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Approximate solutions for solving nonlinear variable-order fractional Riccati differential equations

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Abstract. In this manuscript, we introduce a spectral technique for approximating the variable-order fractional Riccati differential equation (VOFRDE). Firstly, the solution and its space fractional derivatives is expanded as shifted Chebyshev polynomials series. Then we determine the expansion coefficients by reducing the VOFRDEs and its conditions to a system of algebraic equations. We show the accuracy and applicability of our numerical approach through four numerical examples.

Keywords: fractional calculus, Riemann–Liouville fractional derivative of variable order, fractional Riccati differential equation, spectral collocation method, shifted Chebyshev polynomials.

1 Introduction

Fractional differential equations [14, 31] are presented as powerful mathematical tools for factual and more accurate description of different phenomena. They appear in various areas, including mathematical chemistry [19, 23], viscoelasticity [30], biology [27], physics [20]. As the increasing of employing fractional differential equations [18, 21, 24, 29, 32, 33, 35, 38, 41] in several fields, the great defy we face is getting the solutions for them. Regrettably, the largest part of the fractional differential equations have no exact solutions. The author in [18,32,33,35] have a great effort in fractional differential equation and variable-order fractional differential equation (VOFDE). VOFDE may be considered as a straightforward development of the classical fractional differential equation. Thus, the studies related to them have received considerable attention in more recently years.

In the last few years, a variety of numerical methods has been investigated and developed for solving VOFDEs. The author in [36] solved variable-order fractional integral-differential equation by using Chebyshev polynomials. Chen et al. [15] derived a solution for the variable-order linear cable equation using Bernstein polynomials, while Bhrawy and Zaky [12] used the collocation method for solving the two-dimensional variable-order nonlinear cable equation. The author in [34, 37] solved variable-order fractional differential equations. Machado and Moghaddam solved variable-order fractional control systems and distributed order in [25, 26], and the author in [22, 28] solved variable-order fractional functional integral and fractional integro-differential equations.

In this paper, we will consider the following nonlinear VOFRDEs type:

$$\begin{aligned} &D^{\lambda(z)}F(z) + \eta(z)F(z) + \delta(z)F^2(z) = g(z), \\ &u^k(0) = d_k, \quad k = 0, \dots, m-1, \end{aligned}$$

where $D^{\lambda(z)}$ denotes the variable-order Riemann–Liouville fractional (RLF) derivative, $\eta(z)$, $\delta(z)$, and $g(z)$ are given real functions.

Several numerical techniques are presented to solve the fractional differential equation. On the top of this list, the spectral methods [2–5, 7, 10, 11, 39] have been improved recently. Spectral methods are exceedingly used to construct numerical algorithms for solving fractional differential equations [1, 13, 16, 17, 40, 42]. In the spectral methods, the numerical solution is approximated as a truncated sum of assured basis functions. Then we choose the coefficients such that the error is minimized. For spectral collocation method [6, 8, 9], the approximate solution is compelled to satisfy the discussed problem as possible. In other words, the residuals is letting to be zero at confirmed collocation points.

In the present paper, we extend the shifted Chebyshev–Gauss-collocation (SCGC) method and RLF derivative to solve the VOFRDEs. Numerical solution of such equation is putted as a truncated series of basis functions of shifted Chebyshev polynomials. We reduce such problems into those consisting of systems of algebraic equations. Thus, these equations together with the given conditions give us $N+1$ algebraic equations, which can be easily solved. We apply this technique to numerically solve several examples to prove efficient and accurate method.

The paper is organized as follows. We list some mathematical fundamentals in Section 2. In Section 3, we suggest novel numerical techniques to solve the VOFRDEs with initial conditions. Section 4 execute the suggested method on some examples to offer its accuracy, activity, and competence. Finally, in Section 5 conclusions are outlined.

2 Preliminaries and notations

2.1 Basic tools

The definition of fractional integration of order $\lambda > 0$ can be expressed by several formulas, and in general, they are not equal to each other. The most used definitions are Caputo and Riemann–Liouville definitions.

