Kucukarslan Yuzbasi, Z., *et al.*: On Exact Solutions for New Coupled Non-Linear Models ...
THERMAL SCIENCE: Year 2019, Vol. 23, Suppl. 1, pp. S227-S233

S227

ON EXACT SOLUTIONS FOR NEW COUPLED NON-LINEAR MODELS GETTING EVOLUTION OF CURVES IN GALILEAN SPACE

by

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In this work, the new coupled non-linear partial differential equations (CNLPDE) getting the time evolution of the curvatures of the evolving curve are derived in the Galilean space. Exact solutions for these new CNLPDE are obtained. Finally, Lie symmetry analysis is performed on these new CNLPDE and the algebra of Lie point symmetries of these new equations is found.

Key words: *Lie symmetry analysis, CNLPDE, soliton solutions, Bell-Shaped and Kink solitary wave, curve evolution, Galilean space*

Introduction

The issue of how to evolve spatial curves in time is of deep interest and has been examined in different planes and spaces by many researchers. The precursor in these types work is Hasimoto's paper which was first introduced the non-linear Schroedinger equation determining the motion of an isolated non-stretching thin vortex filament [1]. Lamb extended Hasimoto's work to get connect other motions of curves to the modified KdV and sine-Gordon equations [2]. Recently, Abdel-All *et al.* [3, 4] examined the evolution of curves using the velocities of the moving frame and investigated the evolution of plane curves. Also, there are a lot of studies about the motion of curves in [5-7].

In this study, we get CNLPDE getting the time evolution of the curvatures of the curve in Galilean space. We derive the new geometric types of the evolution equations for curvatures from the main CNLPDE in G_3. We get the exact solutions for these new equations and derive two types of non-linear traveling solitary wave, which are well known, bell-shaped and kink solitary waves from these solutions [8-11]. The bell-shaped solitary wave appears in consequence of the balance between non-linearity and dispersion. The balance between non-linearity and dissipation supports known the non-linear wave of stable shape as kink shaped wave.

Moreover, we study this new equation with the aid of Lie symmetry analysis method. This method is one of the most important and efficient techniques to find the exact solutions [12-19].

Time-evolution equations in G_3

A space curve in 3-D Galilean space G_3 is defined in parametric form by $r = r(s)$. Here r be an admissible curve of the class C^∞ in G_3. Then Frenet formulas are given:

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$$\begin{cases} T_s = \kappa N \\ N_s = \tau B \\ B_s = -\tau N \end{cases} \tag{1}$$

where $T = r_s$, N and B are called the vectors of the unit tangent, principal normal and binormal of $r(s)$, respectively. Also κ and τ are geometric parameters that represent, respectively, the curvature and torsion of $r(s)$, [20]. Through this paper, the subscripts describe partial derivatives.

We know that an admissible curve is uniquely detected by two scalar quantities, is said to be the curvature and torsion, as functions. Then we can write the following general theorem:

Theorem 1. (Fundamental existence and uniqueness theorem for space curves). Let $\kappa(s)$ and $\tau(s)$ be smooth functions on $c_1 \leq s \leq c_2$. Then there exists one and only one smooth curve r which is parametrized by arclength has the curvature $\kappa(s)$ and the torsion $\tau(s)$, [21].

If $r(s)$ moves with time t, then eq. (1) are becoming functions of both s and t. We can write the evolution equations of $\{T, N, B\}$ quite generally, in a form similar to eq. (1) as following form [22]:

$$\begin{cases} T_t = \alpha N + \beta B \\ N_t = \gamma B \\ B_t = -\gamma N \end{cases} \tag{2}$$

Clearly α, β, and γ (which are the velocities of the moving frame) detect the motion of the curve. For non-stretching (inextensible) curves, the moving frame must be satisfied the compatibility conditions:

$$T_{st} = T_{ts}, \quad N_{st} = N_{ts} \quad \text{and} \quad B_{st} = B_{ts} \tag{3}$$

Here non-stretching (inextensible) curves imply that the flow described by eq. (2) preserves the curves in arc-length parametrization [23].

