mathematics

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Analysis of Homotopy Perturbation Method for Solving Fractional Order Differential Equations

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Received: 17 September 2018; Accepted: 4 December 2018; Published: 3 January 2019

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Abstract: The analysis of Homotopy Perturbation Method (HPM) for the solution of fractional partial differential equations (FPDEs) is presented. A unified convergence theorem is given. In order to validate the theory, the solution of fractional-order Burger-Poisson (FBP) equation is obtained. Furthermore, this work presents the method to find the solution of FPDEs, while the same partial differential equation (PDE) with ordinary derivative i.e., for $\alpha = 1$, is not defined in the given domain. Moreover, HPM is applied to a complicated obstacle boundary value problem (BVP) of fractional order.

Keywords: Burger-Poisson equation of fractional order; HPM; fractional derivatives

1. Introduction

Fractional models have an important role in many fields of engineering and science, for instance, fluid flows, solute transport, electromagnetic theory, signal processing, biology, economics, physics, and geology, etc. [1–6]. Fractional theory has many applications in wireless networks [7,8]. Moreover, fractional modeling has been applied in micro-grids [9], and decentralized wireless networks [10].

Fractional differential equations (FDEs) involve real or complex order derivatives [11]. Various researchers contributed on fractional derivatives during the 18th and 19th centuries, for example, Abel [12], Caputo [13], Euler [14], Fourier [15], Laplace [16], Liouville [17], or Ross [18]. In 1974, Oldham and Spanier presented fractional operators with mass and heat transfer applications [19]. In this paper, the analytical technique to solve fractional order partial differential equations (PDEs) is described. In order to obtain analytical solution, HPM is used to solve fractional order PDEs. Fractional order PDEs do not have closed form exact solutions in most problems, therefore, it is required to develop efficient and accurate analytical and numerical methods. HPM is well known for its accuracy and simplicity [20,21]. HPM has been widely used to obtain approximate series solutions of fractional order linear and nonlinear PDEs [22–27].

The Burger Poisson equation is widely used to express different physical phenomena, for instance, mathematical models for shallow water and shock waves in a viscous fluid [28]. Tian and Gao proved the existence of the uni-dimensional viscous equation in 2009 [29]. Moreover, Abidi and Omrani obtained the solution of Burger Poisson equation using homotopy analysis method [30].

The obstacle problem plays a role of bridge in the field of variational inequalities and differential equations. It is originated from the study of elasticity theory. In elasticity theory, it is required to obtain the equilibrium position of elastic membrane with fixed boundaries. Obstacle problems occur in diffusion equation and signals processing while determining heat flux at the boundary of semi-infinite rod [31].

The focus of the paper is to generalize the convergence theorem, in the sense that the mapping is nonself and only Y is complete. Theorem is applied to the solution of FBP equation acquired by HPM. Furthermore, this work presents the method to get the solution of FPDEs, while the same PDE with ordinary derivative i.e., for $\alpha = 1$ is not defined in the given domain. Moreover, the proposed HPM method is applied to complex obstacle BVP.

Beside the introduction, the distribution of the article is as under: Section 2 comprises of important definitions and properties, Section 3 contains the implementation of HPM to solve FPDEs and convergence theorem, Section 4 comprises of results and discussion of FBP, Section 5 contains solutions of obstacle BVP, and Section 6 includes the conclusion of the paper.

2. Preliminaries

Fractional calculus is a developing area in mathematical analysis. Several definitions of fractional operators have been propounded like Riemann-Liouvlle, Caputo, and Grunwald-Letnikov [13,17]. These definitions have some limitations, for instance $D_a^\alpha(1) \neq 0$, do not satisfy the product, quotient and chain rules of derivatives. Recently, Khalil et al. published a definition uses for fractional derivatives called conformable [32,33]. Conformable is simpler and natural extension of the usual derivatives as it satisfies the aforementioned properties of derivatives.

Definition 1. *A function $g : [0, \infty) \to \mathbb{R}$. The fractional derivative of g for order α is given below:*

$$D_\alpha(g)(t) = \lim_{\epsilon \to 0} \frac{g(t + \epsilon t^{1-\alpha}) - g(t)}{\epsilon}, \text{ for all } t > 0, \ \alpha \in (0, 1). \tag{1}$$

If g is α-differentiable in $(0, b)$, $b > 0$. If $lim_{t \to 0^+}$ exists, then $g^\alpha(0) = lim_{t \to 0^+} g^\alpha(t)$. The definition given in Equation (1) is known as conformable [33].

Using the above mentioned definition given in Equation (1), we obtain the following useful results: Let g, f are α-differentiable and $\alpha \in (0, 1]$, then

1. If a function $g : [0, \infty] \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$, then g is continuous at t_0.
2. $D_\alpha(cf + dg) = cD_\alpha(f) + dD_\alpha(g)$, for all $c, d \in \mathbb{R}$.
3. $D_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.
4. $D_\alpha(\xi) = 0$, for all constant function $f(t) = \xi$.
5. $D_\alpha(gf) = gD_\alpha(f) + fD_\alpha(g)$.
6. $D_\alpha(\frac{f}{g}) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$.
7. If g is differentiable, then $D_\alpha(g)(t) = t^{1-\alpha} \frac{\mathrm{d}g(t)}{\mathrm{d}t}$.

