gets the Lagrange identity as

$$(Ly, z) - (y, Lz) = [y, \overline{z}]\ |_a^{c-} + [y, \overline{z}]\ |_{c+}^{b}, \tag{1.2}$$

where $[y, \overline{z}]\ |_{t_1}^{t_2} = [y, \overline{z}](t_2) - [y, \overline{z}](t_1)$,

$$[y, z] := y z^{[2]} - y^{[2]} z + i y^{[1]} z^{[1]}, \tag{1.3}$$

and $y^{[r]}$ is the corresponding rth quasi-derivative of y with the rule

$$y^{[0]} = y,\ y^{[1]} = -\frac{1+i}{\sqrt{2}} q_0 y',\ y^{[2]} = i q_0 (q_0 y')' + p_0 y' - i q_1 y.$$

(1.2) implies the existence of the value $[y, \overline{z}]$ at the points a, $c-$, $c+$, and b for $y, z \in D$ that should be finite.

Eq. (1.3) can also be introduced as

$$[y, z] = \begin{bmatrix} z & z^{[2]} \end{bmatrix} E \begin{bmatrix} y \\ y^{[2]} \end{bmatrix} + i y^{[1]} z^{[1]}, \tag{1.4}$$

where

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{1.5}$$

and this representation will allow us to consider more general boundary and transmission conditions.

Eq. (1.1) can be handled as the following first-order system:

$$Y' = [\lambda P + Q]\, Y,\ x \in [a,c) \cup (c,b], \tag{1.6}$$

where Y is a 3×1 vector; P, Q are 3×3 matrices such that

$$Y = \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \end{bmatrix}, \quad P = \begin{bmatrix} & & \\ & & \\ -w & & \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{(1+i) q_0} & 0 \\ -\frac{1+i}{\sqrt{2}} \frac{q_1}{q_0} & i \frac{p_0}{q_0^2} & -\frac{\sqrt{2}}{(1+i) q_0} \\ p_1 & -\frac{1+i}{\sqrt{2}} \frac{q_1}{q_0} & 0 \end{bmatrix}; \tag{1.7}$$

and all the other entries of P are zero. The assumptions on the functions q_0, q_1, p_0, p_1, w, (1.6), and (1.7) imply the following.

Theorem 1.1. *Eq. (1.1) has a unique solution y satisfying the conditions*

$$y^{[r]}(l, \lambda) = l_r,$$

where $l \in [a,b]$, $r = 0, 1, 2,$ and l_r is a complex number. Moreover $y^{[r]}(., \lambda)$ is an entire function in λ.

2. Boundary and transmission conditions

In this section we introduce separated, real-coupled, and complex-coupled boundary and transmission conditions. However, for each problem we need to construct a suitable inner product and corresponding operator associated with the related problem. We shall investigate each problem in the following subsections.

2.1. Separated boundary and transmission conditions

For $y \in D$ the *separated* boundary conditions may be introduced as

$$\begin{aligned}&\cos\alpha y(a) - \sin\alpha y^{[2]}(a) = 0,\\ &(i+\tan\beta)y^{[1]}(a) = \sqrt{\rho_1\rho_2}(1+i\tan\beta)y^{[1]}(b),\\ &\cos\gamma y(b) - \sin\gamma y^{[2]}(b) = 0,\end{aligned} \tag{2.1}$$

and the separated transmission conditions may be introduced as

$$\begin{aligned}&y(c-) = \rho_1 y(c+),\\ &y^{[1]}(c-) = \sqrt{\rho_1\rho_2}y^{[1]}(c+),\\ &y^{[2]}(c-) = \rho_2 y^{[2]}(c+),\end{aligned} \tag{2.2}$$

where $\alpha, \beta, \gamma, \rho_1, \rho_2$ are real numbers such that $\rho_1\rho_2 > 0$.

Note that for $\rho_1 = \rho_2 = 1$, this problem turns out to be a continuous problem and it was studied in [8].

Let $H = L^2_{w_1}[a,c) \oplus L^2_{w_2}(c,b]$ be the Hilbert space with the inner product

$$\langle y, z\rangle = \int_a^c y_1\overline{z}_1 w_1 dx + \rho_1\rho_2 \int_c^b y_2\overline{z}_2 w_2 dx,$$

where

$$y = \begin{cases} y_1, & x\in[a,c) \\ y_2, & x\in(c,b]\end{cases}, z = \begin{cases} z_1, & x\in[a,c) \\ z_2, & x\in(c,b]\end{cases} \in H, \quad w = \begin{cases} w_1, & x\in[a,c) \\ w_2, & x\in(c,b]\end{cases}.$$

Consider the subspace D_s of H consisting of the functions $y \in H$ such that $y^{[r]}$ exists for $0 \le r \le 2$, y satisfying (2.1), (2.2), and $\ell(y) \in H$. Let L_s be the operator on D_s with the rule

$$L_s y = \ell(y),\ y \in D_s.$$

Then we have the following theorem.

Theorem 2.1.1. *The eigenvalues of L_s are real.*

Proof For $y, z \in D_s$ one obtains

$$\langle L_s y, z\rangle - \langle y, L_s z\rangle = [y,\overline{z}]\mid_a^{c-} + \rho_1\rho_2[y,\overline{z}]\mid_{c+}^{b}. \tag{2.3}$$