If we substitute eqs. (1) and (2) into eq. (3), then we get:

$$\kappa_t N + \kappa \gamma B = \alpha_s N + \alpha \tau B + \beta_s B - \tau \beta N$$

$$\tau_t B - \tau \gamma N = \gamma_s B - \tau \gamma N$$

$$-\tau_t N - \tau \gamma B = -\gamma_s N - \tau \gamma B$$

From the previous equations, we obtain:

$$\kappa_t = \alpha_s - \tau \beta, \quad \tau_t = \gamma_s, \quad \gamma = \frac{\tau \alpha + \beta_s}{\kappa} \tag{4}$$

The evolution equations for κ and τ of r can be obtained with respect to $\{\alpha, \beta, \gamma\}$, which can be given as CNLPDE such that:

$$\kappa_t = \alpha_s - \tau \beta, \quad \tau_t = \left(\frac{\tau \alpha + \beta_s}{\kappa} \right)_s \tag{5}$$

The eq. (5) is the main result of this paper. We determine the equations of motion of the curve for a given $\{\alpha, \beta, \gamma\}$. Then, we choose $\{\alpha, \beta, \gamma\}$ in terms of the $\{\kappa, \tau\}$. Moreover, from the previous equations, we take into consideration that γ does not impact the final form of the evolving curve.

Kucukarslan Yuzbasi, Z., *et al.*: On Exact Solutions for New Coupled Non-Linear Models ...
THERMAL SCIENCE: Year 2019, Vol. 23, Suppl. 1, pp. S227-S233

S229

New CNLPDE

Applications for CNLPDE

In this section, our starting point is to give some applications of the new geometric models of the evolution equations for curvatures from eq. (1) in G_3. The set of $\{\kappa, \tau, \alpha, \beta, \gamma\}$ becoming clear in eqs. (1) and (2) fundamentally determines a moving curve.

Type 1. The evolution equations for the curvatures of r in terms of the velocities $\{\alpha, \beta, \gamma\} = \{0, \kappa, (\kappa_s / \kappa)\}$ obtained using eq. (5) as:

$$\kappa_t = -\tau\kappa, \qquad \tau_t = \frac{\kappa\kappa_{ss} - \kappa_s^2}{\kappa^2} \tag{6}$$

So, we can have the general solutions of this system:

$$\kappa = A_1 \mathrm{sech}\left[\lambda(s + vt)\right], \qquad \tau = A_2 \tanh\left[\lambda(s + vt)\right] \tag{7}$$

where A_1, A_2, and λ are arbitrary real constants and v is the velocity of the solitary wave. Also, we see that v is equal to A_2 / λ. In this equation, the bell shaped and kink solitary wave obtained, respectively. Under these conditions, the general solutions are:

$$\{\kappa, \tau, \alpha, \beta, \gamma\} = \{A_1 \mathrm{sech}(\lambda s + A_2 t), A_2 \tanh(\lambda s + A_2 t), 0,$$

$$A_1 \mathrm{sech}(\lambda s + A_2 t), -A_1 \lambda \tanh(\lambda s + A_2 t)\} \tag{8}$$

If we put $A_1 = 5, A_2 = 1$, and $\lambda = 0.5$ in eq. (7), $\kappa(s,t) = 5\mathrm{sech}(0.5s + t)$ and $\tau(s,t) = \tanh(0.2s + t)$. Under these conditions, we see that $\kappa \to 0$ and $\tau \to 0$ as $s \to \pm\infty$, in fig. 1.
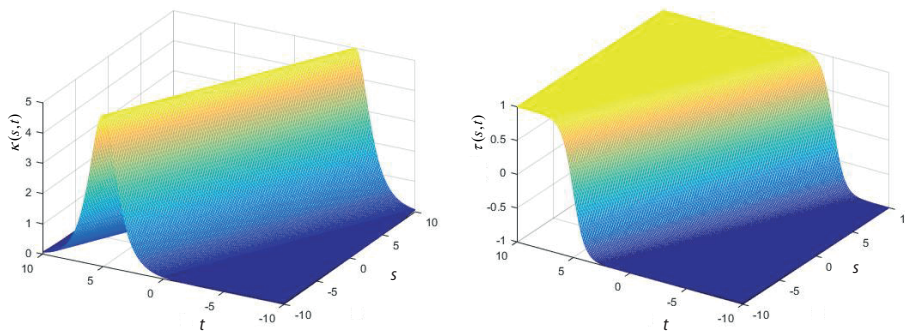
Figure 1. The bell-shaped and kink solitary wave solutions for eq. (7)
(*for color image see journal web site*)