Fractional integral: The fractional integral of order α can be defined below:

$$J_\alpha^a(g)(t) = \int_a^t \frac{g(x)}{x^{1-\alpha}} dx, \text{for all } t > 0, \ \alpha \in (0, 1). \tag{2}$$

Here, J_α is the Riemann improper integral.

3. Application of HPM to FBP Equation

Consider the time-dependent operator equation

$$B\left(w\left(s,t\right)\right)-h\left(s,t\right)=0,$$

where B denotes a differential operator and $w\left(s,t\right)$ is an unknown function. Moreover, we assume $h(s,t)$ is an analytic function, we can decompose the operator B as;

$$B=L+N$$

where L is linear and N is nonlinear or sometimes the complicated part to handle. Let $\Omega\subseteq\mathbb{R}\times\mathbb{R}$ the homotopy $H:\Omega\times[0,1]\rightarrow\mathbb{R}$, is defined as:

$$H\left(w,p\right)=\left(1-p\right)\left[L\left(w\right)-L\left(w_0\right)\right]+p\left[B\left(w\right)-g\right]=0,\ p\in\left[0,1\right],\ w\in\Omega.$$

Note that the function $w_0\left(s,t\right)$ is the initial guess which satisfies the given operator equation. The choice of p from zero to one provide us the deformation from $w_0\left(s,t\right)$ to the solution $w\left(s,t\right)$. Clearly

$$H\left(w,0\right)=0\text{ implies }L\left(w\right)-L\left(w_0\right)=0$$

and

$$H\left(w,1\right)=0\text{ implies }B\left(w\right)-g=0$$

which is the given operator equation. Using perturbation method, we suppose the solution in power series as;

$$\begin{aligned}w\left(s,t\right)&=w_0\left(s,t\right)+p\cdot w_1\left(s,t\right)+p^2\cdot w_2\left(s,t\right)+\ldots\\&=\sum_{n=0}^{\infty}p^n\cdot w_n\left(s,t\right).\end{aligned}$$

For $p=1$,

$$w\left(s,t\right)=w_0\left(s,t\right)+w_1\left(s,t\right)+w_2\left(s,t\right)+\ldots$$

is an approximate solution to the given operator equation.

Remark 1. *The Banach contraction type theorem about the convergence of the solution stated in [34]. Here the generalized, corrected, and unified form is presented in which the completeness of X is not required and Y must be a subset of X. If $Y\not\subseteq X$, we cannot say about the sequence $x_{n+1}=N^n\left(x_0\right)=N\left(x_n\right)$ is contained in Y or not. On the basis of above discussion, a unified theorem is presented as follows:*

Theorem 1. *Let X be a normed space and $Y\subseteq X$ be a Banach space, $N:X\rightarrow Y$ be a mapping such that for all $x,y\in X$*

$$\left\|N\left(x\right)-N\left(y\right)\right\|_Y\le k\left\|x-y\right\|_X \tag{C}$$

for some $k\in[0,1)$, then the sequence

$$x_{n+1}=N^n\left(x_0\right)=N\left(x_n\right)$$

for any $x_0\in X$ converges to a unique fixed point of N.

Proof. We consider the picard sequence $x_{n+1}=N\left(x_n\right)\subseteq Y$, it will be shown that this sequence $\left(x_n\right)$ is Cauchy in Y. For integers $m\ge n$ consider,

$$\left\|x_n-x_m\right\|\le\left\|x_n-x_{n+1}\right\|+\left\|x_{n+1}-x_{n+2}\right\|+\left\|x_{n+2}-x_{n+3}\right\|+\ldots+\left\|x_{m-1}-x_m\right\|.$$

Using the contractive condition (C) , and induction on n, is given as

$$\|x_n - x_{n+1}\| \leq k^n \|x_0 - x_1\|,$$

this implies,

$$\lim_{m\to\infty} \|x_n - x_m\| \leq \frac{k^n}{1+k} \|x_0 - x_1\| \to 0, \text{ as } n \to \infty.$$

This shows that (x_n) is Cauchy sequence in Y, completeness of Y allows us to find $z \in Y$, such that

$$\lim_{n\to\infty} (x_n) = z \in Y.$$

Clearly, (C) ensures the continuity of N, thus

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} N(x_n) = N\left(\lim_{n\to\infty} x_n\right) = N(z).$$

This completes the proof, the uniqueness of z is obvious. The proof of *Theorem* 1 is similar to the proof given in [34,35], but our case is generalized, in the sense that the mapping is nonself and only Y is complete. Now, the extended HPM using conformable is presented to solve space-time FBP equation. □

4. Test Problem 1

The partial differential FBP equation in unidirectional propagation water waves can be described as follows [36,37]:

$$\begin{aligned} &D_t^\alpha w - D_t^\alpha w_{xx} + w_x + ww_x - (3w_x w_{xx} + ww_{xxx}) = 0, \quad \alpha \in (0,1], \\ &\text{with } \ w(x,0) = -x. \end{aligned} \tag{3}$$

In order to apply HPM, the constructed homotopy is given below:

$$(1-p)D_t^\alpha w + p(D_t^\alpha w - D_t^\alpha w_{xx} + w_x + ww_{xx} - 3(w_x w_{xx} + ww_{xxx})) = 0,$$

or

$$D_t^\alpha w + p(-D_t^\alpha w_{xx} + w_x + ww_x - (3w_x w_{xx} + ww_{xxx})) = 0. \tag{4}$$

