DE GRUYTER Demonstr. Math. 2019; 52:249–255

Research Article Open Access

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Further results on the neutrix composition of distributions involving the delta function and the function $\cosh_+^{-1}(x^{1/r}+1)$

https://doi.org/10.1515/dema-2019-0021
Received April 5, 2019; accepted June 4, 2019

Abstract: The neutrix composition $F(f(x))$ of a distribution $F(x)$ and a locally summable function $f(x)$ is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The function $\cosh_+^{-1}(x+1)$ is defined by

$$\cosh_+^{-1}(x+1) = H(x)\cosh^{-1}(|x|+1),$$

where $H(x)$ denotes Heaviside's function. It is then proved that the neutrix composition $\delta^{(s)}[\cosh_+^{-1}(x^{1/r}+1)]$ exists and

$$\delta^{(s)}[\cosh_+^{-1}(x^{1/r}+1)] = \sum_{k=0}^{s-1}\sum_{j=0}^{kr+r-1}\sum_{i=0}^{j}\frac{(-1)^{kr+r+s-j-1}r}{2^{j+2}}\binom{kr+r-1}{j}\binom{j}{i}[(j-2i+1)^s-(j-2i-1)^s]\delta^{(k)}(x),$$

for $r, s = 1, 2, \ldots$. Further results are also proved.
Our results improve, extend and generalize the main theorem of [Fisher B., Al-Sirehy F., Some results on the neutrix composition of distributions involving the delta function and the function $\cosh_+^{-1}(x+1)$, Appl. Math. Sci. (Ruse), 2014, 8(153), 7629–7640].

Keywords: distribution, Dirac-delta function, composition of distributions, neutrix, neutrix limit

MSC: Primary 46F10; Secondary 33B10

1 Introduction

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathcal{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$. We let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and let $\mathcal{D}'[a,b]$ be the space of distributions defined on $\mathcal{D}[a,b]$.

A sequence of functions $\{f_n\}$ is called regular [1] if
(i) $\{f_n\}$ is infinitely differentiable for all n;
(ii) the sequence $\{< f_n, \varphi >\}$ converges to a limit $< f, \varphi >$ for every $\varphi \in \mathcal{D}$;
(iii) $< f, \varphi >$ is continuous in φ in the sense that $\lim_{n\to\infty} < f, \varphi_n >= 0$ for every sequence $\varphi_n \to 0$ in $\mathcal{D}$.

There are several methods for consructing a sequence of regular functions which converges to $\delta(x)$. For example, let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

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(i) $\rho(x) = 0$ for $|x| \geq 1$; (ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$, (iv) $\int_{-1}^{1} \rho(x)\,dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is a distribution in $\mathcal{D}'$ and $F_n(x) = \langle F(x-t), \delta_n(x)\rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

Antosik [2] defined the composition $g(f(x))$ as the limit of the sequence $\{g_n(f_n)\}$ providing the limit exists. By this definition he defined the compositions $\sqrt{\delta} = 0$, $\sqrt{\delta^2+1} = 1+\delta$, $\sin\delta = 0$, $\cos\delta = 1$ etc. Using the definition of Antosik, it is not possible to define the compositions for many pairs of distributions. Fisher gave a general principle, by using the neutrix calculus developed by Van der Corput [3], for the discarding of unwanted infinite quantities from asymptotic expansions and this has been exploited in context of distributions, [4]. The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from divergent integral is referred to as the Hadamard finite part, see [5]. In fact his method can be regarded as a particular application of the neutrix calculus.

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta^{(r)}(f(x))$ by the equation

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{|f'(x)|}\frac{d}{dx}\right]^r \delta(x-x_0),$$

for $r = 0, 1, 2, \dots$, see [6].

In order to give a more general definition for the composition of distributions, the following definition for the neutrix composition of distributions was given in [4] and was originally called the composition of distributions.

Definition 1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \dots$ and N is the neutrix, see [3], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n: \qquad \lambda > 0,\ r = 1, 2, \dots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$.

Note that taking the neutrix limit of a function $f(n)$, is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

It was proved in [7] that if the composition $F(f(x))$ exists by Gel'fand and Shilov's definition, then it exists by Definition 1 and the two are equivalent.

The following theorems were proved in [8], [9] and [10] respectively.