Type 2. The evolution equations for the curvatures of the curve in terms of $\{\alpha, \beta, \gamma\} = \{\kappa_{ss}, 0, \kappa_{ss}\tau / \kappa\}$ obtained using eq. (5):

$$\kappa_t = \kappa_{sss}, \qquad \tau_t = \frac{(\tau\kappa_{sss} + \tau_s\kappa_{ss})\kappa - \tau\kappa_s\kappa_{ss}}{\kappa^2} \tag{9}$$
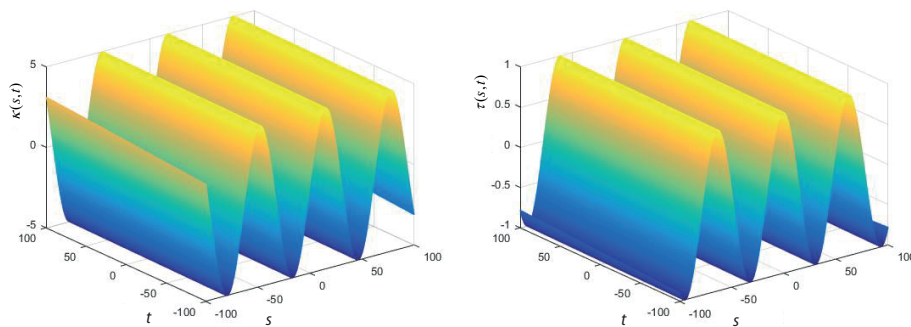
We can get the general solutions of this system:

$$\kappa = B_1 \sin\left[\delta(s + \varpi t)\right], \qquad \tau = B_2 \cos\left[\delta(s + \varpi t)\right] \tag{10}$$

where B_1, B_2, and δ are the arbitrary real constants and ϖ is equal to $-\delta^2$. Under these conditions, the general solutions are:

S230

Kucukarslan Yuzbasi, Z., *et al.*: On Exact Solutions for New Coupled Non-Linear Models ...
THERMAL SCIENCE: Year 2019, Vol. 23, Suppl. 1, pp. S227-S233

$$\{\kappa, \tau, \alpha, \beta, \gamma\} = \left\{ B_1 \sin\left[\delta\left(s - \delta^2 t\right)\right], B_2 \cos\left[\delta\left(s - \delta^2 t\right)\right], \right.$$

$$\left. -B_1 \delta^2 \sin\left[\delta\left(s - \delta^2 t\right)\right], 0, -B_2 \delta^2 \cos\left[\delta\left(s - \delta^2 t\right)\right]\right\} \tag{11}$$

The $\kappa(s,t) = 5\sin[0.1(s - 0.01t)]$ and $\tau(s,t) = 10\cos[0.1(s - 0.01t)]$ trigonometric function solutions are obtained for $B_1 = 5, B_2 = 10$, and $\delta = 0.1$. In that cases, we see that $\kappa \to 0$ and $\tau \to 0$ as $s \to \pm\infty$, in fig. 2.

Figure 2. The trigonometric solutions for eq. (10)
(*for color image see journal web site*)

Type 3. The evolution equations for the curvatures of the curve in terms of $\{\alpha, \beta, \gamma\} = \{\tau, \kappa, (\kappa_s + \tau^2)/\kappa\}$ obtained using eq. (5):

$$\kappa_t = \tau_s - \kappa\tau, \quad \tau_t = \frac{(\kappa_{ss} + 2\tau\tau_s)\kappa - (\kappa_s + \tau^2)\kappa_s}{\kappa^2} \tag{12}$$