Here p is a parameter that lies between 0 and 1. The solution $w(x,t)$ is given as follows:

$$w(x,t) = w_0(x,t) + pw_1(x,t) + p^2 w_2(x,t) + p^3 w_3(x,t) + ... \tag{5}$$

Setting $p = 1$ gives,

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + ...$$

Now, substitute Equation (5) into Equation (3), and collect the similar powers of p, gives

$$\begin{aligned} &p^0 : D_t^\alpha w_0 = 0, w_0(x,0) = -x; \\ &p^1 : D_t^\alpha w_1 - D_t^\alpha w_{0xx} + w_{0x} + w_0 w_{0x} - 3w_{0x} w_{0xx} - w_0 w_{0xxx} = 0, w_1(x,0) = 0; \\ &p^2 : D_t^\alpha w_2 - D_t^\alpha w_{1xx} + w_{1x} + w_0 w_{1x} + w_1 w_{0x} - 3(w_{0x} w_{1xx} + w_{1x} w_{0xx}) \\ &- (w_0 w_{1xxx} + w_1 w_{0xxx}) = 0, w_2(x,0) = 0; \\ &p^3 : D_t^\alpha w_3 - D_t^\alpha w_{2xx} + w_{2x} + (w_0 w_{2x} + w_1 w_{1x} + w_2 w_{0x}) - 3(w_{0x} w_{2xx} + w_{1x} w_{1xx} + w_{2x} w_{0xx}) \\ &- (w_0 w_{2xxx} + w_1 w_{1xxx} + w_2 w_{0xxx}) = 0, w_3(x,0) = 0. \end{aligned} \tag{6}$$

Afterwards, the fractional integral operator J_α with conformable derivative definition (c.f. Equation (2)) is applied on both sides of Equation (6), we have

$$\begin{aligned} w_0(x,t) &= -x; \\ w_1(x,t) &= \frac{t^\alpha}{\alpha}(1-x); \\ w_2(x,t) &= \frac{t^{2\alpha}}{\alpha}(1-x); \\ w_3(x,t) &= \frac{t^{3\alpha}}{\alpha}(1-x) \\ &\vdots \end{aligned} \tag{7}$$

In order to calculate next terms, we have

$$w_l = w_{(l-1)xx} + \int_a^t \frac{1}{t^{1-\alpha}} \left(3\sum_{i=0}^{l-1} w_{ix} w_{(l-i-1)xx} + \sum_{i=0}^{l-1} w_i w_{(l-i-1)xxx} - w_{(l-1)x} - \sum_{i=0}^{l-1} w_i w_{(l-i-1)x} \right) dt \tag{8}$$

$$w_l(x,0) = 0, \text{ for } l \geq 4.$$

If we define $N : C(\Omega) \to C(\Omega)$, with iterative sequence as,

$$N(w_l) = w_{(l-1)xx} + \int_a^t \frac{1}{t^{1-\alpha}} \left(3\sum_{i=0}^{l-1} w_{ix} w_{(l-i-1)xx} + \sum_{i=0}^{l-1} w_i w_{(l-i-1)xxx} - w_{(l-1)x} - \sum_{i=0}^{l-1} w_i w_{(l-i-1)x} \right) dt$$

Then for any $w_0 \in C(\Omega)$, by Theorem 1, the sequence $w_{l+1} = N(w_l)$ converges to the unique solution w of the given FBP.

The Equation (8) can be calculated with the help of symbolic softwares, for instance, Mathematica and Maple. The HPM solution is given below:

$$w(x,t) = -x + \frac{t^\alpha}{\alpha}(1-x) + \frac{t^{2\alpha}}{\alpha^2}(1-x) + \frac{t^{3\alpha}}{\alpha^3}(1-x) + \frac{t^{4\alpha}}{\alpha^4}(1-x) + \ldots \tag{9}$$

$$= \frac{-x\alpha + t^\alpha}{\alpha - t^\alpha} \tag{10}$$

The HPM solution of FBP equation when $\alpha = 1$ is as follows:

$$w(x,t) = -x + (1-x)t + (1-x)t^2 - (1-x)t^3 + \ldots \tag{11}$$

$$= \frac{-x+t}{1-t}. \tag{12}$$

Remark 2. *It is remarked that the exact and HPM solution of FBP equation when $\alpha = 1$, given in Equation (12) does not exist at $t = 1$, while for any $\alpha \in (0,1)$, the solution given in Equation (10) of FBP equation exists. This shows the importance of fractional derivative and its way of dealing these types of the situations where solution of some ordinary PDEs fail to exist.*

Convergence of solution: The FBP equation is as follows:

$$D_t^\alpha w - D_t^\alpha w_{xx} + w_x + w w_x - (3 w_x w_{xx} + w w_{xxx}) = 0, \quad \alpha \in (0,1], \tag{13}$$
$$\text{with } w(x,0) = -x.$$

The approximate first four term solution of FBP equation for $p = 1$ is given by

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t),$$

where,

$$\begin{aligned} w_0(x,t) &= -x; \\ w_1(x,t) &= \frac{t^{\alpha}}{\alpha}(1-x); \\ w_2(x,t) &= \frac{t^{2\alpha}}{\alpha^2}(1-x); \\ w_3(x,t) &= \frac{t^{3\alpha}}{\alpha^3}(1-x). \end{aligned} \tag{14}$$

The sequence generated by HPM will be regarded as

$$V_k = N(V_{k-1}), V_{k-1} = \sum_{i=1}^{k-1} w_i, k = 1,2,3.....$$

We assume that $V_0 = w_0$.

