# APPROXIMATE SOLUTIONS AND CONSERVATION LAWS OF THE PERIODIC BASE TEMPERATURE OF CONVECTIVE LONGITUDINAL FINS IN THERMAL CONDUCTIVITY

by

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*In this paper, the residual power series method is used to study the numerical approximations of a model of oscillating base temperature processes occurring in a convective rectangular fin with variable thermal conductivity. It is shown that the residual power series method is efficient for examining numerical behavior of non-linear models. Further, the conservation of heat is studied using the multiplier method.*

Key words: *residual power series method, numerical approximations, conservation laws*

## Introduction

It is well-known that the majority of the real-world physical phenomena are modeled by mathematical equations, especially PDE [1]. The investigations of the exact and numerical solutions of various PDE have become a very important practice by different scholars. The best test of a numerical method is whether it gives the exact solution at lower cost than its competitors. It is also worthwhile to remember that a single numerical method may not be the best for all problems [2]. So, for assessing the accuracy of a numerical method, comparison with the exact solution of the problem (which includes any errors due to model inaccuracy) is a better test than comparison with experiments. Errors are useful in statistics, computer programming, advanced mathematics and much more [3]. We observe many new progresses in this field [4-19]. The residual power series method (RPSM) is constituted with a repeated series algorithm to derive the residual power series (RPS) solutions of PDE. It has been successfully used to handle the approximate solutions of many non-linear models [4, 5]. The model that will be studied in this paper is given by [6, 7]

$$u_t = -K^2u^2 + \varepsilon u_x^2 + (1+\varepsilon u)u_{xx} \tag{1}$$

where $K$ depends on the physical properties and design parameters, and where $u(x,t)$ has the domain of definition $x \in [0,1]$, $t \in [0,1]$, and subject to a mixed set of homogeneous Neumann

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and inhomogeneous Dirichlet boundary conditions, which includes a sinusoidally varying boundary value:

$$u(1,t) = 1 + s\cos(Bt), \quad u_x(0,t) = 0 \tag{2}$$

The parameter $u$ represent the dimensionless temperature, $x$ – the distance, $t$ – the time, $\varepsilon$ – the thermal conductivity, $K$ – the fin parameter, $S$ – the amplitude of oscillation, and $B$ – the frequency of oscillation [8]. Further details in regard to the derivation and design limitations of the model can be found in [6].

This study is aimed at investigating the numerical approximations to the periodic base temperature of convective longitudinal fins in thermal conductivity using the RPSM [12].

## Numerical approximation using the RPSM

The RPSM is effective and easy to derive power series solutions of non-linear equations. The method does not require perturbation, discretization or linearization from which the numerical results can be investigated. The RPSM converge to the exact solution with only few iterations taken into consideration. To apply the RPSM [4], we consider:

$$u = \sum_{n=0}^{\infty} f_n t^n, \quad 0 \le t \le R, x \in I \tag{3}$$

Let $u_k$ to represent the $k^{\text{th}}$ series of $u$:

$$u_k = \sum_{n=0}^{k} f_n t^n, \quad 0 \le t \le R, \ x \in I \tag{4}$$

with $u_0 = f(x)$.

To derive the value of $f_n(x)$, $n = 1, 2, ..., k$ in series expansion of eq. (1), we use the residual function, $Res$, for eq. (1) as:

$$Res = u_t - \frac{1}{2} u u_{xx} - 2u^2 u_x - (u_x)^2 = 0 \tag{5}$$

and the $k^{\text{th}}$ residual series $Res_k$ is given:

$$Res_k = (u_k)_t - \frac{1}{2} u_k (u_k)_{xx} - 2(u^2)_k (u_k)_x - (u_k^2)_x = 0 \tag{6}$$

With initial condition:

$$u_0 = 1 + s\cos(Bt), \quad u_x(1,t) \tag{7}$$

where $u(1,t)$ is to be obtained from a known exact solution eq. (1).