Theorem 1. *The neutrix composition* $\delta^{(rs+r-1)}[\cosh_+^{-1}(x+1)]^{1/r}$ *exists and*

$$\delta^{(rs+r-1)}[\cosh_+^{-1}(x+1)]^{1/r} = \sum_{k=0}^{s-1}\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{rs+r-j-1}r}{2^{j+2}}\binom{k}{j}\binom{j}{i}\times\frac{[(j-2i+1)^s-(j-2i-1)^s](rs+r-1)!}{k!s!}\delta^{(k)}(x),$$

for $r,s=1,2,\ldots$.

In particular,

$$\delta^{(2r-1)}[\cosh_+^{-1}(x+1)]^{1/r} = \frac{(2r-1)!}{2}\delta(x),$$

for $r=1,2,\ldots$.

Theorem 2. *The neutrix composition* $\delta^{(s)}[(\sinh^{-1}x_+)^{1/r}]$ *exists and*

$$\delta^{(s)}[(\sinh^{-1}x_+)^{1/r}] = \sum_{k=0}^{M-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{i+k}rc_{s,k,i}}{2^{k+1}k!}\delta^{(k)}(x)$$

for $s=0,1,2,\ldots$ *and* $r=1,2,\ldots$, *where M is the smallest positive integer greater than* $(s-r^2+1)/r$ *and*

$$c_{s,k,i} = \begin{cases}\dfrac{[(k-2i+1)^p+(k-2i-1)^p](-1)^s s!}{2p!}, & p=\dfrac{s-r+1}{r}\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

In particular, the neutrix composition $\delta^{(s)}[(\sinh^{-1}x_+)^{1/r}]$ *exists and*

$$\delta^{(s)}[(\sinh^{-1}x_+)^{1/r}] = 0$$

for $s=0,1,2,\ldots,r-1$ *and* $r=2,3,\ldots$.

Theorem 3. *The neutrix composition* $\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)]$ *exists and*

$$\delta^{(s)}[\cosh^{-1}(x_+^{1/r}+1)] = \sum_{k=0}^{M-1}\sum_{i=0}^{kr+r}\binom{k}{i}\frac{(-1)^k rc_{r,s,k}}{(kr+r)k!}\delta^{(k)}(x)$$

for $s=M-1,M,M+1,\ldots$ *and* $r=1,2,\ldots$, *where*

$$c_{r,s,k} = \sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{kr+r+s-i}(2j-i)^{s+1}}{2^{i+1}},$$

M is the smallest integer for which $s-2r+1<2Mr$ *and* $r\leq s/(2M+2)$.

We now need the following lemma, which can be easily proved by induction:

Lemma 1.

$$\int_{-1}^{1} t^i\rho^{(s)}(t)\,dt = \begin{cases}0, & 0\leq i<s,\\ (-1)^s s!, & i=s\end{cases}$$

and

$$\int_{0}^{1} t^s\rho^{(s)}(t)\,dt = \tfrac{1}{2}(-1)^s s!$$

for $s=0,1,2,\ldots$.

In the following we define the functions $\cosh_+^{-1}(x+1)$ and $\cosh_-^{-1}(|x|+1)$ by

$$\cosh_+^{-1}(x+1) = H(x)\cosh^{-1}(|x|+1),$$
$$\cosh_-^{-1}(x+1) = H(-x)\cosh^{-1}(|x|+1).$$

It follows that

$$\cosh^{-1}(|x|+1) = \cosh_+^{-1}(x+1)+\cosh_-^{-1}(x+1).$$

2 Main results

We now prove the following improvement of Theorem 3:

Theorem 4. *The neutrix composition* $\delta^{(s)}[\cosh_+^{-1}(x^{1/r}+1)]$ *exists and*

$$\delta^{(s)}[\cosh_+^{-1}(x^{1/r}+1)] = \sum_{k=0}^{s-1}\sum_{j=0}^{kr+r-1}\sum_{i=0}^{j}\frac{(-1)^{kr+r+s-j-1}r}{2^{j+2}}\binom{kr+r-1}{j}\binom{j}{i} \times[(j-2i+1)^s-(j-2i-1)^s]\delta^{(k)}(x), \tag{2.1}$$

for $r, s = 1, 2, \ldots$.