From (2.2) one gets

$$[y,\overline{z}](c-) = \rho_1\rho_2[y,\overline{z}](c+). \tag{2.4}$$

Moreover, the second condition in (2.1) gives

$$[y,\overline{z}](b)=\frac{1}{\rho_1\rho_2}[y,\overline{z}](a). \tag{2.5}$$

Therefore (2.1)–(2.5) show that

$$\langle L_s y,z\rangle-\langle y,L_s z\rangle=0$$

and this completes the proof. □

2.2. Real-coupled boundary and transmission conditions

For $y\in D$ the real-coupled boundary conditions are given as

$$\begin{bmatrix} y(b)\\ y^{[2]}(b)\end{bmatrix}=\tfrac{1}{\sqrt{\det M}}K\begin{bmatrix} y(a)\\ y^{[2]}(a)\end{bmatrix},\quad y^{[1]}(b)=\tfrac{1}{\sqrt{\det M}}\tfrac{i+m}{1+im}y^{[1]}(a), \tag{2.6}$$

and real-coupled transmission conditions are introduced as

$$\begin{bmatrix} y(c-)\\ y^{[2]}(c-)\end{bmatrix}=M\begin{bmatrix} y(c+)\\ y^{[2]}(c+)\end{bmatrix},\quad y^{[1]}(c-)=\sqrt{\det M}y^{[1]}(c+), \tag{2.7}$$

where $K\in SL_2(\mathbb{R})$, i.e.

$$K=\begin{bmatrix} k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix},\ k_{ij}\in\mathbb{R},\ \det K=1 \tag{2.8}$$

and

$$M=\begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix},\ m_{ij}\in\mathbb{R},\ \det M>0, \tag{2.9}$$

such that

$$K^*EK=E \tag{2.10}$$

and

$$M^*EM=(\det M)\,E. \tag{2.11}$$

Note that for $m_{11}=m_{22}=1$ and $m_{12}=m_{21}=0$ conditions (2.6) and (2.7) were investigated in [8].

Let $H=L^2_{w_1}[a,c)\oplus L^2_{w_2}(c,b]$ be the Hilbert space with the inner product

$$\langle y,z\rangle=\int_a^c y_1\overline{z}_1w_1dx+\det M\int_c^b y_2\overline{z}_2w_2dx,$$

where

$$y=\begin{cases} y_1, & x\in[a,c)\\ y_2, & x\in(c,b]\end{cases},z=\begin{cases} z_1, & x\in[a,c)\\ z_2, & x\in(c,b]\end{cases}\in H,\quad w=\begin{cases} w_1, & x\in[a,c)\\ w_2, & x\in(c,b]\end{cases}.$$

Let D_{rc} be the subspace of H consisting of the functions $y \in H$ such that $y^{[r]}$ exists for $0 \le r \le 2$, y satisfying (2.6), (2.7), and $\ell(y) \in H$. We construct the operator L_{rc} on D_{rc} with the rule

$$L_{rc}y = \ell(y), \ y \in D_{rc}.$$

Then we have the following Theorem.

Theorem 2.2.1. *The eigenvalues of L_{rc} are real.*

Proof For $y, z \in D_{rc}$ we have

$$\begin{aligned}
&\langle L_{rc}y, z\rangle - \langle y, L_{rc}z\rangle \\
&= [y,\overline{z}] \mid_a^{c-} + \det M[y,\overline{z}] \mid_{c+}^{b} = \left[\begin{array}{cc} \overline{z}(c-) & \overline{z^{[2]}}(c-) \end{array}\right] E \left[\begin{array}{c} y(c-) \\ y^{[2]}(c-) \end{array}\right] \\
&+ iy^{[1]}(c-)\overline{z^{[1]}}(c-) - \left[\begin{array}{cc} \overline{z}(a) & \overline{z^{[2]}}(a) \end{array}\right] E \left[\begin{array}{c} y(a) \\ y^{[2]}(a) \end{array}\right] - iy^{[1]}(a)\overline{z^{[1]}}(a) \\
&+ \det M \left(\left[\begin{array}{cc} \overline{z}(b) & \overline{z^{[2]}}(b) \end{array}\right] E \left[\begin{array}{c} y(b) \\ y^{[2]}(b) \end{array}\right] + iy^{[1]}(b)\overline{z^{[1]}}(b) \right. \\
&\left. - \left[\begin{array}{cc} \overline{z}(c+) & \overline{z^{[2]}}(c+) \end{array}\right] E \left[\begin{array}{c} y(c+) \\ y^{[2]}(c+) \end{array}\right] - iy^{[1]}(c+)\overline{z^{[1]}}(c+) \right).
\end{aligned} \tag{2.12}$$

The conditions (2.6) and (2.10) give

$$\begin{aligned}
&\left[\begin{array}{cc} \overline{z}(b) & \overline{z^{[2]}}(b) \end{array}\right] E \left[\begin{array}{c} y(b) \\ y^{[2]}(b) \end{array}\right] = \tfrac{1}{\det M} \left[\begin{array}{cc} \overline{z}(a) & \overline{z^{[2]}}(a) \end{array}\right] E \left[\begin{array}{c} y(a) \\ y^{[2]}(a) \end{array}\right], \\
&y^{[1]}(b)\overline{z^{[1]}}(b) = \tfrac{1}{\det M} y^{[1]}(a)\overline{z^{[1]}}(a).
\end{aligned} \tag{2.13}$$

Moreover, (2.7) and (2.11) imply

$$\begin{aligned}
&\left[\begin{array}{cc} \overline{z}(c-) & \overline{z^{[2]}}(c-) \end{array}\right] E \left[\begin{array}{c} y(c-) \\ y^{[2]}(c-) \end{array}\right] = \det M \left[\begin{array}{cc} \overline{z}(c+) & \overline{z^{[2]}}(c+) \end{array}\right] E \left[\begin{array}{c} y(c+) \\ y^{[2]}(c+) \end{array}\right], \\
&y^{[1]}(c-)\overline{z^{[1]}}(c-) = \det M y^{[1]}(c+)\overline{z^{[1]}}(c+).
\end{aligned} \tag{2.14}$$