We can write the general solutions of this system:

$$\kappa = C_1 \text{sech}\left[\mu(s + vt)\right] + C_2 \tanh\left[\mu(s + vt)\right], \quad \tau = C_3 \text{sech}\left[\mu(s + vt)\right] \tag{13}$$
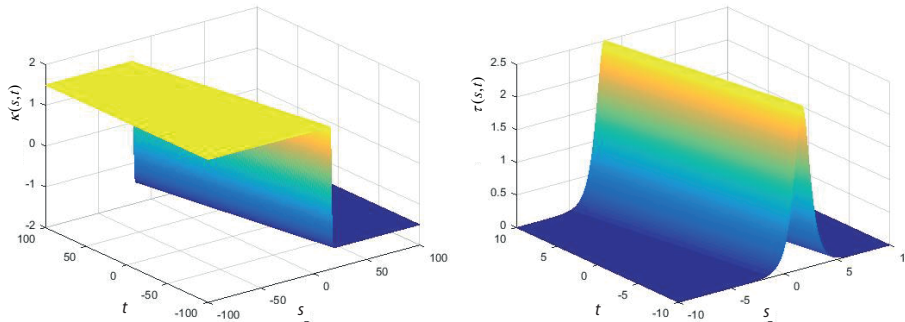
where C_1, C_2, C_3 and μ are arbitrary real constants. In this, μ and v are $C_1^2 - C_2$ and $[(C_2 + \mu)C_3]/C_1\mu$, respectively. So, the general solutions are:

$$\{\kappa, \tau, \alpha, \beta, \gamma\} = \left\{ C_1 \text{sech}\left[\mu(s + vt)\right] + C_2 \tanh\left[\mu(s + vt)\right], C_3 \text{sech}\left[\mu(s + vt)\right], \right.$$

$$C_3 \text{sech}\left[\mu(s + vt)\right], C_1 \text{sech}\left[\mu(s + vt)\right] + C_2 \tanh\left[\mu(s + vt)\right],$$

$$\left. \frac{\text{sech}\left[\mu(s + vt)\right]\left(C_3^2 + C_2\mu - C_1\mu\sinh\left[\mu(s + vt)\right]\right)}{C_1 + C_2\sinh\left[\mu(s + vt)\right]} \right\} \tag{14}$$

If we choose $C_1 = 0.1, C_2 = -1.5$ and $C_3 = 2.5$ in eq. (13),

$$\kappa(s,t) = 0.1\text{sech}[1.51(s + 0.165t)] - 1.5\tanh[1.51(s + 0.165t)] \quad \text{and}$$

$$\tau(s,t) = 2.5\text{sech}[1.51(s + 0.165t)]$$

We see that $\kappa \to \pm 0.1$ and $\tau \to 0$ as $s \to \pm\infty$ in fig. 3.

Kucukarslan Yuzbasi, Z., *et al.*: On Exact Solutions for New Coupled Non-Linear Models ...
THERMAL SCIENCE: Year 2019, Vol. 23, Suppl. 1, pp. S227-S233

S231

Figure 3. The solutions to eq. (13)
(*for color image see journal web site*)

Lie symmetry analysis of CNLPDE

In this section, our purpose is to give Lie symmetry analysis for CNLPDE. Many researchers explain how to derive Lie symmetry analysis in many books, [24, 25].

We will now consider the one parameter group of point transformations of the form

$$\tilde{s} \rightarrow \quad s + \varepsilon\xi(s,t,\kappa,\tau)$$
$$\tilde{t} \rightarrow \quad t + \varepsilon\eta(s,t,\kappa,\tau)$$
$$\tilde{\kappa} \rightarrow \quad \kappa + \varepsilon\zeta_1(s,t,\kappa,\tau)$$
$$\tilde{\tau} \rightarrow \quad \tau + \varepsilon\zeta_2(s,t,\kappa,\tau)$$

where ε is a group parameter. The vector field related to the previous transformations can be given:

$$V = \xi(s,t,\kappa,\tau)\frac{\partial}{\partial s} + \eta(s,t,\kappa,\tau)\frac{\partial}{\partial t} + \zeta_1(s,t,\kappa,\tau)\frac{\partial}{\partial \kappa} + \zeta_2(s,t,\kappa,\tau)\frac{\partial}{\partial \tau} \qquad (15)$$

The Lie point symmetries of the equations are generated by a vector field as the previous form. Through this section $C_1, C_2, C_3,$ and C_4 are arbitrary constants.