– To find first approximation solution $u_1$, we set $k = 1$ in eq. (6):

$$Res_1 = -K^2 u_1^2 - (u_1)_t + \varepsilon (u_1^2)_x + \left(1 + \varepsilon u_1\right)(u_1)_{xx} = 0 \tag{8}$$

where

$$u_1(x,t) = 1 + s\cos(Bt) + tf_1 \tag{9}$$

From eq. (8), we conclude that $\{\partial Res_1 / \partial t\}_{t=0}$ and we get:

$$f_1 = -K^2 (1+s)^2 t \tag{10}$$

The 1st approximate RPS solution is given:

$$u_1(x,t)=1+s\cos(Bt)-K^2(1+s)^2 t \tag{11}$$

– To find first approximation solution $u_2$, we set $k=2$ in eq. (6):

$$Res_2=-K^2u_2^2-(u_2)_t+\varepsilon(u_2^2)_x+(1+\varepsilon u_2)(u_2)_{xx}=0 \tag{12}$$

where

$$u_2(x,t)=1+s\cos(Bt)-K^2(1+s)^2 t+t^2 f_2 \tag{13}$$

From eq. (12), and using the fact that $\{\partial Res_2/\partial t\}_{t=0}$, we get:

$$f_2=\frac{2K^4+B^2s+6K^4s+6K^4s^2+2K^4s^3}{2} \tag{14}$$

The 2nd approximate RPS solution solution is given:

$$u_2(x,t)=1+s\cos(Bt)-K^2(1+s)^2 t+\left(\frac{2K^4+B^2s+6K^4s+6K^4s^2+2K^4s^3}{2}\right)t^2 \tag{15}$$

– To find first approximation solution $u_3$, we set $k=3$ in eq. (6):

$$Res_3=-K^2u_3^2-(u_3)_t+\varepsilon(u_3^2)_x+(1+\varepsilon u_3)(u_3)_{xx}=0 \tag{16}$$

where

$$u_3=1+s\cos(Bt)-K^2\left[(1+s)^2 t+\left(\frac{2K^4+B^2s+6K^4s+6K^4s^2+2K^4s^3}{2}\right)t^2\right]+t^3 f_3 \tag{17}$$

From eq. (16), and using the fact that $\{\partial Res_3/\partial t\}_{t=0}$, we get:

$$f_3=\frac{-6K^6-24K^6s-36K^6s^2-24K^6s^3-6K^6s^4}{6} \tag{18}$$

The 3rd approximate RPS solution solution is given:

$$u_3=1+s\cos(Bt)-K^2(1+s)^2 t+\left(\frac{2K^4+B^2s+6K^4s+6K^4s^2+2K^4s^3}{2}\right)t^2+$$
$$+\left(\frac{-6K^6-24K^6s-36K^6s^2-24K^6s^3-6K^6s^4}{6}\right)t^3 \tag{19}$$

– To find first approximation solution $u_4$, we set $k=4$ in eq. (6):

$$Res_4=-K^2u_4^2-(u_4)_t+\varepsilon(u_4^2)_x+(1+\varepsilon u_4)(u_4)_{xx}=0 \tag{20}$$

where

$$u_4=1+s\cos\left(Bt\right)-K^2(1+s)^2 t+\frac{2K^4(1+s)^2+s\left[B^2+2K^4(1+s)^2\right]t^2}{2}+$$

$$+\left(\frac{-6K^6-24K^6s-36K^6s^2-24K^6s^3-6K^6s^4}{6}\right)t^3+f_4t^4 \tag{21}$$

From eq. (20), and using the fact that $\{\frac{\partial Res_4}{\partial t}\}_{t=0}$, we get:

$$f_4=\left(\frac{24K^8-B^4s+120K^8s+240K^8s^2+240K^8s^3+120K^8s^4+24K^8s^5}{24}\right) \tag{22}$$