Therefore, (2.12)–(2.14) show that

$$\langle L_{rc}y, z\rangle - \langle y, L_{rc}z\rangle = 0$$

and this completes the proof. □

2.3. Complex-coupled boundary and transmission conditions

For $y \in D$ complex-coupled boundary conditions may be given as

$$\left[\begin{array}{c} y(b) \\ y^{[2]}(b) \end{array}\right] = \tfrac{1}{\sqrt{\det M}} e^{i\theta_1} K \left[\begin{array}{c} y(a) \\ y^{[2]}(a) \end{array}\right], \quad y^{[1]}(b) = \tfrac{1}{\sqrt{\det M}} e^{i\theta_2} \tfrac{i+m}{1+im} y^{[1]}(a), \tag{2.15}$$

and complex-coupled transmission conditions may be introduced as

$$\left[\begin{array}{c} y(c-) \\ y^{[2]}(c-) \end{array}\right] = e^{i\eta_1} M \left[\begin{array}{c} y(c+) \\ y^{[2]}(c+) \end{array}\right], \quad y^{[1]}(c-) = \sqrt{\det M} e^{i\eta_2} y^{[1]}(c+), \tag{2.16}$$

where $\theta_1, \theta_2, \eta_1, \eta_2$ are real numbers and matrices K and M are the matrices given by (2.8) and (2.9).

Note that for $\eta_1 = \eta_2 = 0$, $m_{11} = m_{22} = 1$, and $m_{12} = m_{21} = 0$ conditions (2.15) and (2.16) were studied in [8].

Let $H = L^2_{w_1}[a,c) \oplus L^2_{w_2}(c,b]$ be the Hilbert space with the inner product

$$\langle y, z\rangle = \int_a^c y_1\overline{z}_1 w_1 dx + \det M \int_c^b y_2 \overline{z}_2 w_2 dx,$$

where

$$y = \begin{cases} y_1, & x \in [a,c) \\ y_2, & x \in (c,b] \end{cases}, z = \begin{cases} z_1, & x \in [a,c) \\ z_2, & x \in (c,b] \end{cases} \in H, \quad w = \begin{cases} w_1, & x \in [a,c) \\ w_2, & x \in (c,b] \end{cases}.$$

Let D_c be the subspace of H that consists of the functions $y \in H$ such that $y^{[r]}$ exists for $0 \leq r \leq 2$ and y satisfies (2.15), (2.16), and $\ell(y) \in H$. We construct the operator L_c on D_c with the rule

$$L_c y = \ell(y), \ y \in D_c.$$

Then we have the following theorem.

Theorem 2.3.1. *The eigenvalues of L_c are real.*

Proof For $y, z \in D_c$ we have

$$\begin{aligned} &\langle L_c y, z\rangle - \langle y, L_c z\rangle \\ &= [y,\overline{z}]\mid_a^{c-} + \det M [y,\overline{z}]\mid_{c+}^b = \begin{bmatrix} \overline{z}(c-) & \overline{z^{[2]}}(c-) \end{bmatrix} E \begin{bmatrix} y(c-) \\ y^{[2]}(c-) \end{bmatrix} \\ &+ iy^{[1]}(c-)\overline{z^{[1]}}(c-) - \begin{bmatrix} \overline{z}(a) & \overline{z^{[2]}}(a) \end{bmatrix} E \begin{bmatrix} y(a) \\ y^{[2]}(a) \end{bmatrix} - iy^{[1]}(a)\overline{z^{[1]}}(a) \\ &+ \det M \left(\begin{bmatrix} \overline{z}(b) & \overline{z^{[2]}}(b) \end{bmatrix} E \begin{bmatrix} y(b) \\ y^{[2]}(b) \end{bmatrix} + iy^{[1]}(b)\overline{z^{[1]}}(b) \right. \\ &\left. - \begin{bmatrix} \overline{z}(c+) & \overline{z^{[2]}}(c+) \end{bmatrix} E \begin{bmatrix} y(c+) \\ y^{[2]}(c+) \end{bmatrix} - iy^{[1]}(c+)\overline{z^{[1]}}(c+) \right). \end{aligned} \tag{2.17}$$

(2.15) and (2.10) give

$$\begin{aligned} &\begin{bmatrix} \overline{z}(b) & \overline{z^{[2]}}(b) \end{bmatrix} E \begin{bmatrix} y(b) \\ y^{[2]}(b) \end{bmatrix} = \frac{1}{\det M} \begin{bmatrix} \overline{z}(a) & \overline{z^{[2]}}(a) \end{bmatrix} E \begin{bmatrix} y(a) \\ y^{[2]}(a) \end{bmatrix}, \\ &y^{[1]}(b)\overline{z^{[1]}}(b) = \frac{1}{\det M} y^{[1]}(a)\overline{z^{[1]}}(a), \end{aligned} \tag{2.18}$$

and (2.16) and (2.11) give

$$\begin{aligned} &\begin{bmatrix} \overline{z}(c-) & \overline{z^{[2]}}(c-) \end{bmatrix} E \begin{bmatrix} y(c-) \\ y^{[2]}(c-) \end{bmatrix} = \det M \begin{bmatrix} \overline{z}(c+) & \overline{z^{[2]}}(c+) \end{bmatrix} E \begin{bmatrix} y(c+) \\ y^{[2]}(c+) \end{bmatrix}, \\ &y^{[1]}(c-)\overline{z^{[1]}}(c-) = \det M y^{[1]}(c+)\overline{z^{[1]}}(c+). \end{aligned} \tag{2.19}$$

Using (2.17)–(2.19) we get

$$\langle L_c y, z\rangle - \langle y, L_c z\rangle = 0$$

and this completes the proof. □

3. Entire function and resolvent operator

In this section we investigate some spectral properties of each operator L_s, L_{rc}, L_c and therefore corresponding boundary value transmission problems. However, first of all, we shall introduce other representations of the problems.