In particular,

$$\delta[\cosh_+^{-1}(x^{1/r}+1)] = 0 \tag{2.2}$$

and

$$\delta'[\cosh_+^{-1}(x^{1/r}+1)] = \sum_{j=0}^{r-1}\sum_{i=0}^{j}\frac{(-1)^{r-j}r}{2^{j+1}}\binom{r-1}{j}\binom{j}{i}\delta(x), \tag{2.3}$$

for $r = 1, 2, \ldots$.

Proof. It is clear that $\delta^{(s)}[\cosh_+^{-1}(x^{1/r}+1)] = 0$ on any interval not containing the origin and so we only need prove equation (2.1) on the interval $[-1, 1]$. To do this, we will first of all need to evaluate

$$\begin{aligned}\int_{-1}^{1} x^k\delta_n^{(s)}[\cosh_+^{-1}(x^{1/r}+1)]\,dx &= \int_0^1 x^k\delta_n^{(s)}[\cosh^{-1}(x^{1/r}+1)]\,dx + \int_{-1}^{0} x^k\delta_n^{(s)}(0)\,dx \\ &= n^{s+1}\int_0^1 x^k\rho^{(s)}[n(\cosh^{-1}(x^{1/r}+1)]\,dx + n^{s+1}\int_{-1}^{0}x^k\rho^{(s)}(0)\,dx \\ &= I_1 + I_2.\end{aligned} \tag{2.4}$$

Making the substitution $t = n[\cosh^{-1}(x^{1/r}+1)]$ or $x = [\cosh(t/n)-1]^r$, we have

$$\begin{aligned} I_1 &= rn^s\int_0^1[\cosh(t/n)-1]^{kr+r-1}\sinh(t/n)\rho^{(s)}(t)\,dt \\ &= rn^s\sum_{j=0}^{kr+r-1}(-1)^{kr+r-j-1}\binom{kr+r-1}{j}\cosh^j(t/n)\sinh(t/n)\rho^{(s)}(t)\,dt \\ &= rn^s\sum_{j=0}^{kr+r-1}\frac{(-1)^{kr+r-j-1}}{2^{j+1}}\binom{kr+r-1}{j}\int_0^1(e^{t/n}+e^{-t/n})^j(e^{t/n}-e^{-t/n})\rho^{(s)}(t)\,dt \\ &= rn^s\sum_{j=0}^{kr+r-1}\sum_{i=0}^{j}\frac{(-1)^{kr+r-j-1}}{2^{j+1}}\binom{kr+r-1}{j}\binom{j}{i}\int_0^1(e^{(j-2i+1)t/n}-e^{(j-2i-1)t/n})\rho^{(s)}(t)\,dt\end{aligned}$$

and it follows that

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty} I_1 &= \sum_{j=0}^{kr+r-1}\sum_{i=0}^{j}\frac{(-1)^{kr+r-j-1}r}{2^{j+1}}\binom{kr+r-1}{j}\binom{j}{i}\frac{(j-2i+1)^s-(j-2i-1)^s}{s!}\int_0^1 t^s\rho^{(s)}(t)\,dt \\ &= \sum_{j=0}^{kr+r-1}\sum_{i=0}^{j}\frac{(-1)^{kr+r+s-j-1}r}{2^{j+2}}\binom{kr+r-1}{j}\binom{j}{i}[(j-2i+1)^s-(j-2i-1)^s].\end{aligned} \tag{2.5}$$

It is obvious that

$$\operatorname*{N-lim}_{n\to\infty} I_2 = 0 \tag{2.6}$$

and it now follows from equations (2.4), (2.5) and (2.6) that

$$\begin{aligned}\mathop{\mathrm{N-lim}}_{n\to\infty}\int_{-1}^{1} x^k\delta_n^{(s)}[\cosh_+^{-1}(x^{1/r}+1)]\,dx &= \\ &= \sum_{j=0}^{kr+r-1}\sum_{i=0}^{j}\frac{(-1)^{kr+r+s-j-1}r}{2^{j+2}}\binom{kr+r-1}{j}\binom{j}{i}[(j-2i+1)^s-(j-2i-1)^s] \quad (2.7)\end{aligned}$$

for $k = 0, 1, 2, \ldots$.