According to theorem for non-linear mapping N, a sufficient condition for convergence of HPM is strictly contraction N. Therefore, we have

$$\|V_0 - w\| = \left\| -x - \frac{-x\alpha + t^{\alpha}}{\alpha - t^{\alpha}} \right\| = \left\| \frac{(x-1)t^{\alpha}}{\alpha - t^{\alpha}} \right\|$$

Now for $\frac{t^{\alpha}}{\alpha} \leqslant \gamma, 0 < \gamma < 1$, we have

$$\|V_1 - u\| = \|w_0 + w_1 - w\| = \left\| \frac{t^{2\alpha}(x-1)}{\alpha(\alpha - t^{\alpha})} \right\| \leqslant \gamma \left\| \frac{(x-1)t^{\alpha}}{\alpha - t^{\alpha}} \right\| = \gamma \|V_0 - w\|$$

$$\|V_2 - w\| = \|w_0 + w_1 + w_2 - w\| = \left\| \frac{t^{3\alpha}(x-1)}{\alpha^2(\alpha - t^{\alpha})} \right\| \leqslant \gamma^2 \left\| \frac{(x-1)t^{\alpha}}{\alpha - t^{\alpha}} \right\| = \gamma^2 \|V_0 - w\|$$

$$\|V_n - w\| = \left\| \sum_{j=1}^{n} w_j - w \right\| = \left\| \frac{(-1)^n t^{n\alpha+\alpha}(x-1)}{\alpha^n(\alpha - t^{\alpha})} \right\| \leqslant \gamma^n \left\| \frac{(x-1)t^{\alpha}}{\alpha - t^{\alpha}} \right\| = \gamma^n \|V_0 - w\|.$$

Therefore, $\lim_{n\to\infty} \|V_n - w\| \leqslant \gamma^n \|V_0 - w\| = 0$, that is, $w(x,t) = \lim_{n\to\infty} V_n = \frac{-x\alpha + t^{\alpha}}{\alpha - t^{\alpha}}$, which is an exact solution.

Discussion

In this section, results obtained by HPM are discussed. Analytical series solution of space-time FBP equation is given in Equation (9). In Figure 1, a solution is presented for different values of α, *such as* $(0.1, 0.3, 0.5, 0.7)$. Figure 1 shows a big difference in smaller and larger values of α. For larger values of α, $w(x,t)$ attains height and for values closer to 0, height of $w(x,t)$ reduces. In Figure 2, results are presented for $\alpha = 0.8$ and $\alpha = 0.9$, respectively. Exact solution of FBP model for $\alpha = 1$ is given in Equation (12). Equation (12) shows discontinuity at $t = 1$ which is clearly depicted in Figure 2. As value of α approaches to 1, the shock is produced in the vicinity of $t = 1$.

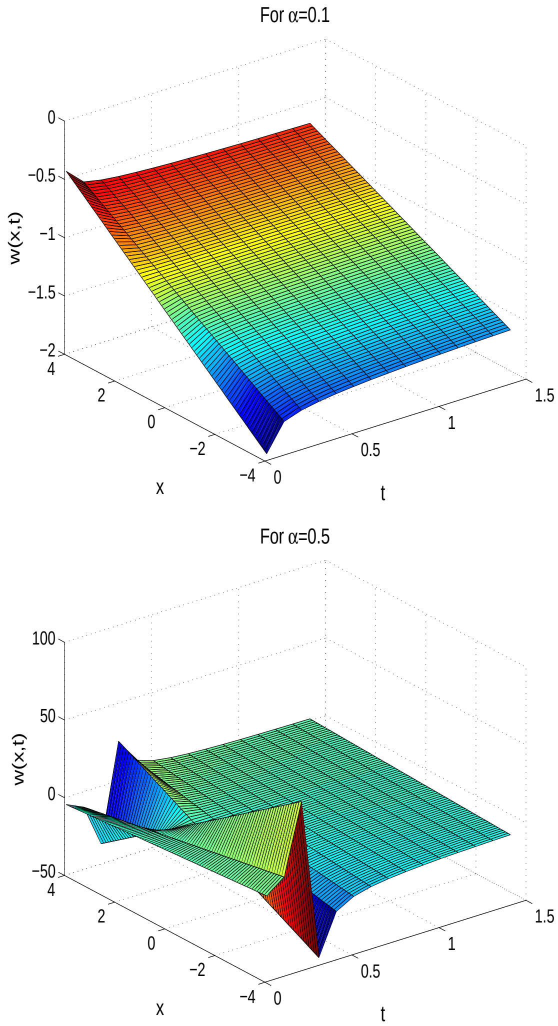

Figure 1. Effects on solution $w(x,t)$ for different values of α.