Conditions (2.1) and (2.2) may also be introduced as

$$Y(a) = \begin{bmatrix} y(a) \\ y^{[1]}(a) \\ y^{[2]}(a) \end{bmatrix} = \begin{bmatrix} \sin\alpha & 0 & 0 \\ 0 & \sqrt{\rho_1\rho_2}\frac{1+i\tan\beta}{i+\tan\beta} & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = A.v,$$
$$Y(b) = \begin{bmatrix} y(b) \\ y^{[1]}(b) \\ y^{[2]}(b) \end{bmatrix} = \begin{bmatrix} \sin\gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = B.v,$$

and

$$Y(c-) = \begin{bmatrix} y(c-) \\ y^{[1]}(c-) \\ y^{[2]}(c-) \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \sqrt{\rho_1\rho_2} & 0 \\ 0 & 0 & \rho_2 \end{bmatrix} \begin{bmatrix} y(c+) \\ y^{[1]}(c+) \\ y^{[2]}(c+) \end{bmatrix} = TY(c+),$$

where v is a 3×1 vector. Similarly, conditions (2.15) and (2.16) may be handled as

$$Y(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = A.v,$$
$$Y(b) = \frac{1}{\sqrt{\det M}} \begin{bmatrix} e^{i\theta_1}k_{11} & 0 & e^{i\theta_1}k_{12} \\ 0 & e^{i\theta_1}\frac{i+m}{1+im} & 0 \\ e^{i\theta_1}k_{21} & 0 & e^{i\theta_1}k_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = B.v,$$

and

$$\begin{aligned} Y(c-) &= \begin{bmatrix} y(c-) \\ y^{[1]}(c-) \\ y^{[2]}(c-) \end{bmatrix} = \begin{bmatrix} e^{i\eta_1}m_{11} & 0 & e^{i\eta_1}m_{12} \\ 0 & \sqrt{\det M}e^{i\theta_1} & 0 \\ e^{i\eta_1}m_{21} & 0 & e^{i\eta_1}m_{22} \end{bmatrix} \begin{bmatrix} y(c+) \\ y^{[1]}(c+) \\ y^{[2]}(c+) \end{bmatrix} \\ &= TY(c+). \end{aligned}$$

Note that for $\theta_1 = \theta_2 = \eta_1 = \eta_2 = 0$ the conditions can be matched with (2.6) and (2.7).

Therefore, each problem may be handled as

$$Y' = [\lambda P + Q]\,Y + F,\ \ x \in [a,c) \cup (c,b], \tag{3.1}$$

with

$$Y(a) = A.v,\ \ Y(b) = B.v,\ \ Y(c-) = TY(c+), \tag{3.2}$$

where

$$F = \begin{bmatrix} 0 \\ 0 \\ wf \end{bmatrix} = \begin{cases} F_1, & x \in [a,c), \\ F_2, & x \in (c,b], \end{cases}$$

and $\det T \neq 0$.

Theorem 3.1. *The eigenvalues of the operators* L_s, L_{rc}, L_c *are real and discrete and can be represented by*

$$\lambda_0, \lambda_1, \lambda_2, ...$$

and the series

$$\sum_{\lambda_j \neq 0} |\lambda_j|^{-1-\epsilon}$$

is convergent for $\epsilon > 0$. *Moreover, each* λ_j *may appear in the sequence at most* 3.

Proof In Section 2 it has been proved that all the eigenvalues are real. For the other assertions we shall construct the fundamental matrix solution $\mathcal{Y}_1(x,\lambda)$ of (1.6) on $[a,c)$ satisfying $\mathcal{Y}_1(a,\lambda) = I$ and $\mathcal{Y}_2(x,\lambda)$ of (1.6) on $(c,b]$ satisfying $\mathcal{Y}_2(c+,\lambda) = T^{-1}\mathcal{Y}_1(c-,\lambda)$, where I is the 3×3 unit matrix and

$$\mathcal{Y}(x,\lambda) = \begin{cases} \mathcal{Y}_1(x,\lambda), & x \in [a,c), \\ \mathcal{Y}_2(x,\lambda), & x \in (c,b]. \end{cases} \tag{3.3}$$

For any other solution

$$Y(x,\lambda) = \begin{cases} Y_1(x,\lambda), & x \in [a,c), \\ Y_2(x,\lambda), & x \in (c,b], \end{cases}$$

of (1.6) on $[a,c) \cup (c,b]$, we let

$$\begin{aligned} Y_1(x,\lambda) &= \mathcal{Y}_1(x,\lambda)Y_1(a,\lambda), \ x \in [a,c), \\ Y_2(x,\lambda) &= \mathcal{Y}_2(x,\lambda)\mathcal{Y}_2^{-1}(c+,\lambda)Y_2(c+,\lambda), \ x \in (c,b]. \end{aligned}$$

Then

$$Y_2(b,\lambda) = B.v = \mathcal{Y}_2(b,\lambda)\mathcal{Y}_2^{-1}(c+,\lambda)Y_2(c+,\lambda).$$

If $Y(x,\lambda)$ satisfies the transmission condition $Y_1(c-,\lambda) = TY_2(c+,\lambda)$ then we may write

$$Y_2(b,\lambda) = B.v = \mathcal{Y}_2(b,\lambda)Y_1(a,\lambda) = \mathcal{Y}_2(b,\lambda)A.v.$$

Therefore, Y satisfies the conditions (3.2) if

$$\{B - \mathcal{Y}_2(b,\lambda)A\}.v = 0 \tag{3.4}$$

and for $v \neq 0$

$$\Delta(\lambda) := \det\{B - \mathcal{Y}_2(b,\lambda)A\} = 0. \tag{3.5}$$

(3.5) shows that if λ is an eigenvalue of problem (1.6) and (3.2) then (3.5) must hold. Conversely, if (3.5) holds then there will be a solution $v \neq 0$ of (3.4) and we can construct a solution of (1.6) satisfying (3.2).

From Theorem 1.1 we may infer that Δ is an entire function. Therefore, the roots of Δ or equivalently the eigenvalues of the problem must be discrete.

The assumptions on the entries of P and Q in (1.6) and Gronwall's inequality prove

$$\mathcal{Y}(b,\lambda) = O\left(\exp(const.|\lambda|\right),$$

which implies the convergence of the series for each $\epsilon > 0$.