The 4th approximate RPS solution solution is given:

$$u_4(x,t)=1+s\cos(Bt)-K^2(1+s)^2t+\frac{2K^4(1+s)^2+s[B^2+2K^4(1+s)^2]t^2}{2}+$$
$$+\left(\frac{2K^6(1+s)^4+2K^2(1+s)\{-B^2s+2K^4(1+s)^2+s[B^2+2K^4(1+s)^2]\}}{6}\right)t^3+$$
$$+\left(\frac{24K^8-B^4s+120K^8s+240K^8s^2+240K^8s^3+120K^8s^4+24K^8s^5}{24}\right)t^4 \tag{23}$$

## Numerical results and discussion

This section provide the solutions by numerical simulations. Table 1 showed the error observed in the numerical computations making comparison with the exact solution eq. (2) and the 4th approximate RPS solution eq. (23) at different times. It is clear that, the RPSM is accurate and provides efficient results and a rapidly convergent series. It is observed that the numerical solutions are in close agreement with the exact solutions. Figures 1-4 showed 3-D and contour surfaces of the fourth iteration $u_4(x,t)$ for the exact solutions and RPS at small time. In the numerical computation, we set the constants $s=0.1$, $K=0.006$, and $B=0.1$. And we considered the test points for $t$ $(0.01, 0.02, 0.03, ..., 0.09)$ to illustrate the convergence of the RPS solutions.

**Table 1. The absolute error**

| $t$ | $\lvert u_{\text{exact}}-u_{\text{RPSM}}\rvert$ |
|---|---|
| 0.01 | $3.856\cdot10^{-7}$ |
| 0.02 | $6.71199\cdot10^{-7}$ |
| 0.03 | $8.56799\cdot10^{-7}$ |
| 0.04 | $9.4298\cdot10^{-7}$ |
| 0.05 | $9.27998\cdot10^{-7}$ |
| 0.06 | $8.13599\cdot10^{-7}$ |
| 0.07 | $5.99202\cdot10^{-7}$ |
| 0.08 | $2.84806\cdot10^{-7}$ |
| 0.09 | $1.29587\cdot10^{-7}$ |