Next, when $k = s$, we note that

$$[\cosh(t/n)-1]^{rs+r-1}\sinh(t/n) = O(n^{-2rs-2r+1})$$

and it follows that

$$\begin{aligned}|I_1| &\le rn^s\int_0^1\big|[\cosh(t/n)-1]^{rs+r-1}\sinh(t/n)\rho^{(s)}(t)\big|\,dt \\ &= O(n^{-2rs-2r+s+1}).\end{aligned}$$

Hence, if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 x^s\delta_n^{(s)}[\cosh^{-1}(x^{1/r}+1)]\psi(x)\,dx = 0, \quad (2.8)$$

for $s = 0, 1, 2, \ldots$.

Further,

$$\int_{-1}^{0} x^s\delta_n^{(s)}(0)\psi(x)\,dx = n^{s+1}\int_{-1}^{0} x^s\rho^{(s)}(0)\psi(x)\,dx$$

and it follows that

$$\mathop{\mathrm{N-lim}}_{n\to\infty}\int_{-1}^{0} x^s\delta_n^{(s)}(0)\psi(x)\,dx = 0. \quad (2.9)$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}[-,]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{s-1}\frac{x^k}{k!}\varphi^{(k)}(0) + \frac{x^s}{s!}\varphi^{(s)}(\xi x),$$

where $0 < \xi < 1$. Then with $s \ge 1$, we have

$$\begin{aligned}\langle\delta_n^{(s)}[\cosh_+^{-1}(x^{1/r}+1)],\varphi(x)\rangle &= \int_{-1}^{1}\delta_n^{(s)}[\cosh_+^{-1}(x^{1/r}+1)]\varphi(x)\,dx \\ &= \sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}x^k\delta_n^{(s)}[\cosh_+^{-1}(x^{1/r}+1)]\,dx \\ &\quad+\int_0^1\frac{x^s}{s!}\delta_n^{(s)}[\cosh_+^{-1}(x^{1/r}+1)]\varphi^{(s)}(\xi x)\,dx+\int_{-1}^{0}\frac{x^s}{s!}\delta_n^{(rs+r-1)}(0)\varphi^{(s)}(\xi x)\,dx.\end{aligned}$$

It now follows from equations (2.7), (2.8) and (2.9) that

$$\mathop{\mathrm{N-lim}}_{n\to\infty}\langle\delta_n^{(s)}[\cosh_+^{-1}(x^{1/r}+1)],\varphi(x)\rangle =$$

$$= \sum_{k=0}^{s-1}\sum_{j=0}^{kr+r-1}\sum_{i=0}^{j} \frac{(-1)^{kr+r+s-k-j-1}r}{2^{j+2}}\binom{kr+r-1}{j}\binom{j}{i}$$
$$\times[(j-2i+1)^s-(j-2i-1)^s]\varphi^{(k)}(0)$$
$$= \sum_{k=0}^{s-1}\sum_{j=0}^{kr+r-1}\sum_{i=0}^{j} \frac{(-1)^{kr+r+s-j-1}r}{2^{j+2}}\binom{kr+r-1}{j}\binom{j}{i}[(j-2i+1)^s-(j-2i-1)^s]\langle\delta(x),\varphi(x)\rangle$$

and equation (2.1) follows.

Equations (2.2) and (2.3) follow immediately from equation (2.1).

Replacing x by $-x$ in Theorem 4, we get

Corollary 4.1 *The neutrix composition* $\delta^{(s)}[\cosh_-^{-1}(|x|^{1/r}+1)]$ *exists and*

$$\delta^{(s)}[\cosh_-^{-1}(|x|^{1/r}+1)]] = \sum_{k=0}^{s-1}\sum_{j=0}^{kr+r-1}\sum_{i=0}^{j} \frac{(-1)^{kr+k+r+s-j-1}r}{2^{j+2}}\binom{kr+r-1}{j}\binom{j}{i}$$
$$\times[(j-2i+1)^s-(j-2i-1)^s]\delta^{(k)}(x), \tag{2.10}$$

for $r, s = 1, 2, \dots$.