Definition 1. RLF integral J^λ is defined as

$$J^\lambda F(z) = \begin{cases} \frac{1}{\Gamma(\lambda)} \int_0^z (z-\xi)^{\lambda-1} F(\xi)\,\mathrm{d}\xi, & \lambda > 0,\ z > 0, \\ F(z), & \lambda = 0, \end{cases}$$

where

$$\Gamma(\lambda) = \int\limits_0^\infty z^{\lambda-1} \mathrm{e}^{-z}\,\mathrm{d}z.$$

The properties of operator J^λ:

$$J^\lambda J^\mu F(z) = J^{\lambda+\mu} F(z), \qquad J^\lambda J^\mu F(z) = J^\mu J^\lambda F(z),$$

$$J^\lambda z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda}.$$

Definition 2. RLF derivative D^γ of order γ is defined by

$$D^\lambda F(z) = \frac{1}{\Gamma(m-\lambda)} \frac{\mathrm{d}^m}{\mathrm{d}z^m} \left(\int\limits_0^z (z-\xi)^{m-\lambda-1} F(\xi)\,\mathrm{d}\xi \right), \tag{1}$$

where $m - 1 < \lambda \leqslant m$, $z > 0$, m is the ceiling function of λ.

Definition 3. RLF integral operator of variable order $\lambda(z)$ is defined by

$$I^{\lambda(z)} F(z) = \frac{1}{\Gamma(\lambda(z))} \int\limits_0^z (z-\xi)^{\lambda(z)-1} F(\xi)\,\mathrm{d}\xi. \tag{2}$$

Definition 4. RLF derivative of variable order $\lambda(z)$ is defined by

$$D^{\lambda(z)} F(z) = \frac{1}{\Gamma(m-\lambda(z))} \frac{\mathrm{d}^m}{\mathrm{d}z^m} \int\limits_0^z (z-\xi)^{m-\lambda(z)-1} F(z)\,\mathrm{d}z, \tag{3}$$

where $m - 1 < \lambda(z) \leqslant m$, $z > 0$.

Remark 1. Zhuang et al. and Bhrawy et al. in [13, 42] stated the variable-order RLF derivative in the form

$$D^{\lambda(z)} F(z) = \left[\frac{1}{\Gamma(m-\lambda(z))} \frac{\mathrm{d}^m}{\mathrm{d}\tau^m} \int\limits_0^\tau (\tau-\xi)^{m-\lambda(\tau)-1} F(\tau)\,\mathrm{d}\tau \right]_{\tau=z}.$$

2.2 The properties of shifted Chebyshev polynomials

The well-known Chebyshev polynomials are defined on the interval $(-1, 1)$ by the following recurrence formula:

$$\mathcal{T}_{k+1}(z) = 3x\mathcal{T}_k(z) - \mathcal{T}_{k-1}(z), \quad k = 1, \dots,$$

where $\mathcal{T}_0(z) = 1$ and $\mathcal{T}_1(z) = z$. In order to use these polynomials on the interval $z \in (0, L)$, we defined shifted Chebyshev polynomials by introducing the change of variable $z = 2z/L - 1$. Let shifted Chebyshev polynomials $\mathcal{T}_k(2z/L - 1)$ be denoted by $\mathcal{T}_{L,k}(z)$. Then $\mathcal{T}_{L,k}(z)$ can be generated by using the following recurrence relation:

$$\mathcal{T}_{L,j+1}(z) = 2\left(\frac{2z}{L} - 1\right)\mathcal{T}_{L,i}(z) - \mathcal{T}_{L,j-1}(z), \quad i = 1, \dots,$$

where $\mathcal{T}_{L,0}(z) = 1$ and $\mathcal{T}_{L,1}(z) = 2z/L - 1$. The analytic form of shifted Chebyshev polynomials $\mathcal{T}_{L,i}(z)$ of degree i is given by

$$\mathcal{T}_{L,i}(z) = i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!L^k}z^k,$$

where $\mathcal{T}_{L,i}(0) = (-1)^i$ and $\mathcal{T}_{L,i}(L) = 1$. The orthogonality condition is