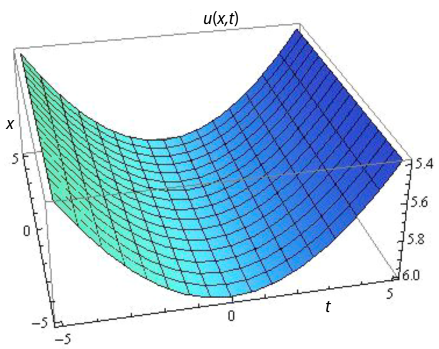


**Figure 1. The 3-D surface of the exact solution eq. (2)**

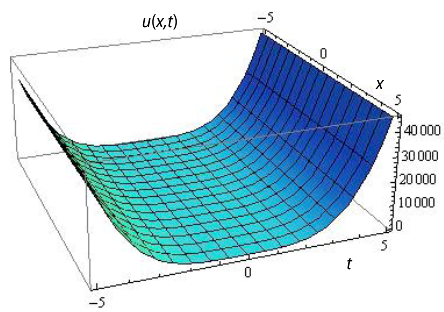


**Figure 2. The 3-D surface of the RPS eq. (23)**

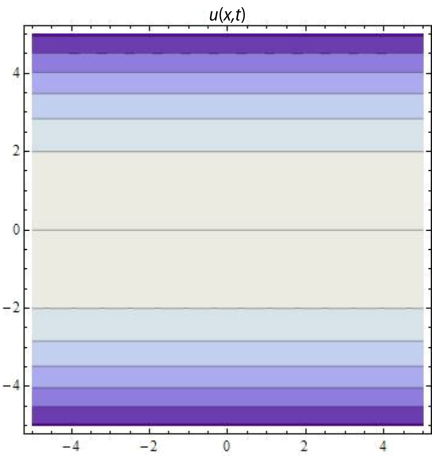


**Figure 3. Contour surface of the exact solution eq. (2)**

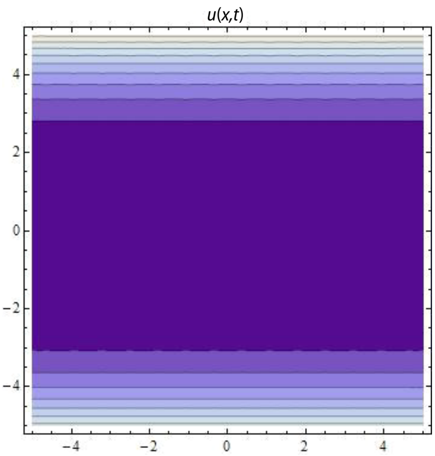


**Figure 4. Contour surface of the RPS eq. (23)**

## Conservation Laws

In this section, the class of eq. (1) will be studied using the multiplier technique [10, 11]. Let $x=(x_1,x_2,...,x_n)$ and $u=(\overline{u}^1,\overline{u}^2,...,\overline{u}^m)$ be a set of $n$ independent variables and $m$ dependent variables. Consider the following $r$ PDE of $k^{\text{th}}$-order [10]:

$$P_\alpha\left[\overline{u}\right]=P_\alpha\left(x,\overline{u},\overline{u}_{(1)},\overline{u}_{(2)},...,\overline{u}_{(k)}\right),\quad \alpha=1,2,...,r \tag{24}$$

with

$$\overline{u}_{(1)}=\left\{\overline{u}^\alpha_{(i)}\right\},\overline{u}_{(2)}=\left\{\overline{u}^\alpha_{(ij)}\right\},\left\{\overline{u}^\alpha_{(i)}\right\}=\frac{\partial\overline{u}^\alpha_i}{\partial x_i},\left\{\overline{u}^\alpha_{(ij)}\right\}=\frac{\partial^2\overline{u}^\alpha}{\partial x_i\partial x_j},...$$

Let $\overline{u}=(\overline{u}^2,\overline{u}^2,...,\overline{u}^N)$ represents functions of the independent variables $x$ and denoting partial derivatives $\partial/\partial x_i$ by subscripts $i$. [10], *i. e*, $\overline{u}^\sigma_i=\partial\overline{u}^\sigma/\partial x_i$, $\overline{u}^\sigma_{ij}=\partial^2\overline{u}^\sigma/\partial x_i x_j$, , *etc.*:

– $$D_i=\frac{\partial}{\partial x_i}+\overline{u}^\alpha_i\frac{\partial}{\partial\overline{u}^\alpha_i}+\overline{u}^\alpha_{ij}\frac{\partial}{\partial\overline{u}^\alpha_i}+\overline{u}^\alpha_{ijk}\frac{\partial}{\partial\overline{u}^\alpha_{jk}}+... \tag{25}$$

where $i,j,k,...=1,2,...,m$

– Multipliers of eq. (24) are the functions $\{\Lambda^\alpha[\overline{u}]\}$ which satisfy:

$$\Lambda^\alpha\left[\overline{u}\right]P_\alpha\left[\overline{u}\right]=D_iT^i\left[\overline{u}\right] \tag{26}$$

for some certain functions $T^i[\overline{u}]$. If $\overline{u}^\sigma=\overline{u}^\sigma(x)$ is solution of eq. (24), from eq. (26), we acquire the class [10]:

$$D_iT^i\left[\overline{u}\right]=0 \tag{27}$$

of eq. (42) and for each $i$, $T^i[\overline{u}]$ is a flux.