In particular,

$$\delta[\cosh_-^{-1}(|x|^{1/r}+1)] = 0 \tag{2.11}$$

and

$$\delta'[\cosh_-^{-1}(|x|^{1/r}+1)] = \sum_{j=0}^{r-1}\sum_{i=0}^{j}\frac{(-1)^{r-j}r}{2^{j+1}}\binom{r-1}{j}\binom{j}{i}\delta(x), \tag{2.12}$$

for $r = 1, 2, \dots$.

Corollary 4.2 *The neutrix composition* $\delta^{(s)}[\cosh^{-1}(|x|^{1/r}+1)]$ *exists and*

$$\delta^{(s)}[\cosh^{-1}(|x|^{1/r}+1)] = \sum_{k=0}^{s-1}\sum_{j=0}^{kr+r-1}\sum_{i=0}^{j} \frac{(-1)^{kr+r+s-j-1}r[1-(-1)^k]}{2^{j+2}}$$
$$\times\binom{kr+r-1}{j}\binom{j}{i}[(j-2i+1)^s-(j-2i-1)^s]\delta^{(k)}(x), \tag{2.13}$$

for $r, s = 1, 2, \dots$.

In particular,

$$\delta[\cosh^{-1}(|x|^{1/r}+1)] = 0 \tag{2.14}$$

and

$$\delta'[\cosh^{-1}(|x|^{1/r}+1)] = \sum_{j=0}^{r-1}\sum_{i=0}^{j}\frac{(-1)^{r-j}r[1-(-1)^k]}{2^{j+1}}\binom{r-1}{j}\binom{j}{i}\delta(x), \tag{2.15}$$

for $r = 1, 2, \dots$.

Proof. Equation (2.13) follows on noting that

$$\delta^{(s)}[\cosh^{-1}(|x|^{1/r}+1)] = \delta^{(s)}[\cosh_+^{-1}(x^{1/r}+1)] + \delta^{(s)}[\cosh_-^{-1}(|x|^{1/r}+1)]].$$

Equations (2.14) and (2.15) follow immediately.

For further related results, see [11], [7], [12], [13] and [14].

References

[1] Temple G., The theory of generalized functions, In: Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, 1955, 228(1173), 175–190

[2] Antosik P., Composition of distributions, Technical Report no.9, University of Wisconsin, Milwaukee, 1988-1989, 1–30

[3] Van der Corput J. G., Introduction to the neutrix calculus, J. Analyse Math., 1959, 7, 291–398

[4] Fisher B., On defining the change of variable in distributions, Rostock. Math. Kolloq., 1985, 28, 75–86

[5] Jones, D. S., Hadamard's Finite Part., Mathematical Methods in the Applied Sciences, 1996, 19(13), 1017–1052

[6] Gel'fand I. M., Shilov G. E., Generalized functions, Vol 1, Academic Press, London and New York, 1964

[7] Fisher B., The composition and neutrix composition of distributions, In: Tas K., Tenreiro Machado J. A., Baleanu D. (Eds.), Mathematical Methods in Engineering, Springer, The Netherlands, 2007, 59–69

[8] Fisher B., Al-Sirehy F., Some results on the neutrix composition of distributions involving the delta function and the function $\cosh_+^{-1}(x+1)$, Appl. Math. Sci. (Ruse), 2014, 8(153), 7629–7640

[9] Fisher B., Kılıçman A., On the composition and neutrix composition of the delta function and powers of the inverse hyperbolic sine function, Integral Transforms Spec. Funct., 2010, 21(12), 935–944

[10] Fisher B., Özçağ E., Al-Sirehy F., On the composition and neutrix composition of the delta function and the function $\cosh^{-1}(|x|^{1/r}+1)$, Internat. J. Anal. Appl., 2017, 13(2), 161–169

[11] Fisher B., The delta function and the composition of distributions, Dem. Math., 2002, 35(1), 117–123

[12] Fisher B., Kılıçman A., On the composition and neutrix composition of the delta function with the hyperbolic tangent and its inverse functions, Journal Appl. Math., 2011, Article ID 846736, http://dx.doi.org/10.1155/2011/846736

[13] Fisher B., Özçağ E., Some results on the neutrix composition of the delta function, Filomat, 2012, 26(6), 1247–1256

[14] Kraiweeradechachai T., Orankitjaroen S., Fisher B., Özçağ E., Further results on the neutrix composition of the delta function, East-West J. Math., 2009, 11(2), 151–164