Finally, since the number of linearly independent solutions v of (3.4) is at most 3, the proof is completed. $\square$

Now we shall introduce a method to solve problem (3.1) and (3.2).

Let

$$Y(x,\lambda)=\begin{cases} Y_1(x,\lambda), & x\in[a,c),\\ Y_2(x,\lambda), & x\in(c,b]\end{cases}$$

be the solution of (3.1) satisfying (3.2). We construct the functions $Y_1(x,\lambda)=\mathcal{Y}_1(x,\lambda)\tau_1(x,\lambda),\ x\in[a,c)$, and $Y_2(x,\lambda)=\mathcal{Y}_2(x,\lambda)\tau_2(x,\lambda),\ x\in(c,b]$, where $\mathcal{Y}(x,\lambda)$ is defined by (3.3) and τ_1 and τ_2 are 3×1 vector functions. Then we get

$$\begin{array}{l} Y_1'=[\lambda P+Q]\,\mathcal{Y}_1'\tau_1+\mathcal{Y}_1\tau_1',\ x\in[a,c),\\ Y_2'=[\lambda P+Q]\,\mathcal{Y}_2'\tau_2+\mathcal{Y}_2\tau_2',\ x\in(c,b],\end{array}$$

and therefore

$$\begin{array}{l} \tau_1'=\mathcal{Y}_1^{-1}F_1,\ x\in[a,c),\\ \tau_2'=\mathcal{Y}_2^{-1}F_2,\ x\in(c,b].\end{array}$$

The conditions $Y_1(a)=\tau_1(a)=A.v$ and $Y_2(b)=\mathcal{Y}_2(b)\tau_2(b)=B.v$ imply

$$\begin{array}{l} \tau_1(x)=A.v+\int\limits_a^x\mathcal{Y}_1^{-1}(t)F_1(t)dt,\ x\in[a,c),\\ \tau_2(x)=\mathcal{Y}_2^{-1}(b)B.v-\int\limits_x^b\mathcal{Y}_2^{-1}(t)F_2(t)dt,\ x\in(c,b],\end{array}$$

and therefore

$$\begin{array}{l} Y_1(x)=\mathcal{Y}_1(x)A.v+\int\limits_a^x\mathcal{Y}_1(x)\mathcal{Y}_1^{-1}(t)F_1(t)dt,\ x\in[a,c),\\ Y_2(x)=\mathcal{Y}_2(x)\mathcal{Y}_2^{-1}(b)B.v-\int\limits_x^b\mathcal{Y}_2(x)\mathcal{Y}_2^{-1}(t)F_2(t)dt,\ x\in(c,b].\end{array}$$

Transmission condition $Y_1(c-)=TY_2(c+)$ gives

$$v=[B-\mathcal{Y}_2(b)A]^{-1}\,\mathcal{Y}_2(b)\int_a^b\mathcal{Y}^{-1}(t)F(t)dt$$

provided that λ is not an eigenvalue of the problem. Therefore,

$$\begin{aligned} Y_1(x)\quad &=\mathcal{Y}_1(x)A\,[B-\mathcal{Y}_2(b)B]^{-1}\,\mathcal{Y}_2(b)\int\limits_a^b\mathcal{Y}^{-1}(t)F(t)dt\\ &+\int\limits_a^x\mathcal{Y}_1(x)\mathcal{Y}_1^{-1}(t)F_1(t)dt,\ x\in[a,c),\end{aligned}$$

and

$$\begin{aligned} Y_2(x) &= \mathcal{Y}_2(x)\mathcal{Y}_2^{-1}(b)B\left[B-\mathcal{Y}_2(b)A\right]^{-1}\mathcal{Y}_2(b)\int\limits_a^b \mathcal{Y}^{-1}(t)F(t)dt \\ &- \int\limits_x^b \mathcal{Y}_2(x)\mathcal{Y}_2^{-1}(t)F_2(t)dt,\ x\in(c,b]. \end{aligned}$$

Consequently the solution $Y(x) := Y(x,\lambda)$ of (3.1) and (3.2) can be introduced as

$$Y(x) = \begin{cases} \mathcal{Y}_1(x)A\left[B-\mathcal{Y}_2(b)A\right]^{-1}\mathcal{Y}_2(b)\int\limits_x^c \mathcal{Y}_1^{-1}(t)F_1(t)dt \\ +\mathcal{Y}_1(x)\mathcal{Y}_2^{-1}(b)B\left[B-\mathcal{Y}_2(b)B\right]^{-1}\mathcal{Y}_2(b)\int\limits_a^x \mathcal{Y}_1^{-1}(t)F_1(t)dt \\ +\mathcal{Y}_1(x)A\left[B-\mathcal{Y}_2(b)A\right]^{-1}\mathcal{Y}_2(b)\int\limits_c^b \mathcal{Y}_2^{-1}(t)F_2(t)dt,\ x\in[a,c), \\ +\mathcal{Y}_2(x)\mathcal{Y}_2^{-1}(b)B\left[B-\mathcal{Y}_2(b)B\right]^{-1}\mathcal{Y}_2(b)\int\limits_a^c \mathcal{Y}_1^{-1}(t)F_1(t)dt \\ +\mathcal{Y}_2(x)\mathcal{Y}_2^{-1}(b)B\left[B-\mathcal{Y}_2(b)B\right]^{-1}\mathcal{Y}_2(b)\int\limits_c^x \mathcal{Y}_2^{-1}(t)F(t)dt \\ +\mathcal{Y}_2(x)A\left[B-\mathcal{Y}_2(b)B\right]^{-1}\mathcal{Y}_2(b)\int\limits_x^b \mathcal{Y}_2^{-1}(t)F_2(t)dt,\ x\in(c,b]. \end{cases} \tag{3.6}$$

Hence, we may introduce the following.