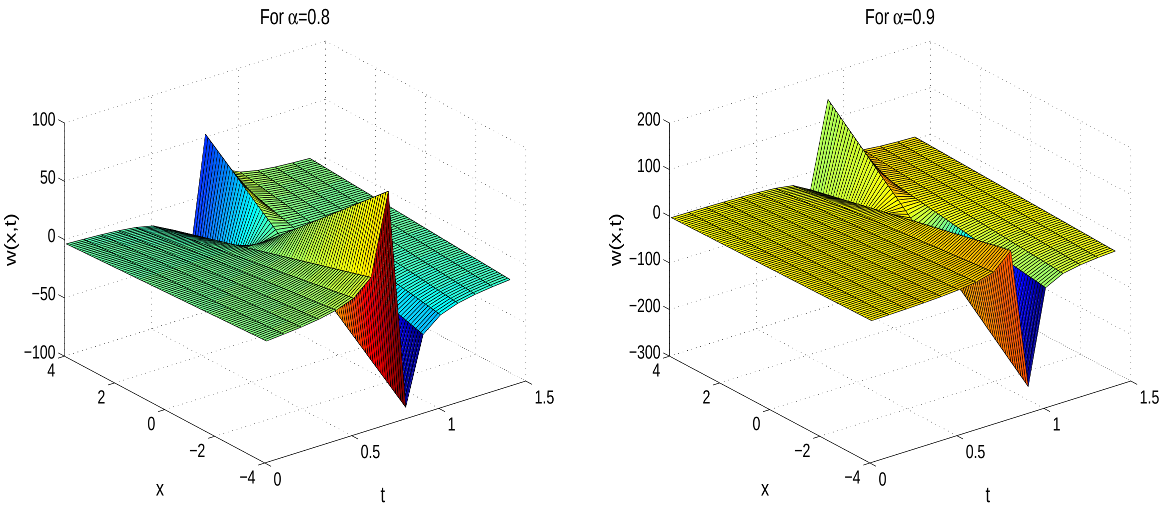

Figure 2. Effects on solution $w(x,t)$ for $\alpha = 0.8$ and $\alpha = 0.9$.

5. Test Problem 2

Consider the fractional second-order obstacle BVP:

$$D^\alpha v = \begin{cases} \dfrac{v^3}{3!} + \dfrac{v^2}{2!} + v + 1, & for\ 1 \leq x < \dfrac{3}{2}\ and\ \dfrac{5}{2} \leq x \leq 3 \\ \dfrac{v^3}{3!} + \dfrac{v^2}{2!} + 2v, & for\ \dfrac{3}{2} \leq x < \dfrac{5}{2} \end{cases} \tag{15}$$

with boundary conditions

$$v(-1) = v(1) = 0$$

In order to apply HPM, we construct homotopy in three different domains:
Case I: $1 \leq x < \dfrac{3}{2}$

$$(1-p)D^\alpha v + p\left(D^\alpha v - \left(\frac{v^3}{3!} + \frac{v^2}{2!} + v + 1\right)\right) = 0$$

or

$$D^\alpha v + p\left(\frac{v^3}{3!} + \frac{v^2}{2!} + v + 1\right) = 0 \tag{16}$$

Here p is parameter that lies between 0 and 1. The solution $v(x)$ is given as follows:

$$v(x) = v_0(x) + pv_1(x) + p^2 v_2(x) + p^3 v_3(x) + \ldots \tag{17}$$

Setting $p = 1$ provides,

$$v(x) = v_0(x) + v_1(x) + v_2(x) + v_3(x) + \ldots \tag{18}$$

Now, substitute Equation (3) in Equation (2), and comparing the coefficients of similar powers of p, we get

$$\begin{aligned} p^0 :\ & D^\alpha v_0 = 0,\ v_0(0) = 0 \\ p^1 :\ & D^\alpha v_1 + \frac{v_0^3}{3!} + \frac{v_0^2}{2!} + v_0 + 1 = 0,\ v_1(0) = 0 \\ p^2 :\ & D^\alpha v_2 + \frac{3v_0^2 v_1}{3!} + \frac{2v_0 v_1}{2!} + v_1 = 0,\ v_2(0) = 0 \\ p^3 :\ & D^\alpha v_3 + \frac{3v_0 v_1^2}{3!} + \frac{3v_0^2 v_2}{3!} + \frac{v_1^2}{2!} + \frac{2v_0 v_2}{2!} + v_2 = 0,\ v_3(0) = 0 \end{aligned} \tag{19}$$

Now, the fractional integral operator J_α with comfortable derivative definition is applied on both sides of Equation (5), we get

$$\begin{aligned} v_0(x) &= c_1 x + c_2, \\ v_1(x) &= c_1 x + c_2 + \frac{1}{24}x^{4+\alpha}c_1^3 + \frac{1}{6}x^{3+\alpha}c_1^2 + \frac{1}{6}x^{3+\alpha}c_1^2 c_2 + \frac{1}{2}x^{2+\alpha}c_1 c_2 + \frac{1}{2}x^{2+\alpha}c_1 \\ &\quad + \frac{1}{4}x^{2+\alpha}c_1 c_2^2 + \frac{1}{2}x^{1+\alpha}c_2^2 + x^{1+\alpha} + \frac{1}{6}x^{1+\alpha}c_2^3 + \frac{1}{6}x^{4+\alpha}c_1^3 + x^{1+\alpha}c_2, \\ &\ \ \vdots \end{aligned} \tag{20}$$