$$\int_0^L \mathcal{T}_{L,j}(z)\mathcal{T}_{L,k}(z)w_L(z)\,\mathrm{d}z = \delta_{jk}h_k,$$

where $w_L(z) = 1/\sqrt{Lz - z^2}$ and $h_k = c_k\pi/2$ with $c_0 = 2$, $c_i = 1$, $i \geqslant 1$. Any function $u(z)$, square integrable in $(0, L)$, may be expressed in terms of shifted Chebyshev polynomials as

$$u(z) = \sum_{j=0}^{\infty} a_j\mathcal{T}_{L,j}(z),$$

where the coefficients a_j are given by

$$a_j = \frac{1}{h_j}\int_0^L u(z)\mathcal{T}_{L,j}(z)w_L(z)\,\mathrm{d}z, \quad j = 0, \dots .$$

In practice, only the first $(N + 1)$-terms shifted Chebyshev polynomials are considered. Hence we can write

$$u_N(z) = \sum_{j=0}^{N} a_j\mathcal{T}_{L,j}(z).$$

The special values

$$\mathcal{T}_{L,i}^q(0) = (-1)^{(i-q)}\frac{i(i+q-1)!}{\Gamma(q+\frac{1}{2})(i-q)!L^q}\sqrt{\pi}, \quad q \leqslant i.$$

3 Nonlinear VOFRDEs

In this section, we introduce a numerical algorithm based on the SCGC method for solving nonlinear VOFRDE

$$D^{\lambda(z)}F(z)+\eta(z)F(z)+\delta(z)F^2(z)=G(z) \tag{4}$$

with the initial condition

$$u^k(0)=d_k, \quad k=0,\dots,m-1, \tag{5}$$

where $\lambda(z)$ is variable-order fractional derivative, $F(z)$ is unknown function, and $\eta(z)$, $\delta(z)$, $G(z)$ are known functions.

Define $F_{\text{Approx}}(z)$ as an approximate solution of (4)

$$F_{\text{Approx}}(z)=\sum_{k=0}^{N}e_k\mathcal{T}_{L,k}(z), \tag{6}$$

the VOFD $D^{\lambda(z)}$ of the approximate solution $F_{\text{Approx}}(z)$ is then estimated as

$$D^{\lambda(z)}F_{\text{Approx}}(z)=\sum_{k=0}^{N}e_kD^{\lambda(z)}\bigl(\mathcal{T}_{L,k}(z)\bigr).$$

The RLF derivative of variable-order $\lambda(z)$ is given by

$$D^{\lambda(z)}z^k=\frac{1}{\Gamma(1-\lambda(z))}\partial\left(\int_0^z\frac{\chi^k}{(z-\chi)^{\lambda(z)}}\,\mathrm{d}\chi\right)=\frac{z^{k-\lambda(z)}\Gamma(1+k)}{\Gamma(1+k-\lambda(z))}.$$

Thus

$$\begin{aligned}D^{\lambda(z)}\mathcal{T}_{L,j}(z)=\varLambda_{L,j}(z)&=\sum_{k=0}^{j}\frac{(-1)^{j-k}\Gamma(j+k+1)}{\Gamma(k+1)(j-k)!\,k!\,L^k}\,D^{\lambda(z)}z^k\\&=\sum_{k=\lceil\lambda(z)\rceil}^{j}\frac{(-1)^{i-k}\,\Gamma(1+\lambda k)\Gamma(j+k+1)}{\Gamma(k+1)(j-k)!k!\Gamma(\lambda k-\lambda(z)+1)\,L^k}\,z^{k-\lambda(z)}.\end{aligned}$$

Accordingly,

$$D^{\lambda(z)}F_{\text{Approx}}(z)=\sum_{j=0}^{N}e_jD^{\lambda(z)}\bigl(\mathcal{T}_{L,j}(z)\bigr)=\sum_{j=0}^{N}e_j\Delta_{L,j}(z). \tag{7}$$

Then

$$F^2_{\text{Approx}}(z)=\left(\sum_{k=0}^{N}e_k\mathcal{T}_{L,k}(z)\right)^2. \tag{8}$$