Theorem 3.2. *The solution $Y(x,\lambda)$ of (3.1) and (3.2) can be represented by (3.6) provided that λ is not an eigenvalue of the problem.*

4. Banach space

In this section we investigate the dependence property of the eigenvalues and eigenfunctions of the problems on some elements of data.

We can consider conditions (3.2) because each boundary value transmission problem can be embedded into (3.1),(3.2). However, we should note that for the intervals $(a',c)\cup(c,b')\supset[a,c)\cup(c,b]$ we will consider the following:

$$\widetilde{p}_0 = \begin{cases} p_0 \text{ on } [a,c)\cup(c,b] \\ 0 \text{ otherwise} \end{cases},\ \widetilde{p}_1 = \begin{cases} p_1 \text{ on } [a,c)\cup(c,b] \\ 0 \text{ otherwise} \end{cases},\ \widetilde{w} = \begin{cases} w \text{ on } [a,c)\cup(c,b] \\ 0 \text{ otherwise} \end{cases}.$$

Let us consider the Banach space

$$X = \mathbb{R}\times\mathbb{R}\times M_{3,3}(\mathbb{C})\times M_{3,3}(\mathbb{C})\times M_{3,3}(\mathbb{C})\times L^1(a',b')\times L^1(a',b')\times L^1(a',b')$$

with the norm

$$\|\omega_1\| = |a|+|b|+\|A\|+\|B\|+\|T\|+\int\limits_{a'}^{b'}\left(|\widetilde{p}_0|+|\widetilde{p}_1|+|\widetilde{w}|\right),$$

where $M_{3,3}(\mathbb{C})$ denotes the set consisting of all 3×3 matrices with complex entries and

$$\omega_1 = (a, b, A, B, T, \widetilde{p}_0, \widetilde{p}_1, \widetilde{w}).$$

Construct the subspace Λ_1 of X consisting of all elements ω_1. We identify the set Λ consisting of all elements ω such that

$$\omega = (a, b, A, B, T, p_0, p_1, w)$$

with Λ_1 as a subset of X to inherit the norm from X and convergence in Λ that is determined by this norm.

Lemma 4.1. *Consider the solution y of (1.1) and (3.2) satisfying*

$$y^{[r]}(l, \lambda) = l_j, \ l \in (a', b'), \ 0 \le r \le 2.$$

Then the solution $y = y(., l, l_0, l_1, l_2, p_0, p_1, w)$ is continuous of all its variables.

Proof The proof follows from Eq. (1.6) and Theorem 2.7 in [4]. □

Theorem 4.2. *Let $\lambda = \lambda(\omega)$ be an eigenvalue of (1.1) and (3.2). Then λ is continuous at $\omega_0 \in \Lambda$.*

Proof For $\omega \in \Lambda$, consider the following:

$$\Theta(\omega, \lambda) := \det\left\{B - \mathcal{Y}_2(a, b, m, p_0, p_1, w, \lambda)A\right\}.$$

If $\Theta(\omega, \lambda) = 0$, then $\lambda(\omega)$ becomes an eigenvalue of (1.1) and (3.2). Let $\Theta(\omega_0, \psi) = 0$. Then $\Theta(\omega_0, \lambda)$ is not constant in λ because Θ is an entire function in λ. Therefore, for

$$\lambda \in \{\lambda \in \mathbb{C} : |\lambda - \psi| = \sigma\}, \ \sigma > 0,$$

$\Theta(\omega_0, \lambda) \neq 0$. Therefore, the result follows from the well-known theorem on continuity of roots of an equation as a function of parameters [3]: □

Remark 4.3. *According to Theorem 4.2 one may infer for any fixed eigenvalue μ with $\omega = \omega_0$ that there exists a continuous eigenvalue branch $\lambda(\omega)$ such that $\lambda(\omega_0) = \mu$. However, this result does not imply that the nth eigenvalue $\lambda_n(\omega)$ for fixed n is always continuous in ω. Therefore, we will consider that each eigenvalue $\lambda(\omega)$ of (1.1) and (3.2) for $\omega \in \Lambda$ is embedded in a continuous branch.*

Lemma 4.4. *Let $\lambda = \lambda(\omega)$ be an eigenvalue of (1.1), (3.2) and $\omega_0 \in \Lambda$. Then:*

(i) If $\lambda = \lambda(\omega_0)$ is simple for some $\omega_0 \in \Lambda$ and $u(., \omega_0)$ is the normalized eigenfunction of $\lambda(\omega_0)$, then there exist normalized eigenfunctions $u(., \omega)$ of $\lambda(\omega)$ for $\omega \in \Lambda$ such that

$$u^{[j]}(., \omega) \to u^{[j]}(., \omega_0) \text{ as } \omega \to \omega_0 \text{ in } \Lambda, \ j = 0, 1, 2,$$

both uniformly on any compact subintervals of (a', b').

(ii) If $\lambda = \lambda(\omega_0)$ is double for some $\omega_0 \in \Lambda$ and $u(., \omega_0)$ is the normalized eigenfunction of $\lambda(\omega_0)$, then there exist normalized eigenfunctions $u(., \omega)$ of $\lambda(\omega)$ for $\omega \in \Lambda$ such that

$$u^{[j]}(., \omega) \to u^{[j]}(., \omega_0) \text{ as } \omega \to \omega_0 \text{ in } \Lambda, \ j = 0, 1, 2,$$

both uniformly on any compact subintervals of (a', b'). *In this case, given two linearly independent normalized eigenfunctions* u_j *of* $\lambda(\omega_0)$ *there exist a pair of linearly independent normalized eigenfunctions of* $\lambda(\omega)$, *one of which converges to* u_1 *and the other to* u_2 *as* $\omega \to \omega_0$ *in* Λ.