Case II: $\frac{3}{2} \leq x < \frac{5}{2}$

$$(1-p)D^{\alpha}v + p\Big(D^{\alpha}v - \Big(\frac{v^3}{3!} + \frac{v^2}{2!} + 2v\Big)\Big) = 0$$

or

$$D^{\alpha}v + p\Big(\frac{v^3}{3!} + \frac{v^2}{2!} + 2v\Big) = 0 \tag{21}$$

Now, substitute Equation (3) in Equation (7), and comparing the coefficients of similar powers of p, we get

$$\begin{aligned} p^0: \;\; D^{\alpha}v_0 = 0, \;\; v_0(0) = 0 \\ p^1: \;\; D^{\alpha}v_1 + \frac{v_0^3}{3!} + \frac{v_0^2}{2!} + 2v_0 = 0, \;\; v_1(0) = 0 \\ p^2: \;\; D^{\alpha}v_2 + \frac{3v_0^2 v_1}{3!} + \frac{2v_0 v_1}{2!} + 2v_1 = 0, \;\; v_2(0) = 0 \\ p^3: \;\; D^{\alpha}v_3 + \frac{3v_0 v_1^2}{3!} + \frac{3v_0^2 v_2}{3!} + \frac{v_1^2}{2!} + \frac{2v_0 v_2}{2!} + 2v_2 = 0, \;\; v_3(0) = 0 \end{aligned} \tag{22}$$

Now, the fractional integral operator J_{α} with comfortable derivative definition is applied on both sides of Equation (8), we get

$$\begin{aligned} v_0(x) &= c_3 x + c_4 \\ v_1(x) &= c_3 x + c_4 + \frac{1}{24}x^{4+\alpha}c_3^3 + \frac{1}{6}x^{3+\alpha}c_3^2 + \frac{1}{6}x^{3+\alpha}c_3^2 c_4 + \frac{1}{2}x^{2+\alpha}c_3 c_4 + \frac{1}{2}x^{2+\alpha}c_3 \\ &\quad + \frac{1}{4}x^{2+\alpha}c_3 c_4^2 + \frac{1}{2}x^{1+\alpha}c_4^2 + x^{1+\alpha} + \frac{1}{6}x^{1+\alpha}c_4^3 + \frac{1}{6}x^{4+\alpha}c_3^3 + x^{1+\alpha}c_4 \\ &\;\; . \\ &\;\; . \\ &\;\; . \end{aligned} \tag{23}$$

Case III: $\frac{5}{2} \leq x \leq 3$

In this case, the constructed homotopy will be same as in Case I. After substituting Equation (3) in Equation (2), we get

$$\begin{aligned} v_0(x) &= c_5 x + c_6 \\ v_1(x) &= c_5 x + c_6 + \frac{1}{24}x^{4+\alpha}c_5^3 + \frac{1}{6}x^{3+\alpha}c_5^2 + \frac{1}{6}x^{3+\alpha}c_5^2 c_6 + \frac{1}{2}x^{2+\alpha}c_5 c_6 + \frac{1}{2}x^{2+\alpha}c_5 \\ &\quad + \frac{1}{4}x^{2+\alpha}c_5 c_6^2 + \frac{1}{2}x^{1+\alpha}c_6^2 + x^{1+\alpha} + \frac{1}{6}x^{1+\alpha}c_6^3 + \frac{1}{6}x^{4+\alpha}c_5^3 + x^{1+\alpha}c_6 \\ &\;\; . \\ &\;\; . \\ &\;\; . \end{aligned} \tag{24}$$

We calculate the results by taking $\alpha = 1.25, 1.5$ and 1.75.

1. For $\alpha = 1.25$

$$v(x) = \begin{cases} c_1x + c_2 + 0.04166666667x^{\frac{21}{4}}c_1^3 + 0.1666666666x^{\frac{17}{4}}c_1^2 + 0.1666666666x^{\frac{17}{4}}c_1^2c_2 \\ +0.500000000x^{\frac{13}{4}}c_1c_2 + 0.500000000x^{\frac{13}{4}}c_1 + .250000000x^{\frac{13}{4}}c_1c_2^2 + 0.5000000000x^{\frac{9}{4}}c_2^2 \\ +x^{\frac{9}{4}} + 0.1666666667x^{\frac{9}{4}}c_2^3 + x^{\frac{9}{4}}c_2, \qquad for\ 1 \le x < \dfrac{3}{2} \\ c_3x + c_4 + 0.04166666667x^{\frac{21}{4}}c_3^3 + 0.1666666666x^{\frac{17}{4}}c_3^2 + 0.1666666666x^{\frac{17}{4}}c_3^2c_4 \\ +0.500000000x^{\frac{13}{4}}c_3c_4 + 1.000000000x^{\frac{13}{4}}c_3 + 0.250000000x^{\frac{13}{4}}c_3c_4^2 + 0.5000000000x^{\frac{9}{4}}c_4^2 \\ +0.1666666667x^{\frac{9}{4}}c_4^3 + 2x^{\frac{9}{4}}c_4, \qquad for\ \dfrac{3}{2} \le x < \dfrac{5}{2} \\ c_5x + c_6 + 0.04166666667x^{\frac{21}{4}}c_5^3 + 0.1666666666x^{\frac{17}{4}}c_5^2 + 0.1666666666x^{\frac{17}{4}}c_5^2c_6 \\ +0.500000000x^{\frac{13}{4}}c_5c_6 + 0.500000000x^{\frac{13}{4}}c_5 + 0.250000000x^{\frac{13}{4}}c_5c_6^2 + 0.5000000000x^{\frac{9}{4}}c_6^2 \\ +x^{\frac{9}{4}} + 0.1666666667x^{\frac{9}{4}}c_6^3 + x^{\frac{9}{4}}c_6, \qquad for\ \dfrac{5}{2} \le x \le 3 \end{cases} \tag{25}$$