– The Euler operators w.r.t the differential function $U^j$ and the derivatives $\overline{u}^i_j$, $\overline{u}^j_{i_1i_2}$... are defined:

$$E_{\bar{u}}^{j}=\frac{\partial}{\partial \bar{u}^{j}}-D_{i}\frac{\partial}{\partial \bar{u}_{i}^{j}}+\ldots+(-1)^{s}D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial \bar{u}_{i_1\ldots i_s}^{j}} \tag{28}$$

for each $j=1,2,\ldots,m,\ \{\Lambda^{\alpha}[\bar{u}]\}$ gives a the multipliers of class of eq. (24) iff each operator in eq. (29) annihilates the left-hand side of eq. (26):

$$E_{\bar{u}}^{j}\left(\Lambda^{\alpha}\left[\bar{u}\right]P_{\alpha}\left[\bar{u}\right]\right)\equiv 0, \quad j=1,\ldots,n \tag{29}$$

for arbitrary $\bar{u},\bar{u}_i,\bar{u}_{ij}\ldots$ *etc.*

To construct the class of eq. (1) using the previous described technique, we apply eq. (26) to get the following determining equations:

$$\Lambda_{xx}=\frac{2\Lambda K^2}{\varepsilon}, \quad \Lambda_t=\frac{-2\Lambda K^2}{\varepsilon}, \quad \Lambda=0 \tag{30}$$

Solving eq. (30), we acquire the following multiplier $\Lambda(x,t,u)$ given:

$$\Lambda=\left\{c_1\mathrm{e}^{\sqrt{\frac{2}{\varepsilon}}Kx}+c_2\mathrm{e}^{-\sqrt{\frac{2}{\varepsilon}}Kx}\right\}\mathrm{e}^{\frac{-2K^2t}{\varepsilon}} \tag{31}$$

where $c_1$ and $c_2$ are arbitrary constants. We derive the following multipliers for four fluxes based on the constants $c_1$ and $c_2$ follows:

– If $c_1=1, c_2=0$, then we have the following multipliers:

$$\Lambda=\mathrm{e}^{-\frac{2K\sqrt{\varepsilon}t-\sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}}$$

Subsequently, we obtain the following fluxes:

$$T_1^x=-(u\varepsilon+1)u_x\mathrm{e}^{-\frac{2K\sqrt{\varepsilon}t-\sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}}$$
$$T_1^t=-\frac{1}{2K}\left(\sqrt{2}\varepsilon^{\frac{3}{2}}uu_x+Ku^2\varepsilon+\sqrt{2}\varepsilon u_x\right)\mathrm{e}^{-\frac{2K\sqrt{\varepsilon}t-\sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}} \tag{32}$$

– If $c_2=1, c_1=0$, then we have the following multipliers:

$$\Lambda=\mathrm{e}^{-\frac{2K\sqrt{\varepsilon}t+\sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}}$$

Subsequently, we obtain the following fluxes:

$$T_2^x=-(u\varepsilon+1)u_x\mathrm{e}^{-\frac{2K\sqrt{\varepsilon}t+\sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}}$$
$$T_2^t=-\frac{1}{2K}\left(-\sqrt{2}\varepsilon^{\frac{3}{2}}uu_x+Ku^2\varepsilon-\sqrt{2}\varepsilon u_x\right)\mathrm{e}^{-\frac{2K\sqrt{\varepsilon}t+\sqrt{2}\varepsilon x}{\varepsilon^{\frac{3}{2}}}} \tag{33}$$

## Concluding remarks

In this paper, we have successfully applied the RPSM to study the numerical approximations to a model of oscillating base temperature processes occurring in a convective rectangular fin with variable thermal conductivity. We showed that the RPSM is efficient for examining numerical behavior of non-linear models. Some interesting figures are shown to show the reliability of the method. We have confirmed the conservation of heat and temperature using the multiplier method of conservation laws.