(*iii*) *If* $\lambda = \lambda(\omega_0)$ *is triple for some* $\omega_0 \in \Lambda$ *and* $u(., \omega_0)$ *is the normalized eigenfunction of* $\lambda(\omega_0)$, *then there exist normalized eigenfunctions* $u(., \omega)$ *of* $\lambda(\omega)$ *for* $\omega \in \Lambda$ *such that*

$$u^{[j]}(., \omega) \to u^{[j]}(., \omega_0) \text{ as } \omega \to \omega_0 \text{ in } \Lambda,\ j = 0, 1, 2,$$

both uniformly on any compact subintervals of (a', b'). *In this case, given three linearly independent normalized eigenfunctions* u_j *of* $\lambda(\omega_0)$ *there exist a triple of linearly independent normalized eigenfunctions of* $\lambda(\omega)$, *one of which converges to* u_1, *the other to* u_2, *and the last to* u_3 *as* $\omega \to \omega_0$ *in* Λ.

Proof The proof can be given similarly as in [4], pp. 11–12, with the aid of Lemma 4.1 and Theorem 4.2. □

5. Derivatives of eigenvalues

In this section we introduce the derivatives of eigenvalues with respect to some elements of data.

We shall recall that a map A from a Banach space K into another Banach space $\widetilde{K}$ is differentiable at $x \in K$ if there exists a bounded linear operator $dA_x : K \to \widetilde{K}$ such that for $h \in K$

$$\|A(x+h) - A(x) - dA_x(h)\| = o(h) \text{ as } h \to 0.$$

Then we may introduce the following theorem.

Theorem 5.1. *Let* $\lambda = \lambda(\omega)$ *be a simple, double, or triple eigenvalue in some neighborhood of* ω *in* Λ *and let* $u = u(., \omega)$ *be a normalized eigenfunction of* $\lambda(\omega)$.

(*i*) *For* $\lambda = \lambda(\alpha)$ *and* $u = u(., \alpha)$ *we have*

$$\lambda'(\alpha) = -|u(a)| - \left|u^{[2]}(a)\right|.$$

(*ii*) *For* $\lambda = \lambda(\beta)$ *and* $u = u(., \beta)$ *we have*

$$\lambda'(\beta) = 2\left|u^{[1]}(a)\right| = 2\left|u^{[1]}(b)\right|.$$

(*iii*) *For* $\lambda = \lambda(\gamma)$ *and* $u = u(., \gamma)$ *we have*

$$\lambda'(\gamma) = \rho_1\rho_2\left(\left|u^{[2]}(b)\right| + |u(b)|^2\right).$$

(*iv*) *For* $\lambda = \lambda(\theta_1)$ *and* $u = u(., \theta_1)$ *we have*

$$\lambda'(\theta_1) = -2\mathrm{Im}\left(u(a)\overline{u^{[2]}(a)}\right).$$

(*v*) *For* $\lambda = \lambda(\theta_2)$ *and* $u = u(., \theta_2)$ *we have*

$$\lambda'(\theta_2) = -\left|u^{[1]}(a)\right|^2 = -\left|u^{[1]}(b)\right|^2.$$

(vi) For $\lambda = \lambda(\eta_1)$ *and* $u = u(.,\eta_1)$ *we have*

$$\lambda'(\eta_1) = -2\det M \mathrm{Im}\left(u(c+)\overline{u^{[2]}(c+)}\right).$$

(vii) For $\lambda = \lambda(\eta_2)$ *and* $u = u(.,\eta_2)$ *we have*

$$\lambda'(\eta_1) = -\det M\left|u^{[1]}(c+)\right|^2.$$

(viii) For $\lambda = \lambda(K)$ *and* $u = u(.,K)$ *we have*

$$d\lambda_K(H) = -\left[\begin{array}{cc} \overline{u(a)} & \overline{u^{[2]}(a)} \end{array}\right] H^* E K \left[\begin{array}{c} u(a) \\ u^{[2]}(a) \end{array}\right],$$

where $H \in M_{2,2}(\mathbb{R})$ *such that* $K + H \in SL_2(\mathbb{R})$.

(ix) For $\lambda = \lambda(M)$ *and* $u = u(.,M)$ *we have*

$$d\lambda_M(H) = -\left[\begin{array}{cc} \overline{u(c+)} & \overline{u^{[2]}(c+)} \end{array}\right] H^* E M \left[\begin{array}{c} u(c+) \\ u^{[2]}(c+) \end{array}\right],$$

where $M \in M_{2,2}(\mathbb{R})$ and $\det(M + H) = \det M$.

(x) For $\lambda = \lambda(p_0)$ *and* $u = u(.,p_0)$ *we have*

$$d\lambda_{p_0}(h) = -\int_a^c |u'|^2 h - \det M \int_a^c |u'|^2 h,\ h \in L^1(a,b).$$

(xi) For $\lambda = \lambda(p_1)$ *and* $u = u(.,p_1)$ *we have*

$$d\lambda_{p_1}(h) = \int_a^c |u'|^2 h + \det M \int_a^c |u'|^2 h,\ h \in L^1(a,b).$$

(xii) For $\lambda = \lambda(w)$ *and* $u = u(.,w)$ *we have*

$$d\lambda_w(h) = \lambda\left(\int_a^c |u|^2 h + \det M \int_a^c |u|^2 h\right),\ h \in L^1(a,b).$$

Proof We should stress that we want to mean by $\lambda(\omega)$ a continuous eigenvalue branch. Moreover, a normalized eigenfunction $u(.,\omega)$ should be understood as a uniformly convergent normalized eigenfunction branch.

For the proofs of (i)–(v), $(viii)$, and (x)–(xii) we refer to [8]. However, the other assertions need to be explained in detail.