Now by applying the continuity conditions at $x = \dfrac{3}{2}$ and $x = \dfrac{5}{2}$ and BCs, we get a system of six nonlinear equations. By using Newton's method for nonlinear system, we obtain the values of constants:

$$c_1 = -1.3295886370, \quad c_2 = 0.4580256818, \quad c_3 = -0.00096246893$$

$$c_4 = 0.0247772323, \quad c_5 = -0.4962384877, \quad c_6 = -0.5257196774 \tag{26}$$

By substituting values of constants from Equation (12) into Equation (11), we get the analytical solution of system of second-order fractional BVPs subject to obstacle problem given in Equation (1).

$$v(x) = \begin{cases} -1.3295886370x + 0.4580256818 - 0.09793561230x^{\frac{21}{4}} + 0.4295844110x^{\frac{17}{4}} \\ -1.039019967x^{\frac{13}{4}} + 1.578934123x^{\frac{9}{4}}, \qquad for\ 1 \le x < \dfrac{3}{2} \\ -0.0096246893x + 0.0247772323 - 3.714915284 \times 10^{-8}x^{\frac{21}{4}} \\ +1.582164570 \times 10^{-5}x^{\frac{17}{4}} - 0.009745403058x^{\frac{13}{4}} \\ +0.0498639553x^{\frac{9}{4}}, \qquad for\ \dfrac{3}{2} \le x < \dfrac{5}{2} \\ -0.4962384877x - 0.5257196774 - 0.005091668167x^{\frac{21}{4}} + 0.01946546333x^{\frac{17}{4}} \\ -0.1519658196x^{\frac{13}{4}} + 0.5882544081x^{\frac{9}{4}}, \qquad for\ \dfrac{5}{2} \le x \le 3 \end{cases} \tag{27}$$

In similar manners, we can find the solution of problem mentioned in Equation (1) for $\alpha = 1.5$ and $\alpha = 1.75$.

2. For $\alpha = 1.5$

We get the following analytical solution of system given in Equation (1) for $\alpha = 1.5$.

$$v(x) = \begin{cases} -1.5493863800 + 0.6139530788 - 0.15497725350x^{\frac{11}{2}} + 0.6457421302x^{\frac{9}{2}} \\ -1.396324258x^{\frac{7}{2}} + 1.840992684x^{\frac{5}{2}}, \qquad for\ 1 \le x < \dfrac{3}{2} \\ -0.0191553809x + 0.0400302957 - 2.928607263 \times 10^{-7}x^{\frac{11}{2}} \\ +6360281305 \times 10^{-5}x^{\frac{9}{2}} - 0.01954645244x^{\frac{7}{2}} \\ +0.08087249461x^{\frac{5}{2}}, \qquad for\ \dfrac{3}{2} \le x < \dfrac{5}{2} \\ -0.4940057367x - 0.5879890892 - 0.005023249330x^{\frac{11}{2}} + 0.01675797164x^{\frac{9}{2}} \\ -0.1444661720x^{\frac{7}{2}} + 0.5509954694x^{\frac{5}{2}}, \qquad for\ \dfrac{5}{2} \le x \le 3 \end{cases} \tag{28}$$

3. For $\alpha = 1.75$

We obtain the following analytical solution of the considered problem given in Equation (1) for $\alpha = 1.75$ is as follows:

$$v(x) = \begin{cases} -1.7503932530x + 0.7531819483 - 0.2234578677x^{\frac{23}{4}} + 0.8952555066x^{\frac{19}{4}} \\ -1.782621032x^{\frac{15}{4}} + 2.108034697x^{\frac{11}{4}}, \qquad for\ 1 \le x < \frac{3}{2} \\ -0.0345683146x + 0.0637056320 - 1.721168446 \times 10^{-6}x^{\frac{23}{4}} \\ +0.000211849098x^{\frac{19}{4}} - 0.03570448584x^{\frac{15}{4}} \\ +0.1294835584x^{\frac{11}{4}}, \qquad for\ \frac{3}{2} \le x < \frac{5}{2} \\ -0.4976869784x - 0.6267857182 - 0.005136385271x^{\frac{23}{4}} + 0.01540705241x^{\frac{19}{4}} \\ -0.1417523126x^{\frac{15}{4}} + 0.5286045753x^{\frac{11}{4}}, \qquad for\ \frac{5}{2} \le x \le 3 \end{cases} \tag{29}$$

Discussion

Figures 3–5 present the solution of obstacle problem for different values of $\alpha = 1.25$, $\alpha = 1.50$, $\alpha = 1.75$, respectively. Figure 6 shows the comparison between different values of α. In Figure 6, obstacle achieves more height for larger values of α and vice versa.

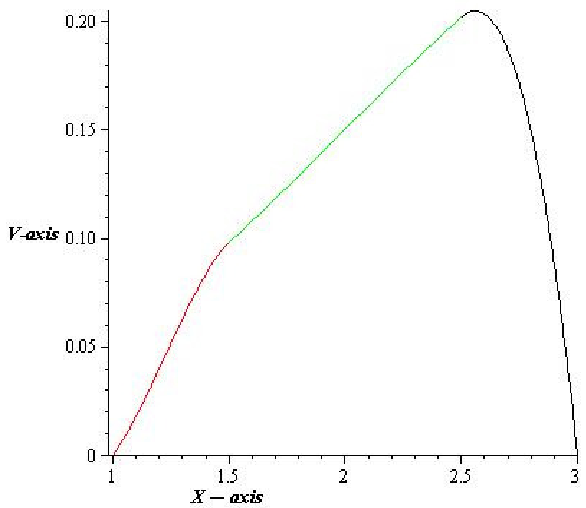

Figure 3. Analytical solution of the problem (c.f. Equation (1)) by homotopy perturbation method (HPM) for $\alpha = 1.25$.