## References

[1] Whitham, G. B., *Linear and Nonlinear Waves*, John Whiley, New York, USA, 1974
[2] Agrawal, G. P., *Nonlinear Fiber Optics,* 5th ed., Academic Press, New York, USA, 2013
[3] Ferziger, J. H., A Note on Numerical Accuracy, *Int. J. Num. M. in Fluids, 8* (1988), 9, pp. 995-996
[4] Inc, M., *et al.*, A New Method for Approximate Solutions of Some Nonlinear Equations: Residual Power Series Method, *Advances in Mechanical Engineering, 8* (2016), Apr., pp.1-7
[5] Maysaa. A. Q., *et al.*, Approximate Solitons of Bright and Dark Optical Solitons in Birefrigent Fibers, *Optik, 140* (2017), July, pp.45-61
[6] Yang., Y. T., *et al.*, A Double Decomposition Method for Solving the Annular Hyperbolic Profile Fins with Variable Thermal Conductivity, *Heat Transfer Eng, 31* (2010), 14, pp.1165-1172
[7] Yang. Y. T., *et al.*, A Double Decomposition Method for Solving the Periodic Base Temperature in Convective Longitudinal Fins, *Energy Conversion and Management*, *49* (2008), 10, pp. 2910-2916
[8] Duan., J. S., *et al.*, A New Modification of the Adomian Decomposition Method for Solving Boundary Value Problems for Higher Order Nonlinear Differential Equations*, Applied Mathematics and Computation, 218* (2011), 8, pp. 4090-4118
[9] Buhe, E., *et al.*, Symmetry Reductions, Exact Solutions, and Conservation Laws of the Generalized Zakharov Equations, *Journal of Mathematical Physics, 56* (2015), 10, pp. 101501
[10] Anco., S. C., *et al.*, Direct Construction Method for Conservation Laws of Partial Differential Equations, Part II: General Treatment, *European Journal of Applied Mathematics*, *13* (2002), 5, pp. 567-585
[11] Inc, M., *et al.*, Exact Solutions and Conservation Laws of the Bogoyavlenskii Equation, *Acta Physica Polonica A*, *133* (2018), 5, pp. 1133-1137
[12] Seadawy., A. R., Travelling-Wave Solutions of a Weakly Nonlinear Two-Dimensional Higher-Order Kadomtsev-Petviashvili Dynamical Equation for Dispersive Shallow-Water Waves,*, Eur. Phys. J. Plus*, *132* (2017), 1, pp. 29
[13] Seadawy., A. R, Ionic Acoustic Solitary Wave Solutions of Two-Dimensional Nonlinear Kadomtsev-Petviashvili-Burgers Equations in Quantum Plasma, *Mathematical Methods and Applied Sciences*, *40* (2017), 5, pp. 1598-1607
[14] Seadawy., A. R., Three-Dimensional Nonlinear Modified Zakharov-Kuznetsov Equation of Ion-Acoustic Waves in a Magnetized Plasma, *Comput. Math. Appl*, *71* (2016), 1, pp. 201-2012
[15] Seadawy., A. R., Approximation Solutions of Derivative Nonlinear Schrodinger Equation with Computational Applications by Variational Method*, Eur. Phys. J. Plus*, *132* (2017), Sept., 182
[16] Aliyu., A. I., *et al.*, Symmetry Analysis, Explicit Solutions, and Conservation Laws of a Sixth-Order Nonlinear Ramani Equation, *Symmetry*, *10* (2018), 8, pp. 341
[17] Chunyu., Y., *et al.*, Amplification, Reshaping, Fission and Annihilation of Optical Solitons in Dispersion-Decreasing Fiber, *Nonlinear Dynamics*, *92* (2018), 2, pp. 203-213
[18] Wenyi., L., *et al.,* Soliton Structures in the (1+1)-Dimensional Ginzburg-Landau Equation, with a Parity-Time-Symmetric Potential in Ultrafast Optics, *Chinese Physical B, 27* (2018), 3, pp. 030504
[19] Weitian., Y *et al.,* Interactions of Solitons, Dromion-Like Structures and Butterfly-Shaped Pulses for Variable Coefficient Nonlinear Schroedinger Equation, *Optik, 159* (2018), Apr., pp. 21-30

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