(vi) Consider that $\lambda = \lambda(\eta_1)$ and $u = u(.,\eta_1)$, and fix all the other elements of the data. Setting

$v = u(., \eta_1 + h),\ h > 0,$ we get

$$
\begin{aligned}
&\langle \ell(u), v\rangle - \langle u, \ell(v)\rangle = (\lambda(\eta_1) - \lambda(\eta_1 + h))\langle u, v\rangle = [u, \overline{v}]\ |_a^{c-} + \det M[u, \overline{v}]\ |_{c+}^b \\
&= \begin{bmatrix} \overline{v}(c-) & \overline{v^{[2]}}(c-) \end{bmatrix} E \begin{bmatrix} u(c-) \\ u^{[2]}(c-) \end{bmatrix} - \det M \begin{bmatrix} \overline{v}(c+) & \overline{v^{[2]}}(c+) \end{bmatrix} E \begin{bmatrix} u(c+) \\ u^{[2]}(c+) \end{bmatrix} \\
&= \begin{bmatrix} \overline{v}(c+) & \overline{v^{[2]}}(c+) \end{bmatrix} e^{-i(\eta_1+h)} M^* E M e^{i\eta_1} \begin{bmatrix} u(c+) \\ u^{[2]}(c+) \end{bmatrix} \\
&- \det M \begin{bmatrix} \overline{v}(c+) & \overline{v^{[2]}}(c+) \end{bmatrix} E \begin{bmatrix} u(c+) \\ u^{[2]}(c+) \end{bmatrix} \\
&= (e^{-ih} - 1) \det M \begin{bmatrix} \overline{v}(c+) & \overline{v^{[2]}}(c+) \end{bmatrix} E \begin{bmatrix} u(c+) \\ u^{[2]}(c+) \end{bmatrix}.
\end{aligned} \tag{5.1}
$$

Dividing by h of the second and the last expressions of (5.1) and passing to the limit as $h \to 0$, we obtain the result.

(vii) Consider that $\lambda = \lambda(\eta_2)$ and $u = u(., \eta_2)$, and fix all the other elements of the data. Setting $v = u(., \eta_2 + h),\ h > 0,$ we obtain

$$
\begin{aligned}
&\langle \ell(u), v\rangle - \langle u, \ell(v)\rangle = (\lambda(\eta_2) - \lambda(\eta_2 + h))\langle u, v\rangle = [u, \overline{v}]\ |_a^{c-} + \det M[u, \overline{v}]\ |_{c+}^b \\
&= iu^{[1]}(c-)\overline{v^{[1]}}(c-) - i \det M u^{[1]}(c+)\overline{v^{[1]}}(c+) \\
&= i \det M \left(e^{i\eta_2} e^{-i(\eta_2+h)} - 1\right) u^{[1]}(c+)\overline{v^{[1]}}(c+) \\
&= i \det M \left(e^{-ih} - 1\right) u^{[1]}(c+)\overline{v^{[1]}}(c+).
\end{aligned} \tag{5.2}
$$

Now a similar discussion that has been shared for (vii) implies for (5.2) the result.

(ix) Consider that $\lambda = \lambda(M)$ and $u = u(., M)$, and fix all the other elements of the data. Setting $v = u(., M + H)$ we obtain

$$
\begin{aligned}
&\langle \ell(u), v\rangle - \langle u, \ell(v)\rangle = (\lambda(M) - \lambda(M + H))\langle u, v\rangle = [u, \overline{v}]\ |_a^{c-} + \det M[u, \overline{v}]\ |_{c+}^b \\
&= \begin{bmatrix} \overline{v}(c-) & \overline{v^{[2]}}(c-) \end{bmatrix} E \begin{bmatrix} u(c-) \\ u^{[2]}(c-) \end{bmatrix} - \det M \begin{bmatrix} \overline{v}(c+) & \overline{v^{[2]}}(c+) \end{bmatrix} E \begin{bmatrix} u(c+) \\ u^{[2]}(c+) \end{bmatrix} \\
&= \begin{bmatrix} \overline{v}(c+) & \overline{v^{[2]}}(c+) \end{bmatrix} (M + H)^* E M \begin{bmatrix} u(c+) \\ u^{[2]}(c+) \end{bmatrix} \\
&- \det M \begin{bmatrix} \overline{v}(c+) & \overline{v^{[2]}}(c+) \end{bmatrix} E \begin{bmatrix} u(c+) \\ u^{[2]}(c+) \end{bmatrix} \\
&= - \begin{bmatrix} \overline{u}(c+) & \overline{u^{[2]}}(c+) \end{bmatrix} H^* E M \begin{bmatrix} u(c+) \\ u^{[2]}(c+) \end{bmatrix},
\end{aligned}
$$

where $M \in M_{2,2}(\mathbb{R})$ and $\det(M + H) = \det M$. The result follows from the last equation. □

References

[1] Akdoğan Z, Demirci M, Mukhtarov OSh. Green function of discontinuous boundary-value problem with transmission conditions. Mathematical Methods in the Applied Sciences 2007; 30: 1719-1738.

[2] Aydemir K, Mukhtarov OSh. Spectrum and Green's function of a many-interval Sturm-Liouville problem. Zeitschrift für Naturforschung A 2015; 70: 301-308.

[3] Dieudonne J. Foundations of Modern Analysis. New York, NY, USA: Academic Press, 1969.

[4] Kong Q, Zettl A. Linear ordinary differential equations in “inequalities and applications”. In: Agarwal RP (editor). Vol. 3 - Inequalities and Applications. Singapore, World Scientific, 1994, pp. 381-397.

[5] Mukhtarov OSh, Aydemir K. New type of Sturm-Liouville problems in associated Hilbert spaces. Journal of Function Spaces 2014; 2014: 606815. doi: http://dx.doi.org/10.1155/2014/606815

[6] Mukhtarov OSh, Olgar H, Aydemir K. Resolvent operator and spectrum of new type boundary value problems. Filomat 2015; 29: 1671-1680.

[7] Mukhtarov OSh, Olgar H, Aydemir K, Jabbarov I. Operator-pencil realization of one Sturm-Liouville problem with transmission conditions. Applied and Computational Mathematics 2018; 17: 284-294.

[8] Uğurlu E. Regular third-order boundary value problems. Applied Mathematics and Computations 2019; 343: 247-257.