Type 1. We can show it considering a well established procedure that eq. (6) admits the following infinitesimals:

$$\xi(s,t,\kappa,\tau) = 2C_2 s + C_4$$
$$\eta(s,t,\kappa,\tau) = 2C_2 t + C_3$$
$$\zeta_1(s,t,\kappa,\tau) = \kappa\left[C_1\ln\kappa + C_2\ln\kappa + F(s,t)\right]$$
$$\zeta_2(s,t,\kappa,\tau) = C_1\tau - C_2\tau - F_t(s,t)$$

where $C_1, C_2, C_3,$ and C_4 are arbitrary constants and $F(s,t)$ is an arbitrary function such that $F_{ss} = -F_{tt}$. The algebra of Lie point symmetries of eq. (6) is generated by five vector fields:

$$V_1 = \kappa\ln\kappa\frac{\partial}{\partial\kappa} + \tau\frac{\partial}{\partial\tau}, \quad V_2 = \kappa\ln\kappa\frac{\partial}{\partial\kappa} + 2t\frac{\partial}{\partial t} - \tau\frac{\partial}{\partial\tau} + 2s\frac{\partial}{\partial s}$$

$$V_3 = \frac{\partial}{\partial t}, \quad V_4 = \frac{\partial}{\partial s}, \quad V_5 = \kappa F(s,t)\frac{\partial}{\partial\kappa} - F_t(s,t)\frac{\partial}{\partial\tau}$$

Type 2. We can present it considering a well established procedure that eq. (9) admits the following infinitesimals:

$$\xi(s,t,\kappa,\tau) = C_2 s + C_4, \qquad \eta(s,t,\kappa,\tau) = 3C_2 t + C_3$$
$$\zeta_1(s,t,\kappa,\tau) = \kappa C_1 \qquad \zeta_2(s,t,\kappa,\tau) = G(\kappa,\tau)$$

where $C_i (i = 1,2,3,4)$ are arbitrary constants and $G(\kappa,\tau)$ satisfies the equation:

$$G = \kappa G_\kappa + \tau G_\tau$$

The algebra of Lie point symmetries of eq. (9) is constituted by five vector fields:

$$V_1 = \kappa \frac{\partial}{\partial \kappa}, \quad V_2 = 3t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s}, \quad V_3 = \frac{\partial}{\partial t}$$
$$V_4 = \frac{\partial}{\partial s}, \quad V_5 = G(\kappa,\tau) \frac{\partial}{\partial \tau}$$

Type 3. We can show it considering a well established procedure that eq. (12) admits the following infinitesimals:

$$\xi(s,t,\kappa,\tau) = -C_1 s + C_3, \quad \eta(s,t,\kappa,\tau) = -C_1 t + C_2$$
$$\zeta_1(s,t,\kappa,\tau) = \kappa C_1, \quad \zeta_2(s,t,\kappa,\tau) = C_1 \tau$$

where $C_i (i = 1,2,3)$ are arbitrary constants. The algebra of Lie point symmetries of eq. (12) is generated by three vector fields:

$$V_1 = \kappa \frac{\partial}{\partial \kappa} - t \frac{\partial}{\partial t} - s \frac{\partial}{\partial s} + \tau \frac{\partial}{\partial \tau}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial s}$$

Conclusion

In this work, we have derived the geometrical types for the motion of spatial curves in Galilean space. For each type, we get the evolution equations in terms of the curvatures. Solutions of these equations lead to the exact solutions as two types of non-linear traveling solitary wave, which are well known, bell-shaped and kink solitary waves traveling wave, the trigonometric function solutions, and other similarity solutions. Moreover, we illustrated all the solutions by using MATLAB in figs. 1-3. Finally, we applied Lie symmetry analysis to every geometric model.

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Kucukarslan Yuzbasi, Z., *et al*.: On Exact Solutions for New Coupled Non-Linear Models ...
THERMAL SCIENCE: Year 2019, Vol. 23, Suppl. 1, pp. S227-S233

S233

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