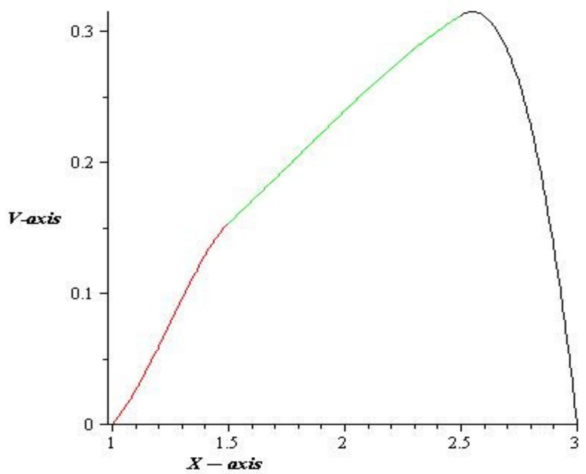

Figure 4. Analytical solution of the problem (c.f. Equation (1)) by HPM for $\alpha = 1.50$.

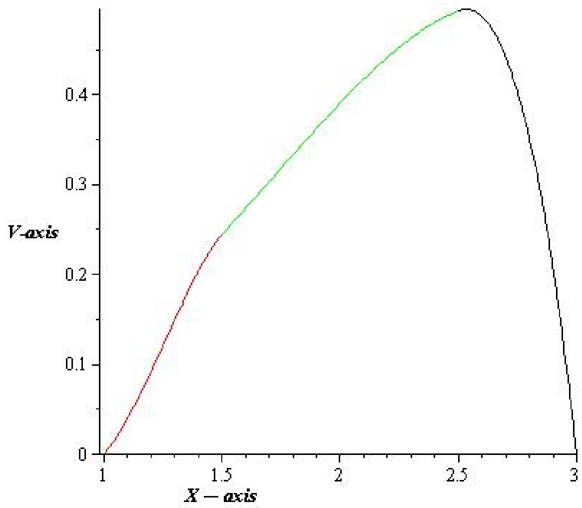

Figure 5. Analytical solution of the problem (c.f. Equation (1)) by HPM for $\alpha = 1.75$.

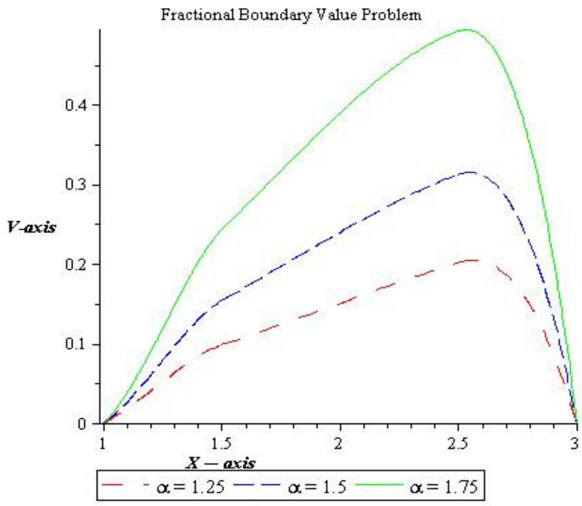

Figure 6. Analytical solution of the problem (c.f. Equation (1)) for different values of α by using HPM.

6. Conclusions

The analysis of HPM for the solution of FPDEs was given. A unified convergence theorem was proved and results were validated for the solution of FBP equation. The method to solve FPDEs was presented, while the same partial differential equation with ordinary derivative i.e., for $\alpha = 1$ fails to exist. This study demonstrated the importance of fractional derivative and the technique of dealing with these types of PDEs where solution of some ordinary PDEs does not exist. Moreover, HPM was applied to solve complex obstacle BVP. The suggested method can be applied to find solutions of other PDEs (both linear and nonlinear) of fractional order. The present study can be useful to analyze other traditional analytical techniques, such as Adomian Decomposition Method and Homotopy Analysis Method, to solve nonlinear differential equation of non- integer order. Furthermore, the present work may be extended to solve practical fractional models, for example wireless networks and nonlinear obstacle problems.

Author Contributions: Conceptualization, S.J; Methodology, S.J. and A.W; Software, S.J. and A.W; Validation, D.B and H.A.; Writing—original draft preparation, S.J, M.S.K and H.A.; Writing—review and editing, D.B.; Visualization, M.S.K; Supervision, D.B.

Funding: This research received no external funding.

Acknowledgments: The authors gratefully acknowledge Muhammad Asif Javed for helping in grammatical and stylistic editing of the article. The authors are thankful to Aqsa Mumtaz for helping in applying the code in Maple software.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

D_α	conformable derivative
J_α	conformable integral
HPM	homotopy perturbation method
FPDEs	fractional partial differential equations
BVP	boundary value problem
FDEs	fractional differential equations
PDEs	partial differential equations

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