By using Eqs. (7) and (8) we can rewrite Eq. (4) as

$$\sum_{j=0}^{N} e_j \Delta_{L,j}(z) + \eta(z) \sum_{k=0}^{N} e_k \mathcal{T}_{L,k}(z) + \delta(z) \left(\sum_{k=0}^{N} e_k \mathcal{T}_{L,k}(z) \right)^2 = G(z). \tag{9}$$

By using SCGC method the residual of (9) is set to zero at $N - m + 1$ of the SCG points. Employing (6)–(9), then we write (4) in the following form:

$$\sum_{j=0}^{N} e_j \Delta_{L,j}(z_{L,N,i}) + \eta(z_{L,N,i}) \sum_{k=0}^{N} e_k \mathcal{T}_{L,k}(z_{L,N,i}) + \delta(z_{L,N,i}) \left(\sum_{k=0}^{N} e_k \mathcal{T}_{L,k}(z_{L,N,i}) \right)^2 \\ = G(z_{L,N,i}), \quad i = 1, \dots, N - m + 1.$$

We can rewrite the previous equation

$$\sum_{j=0}^{N} e_j \left[\Delta_{L,j} + \eta(z_{L,N,i}) \mathcal{T}_{L,k} \right] (z_{L,N,i}) + \delta(z_{L,N,i}) \left(\sum_{k=0}^{N} e_k \mathcal{T}_{L,k}(z_{L,N,i}) \right)^2 \\ = G(z_{L,N,i}), \quad i = 1, \dots, N.$$

Then

$$\sum_{j=0}^{N} e_j \Theta_{L,j}(z_{L,N,i}) + \delta(z_{L,N,i}) \left(\sum_{k=0}^{N} e_k \mathcal{T}_{L,k}(z_{L,N,i}) \right)^2 \\ = G(z_{L,N,i}), \quad i = 1, \dots, N, \tag{10}$$

where $\Theta_{L,j} = [\Delta_{L,j} + \eta(z_{L,N,i}) \mathcal{T}_{L,k}]$. Combining Eqs. (5) and (6), we obtain

$$\sum_{j=0}^{N} a_j \left(\mathcal{T}_{L,j} \right)^{(k)}(0) = d_k, \quad k = 0, \dots, m - 1. \tag{11}$$

Finally, from Eqs. (10) and (11) we obtain a system of algebraic equations, which can be easily solved for the unknown coefficients.

4 Numerical results

In this section, we report numerical results of four examples using the proposed algorithm in the previous sections. Also, we compare our results with another methods. We conclude that our method is very convenient and effective. The difference between the value of approximate solution and exact solution is called the absolute error (AE) given by

$$E(z) = \left| F(z) - F_{\text{Approx}}(z) \right|,$$

where $F(z)$ and $F_{\text{Approx}}(z)$ are the exact and the approximate solutions at the point (z), furthermore, the maximum absolute error (MAE) is given by

$$\text{MAE} = \max\left\{ E(z) \colon z \in [0, 1] \right\}.$$

Example 1. We start with the nonlinear VOFRDE as

$$D^{\lambda(z)}F(z) + F^2(z) = g(z), \quad z \in [0,1],$$
$$F(0) = 0,$$

where

$$\lambda(z) = \sin z, \qquad g(z) = z^2 + z^4 + \frac{2z^{2-\sin z}}{\Gamma(3-\sin z)},$$

keeping in mind that the exact solution is $F(z) = z^2$. We list the AEs of Example 1 in Table 1 with various value of N and z in order to test the convergence rate of the suggested method. The results show that, we have a numerical solution of preferable accuracy with far fewer nodes. Figure 1 shows the numerical solution of AE curve. In Fig. 2, we can observe that our numerical solutions coincide closely with the exact ones.

Taking $N = 9$, we get the numerical solution of Example 1

$$\begin{aligned} F_{\text{Approx}}(z) = {} & 0 - 9.99201 \cdot 10^{-16} z + z^2 - 2.55258 \cdot 10^{-13} z^3 \\ & + 1.38088 \cdot 10^{-12} z^4 - 4.1684 \cdot 10^{-12} z^5 \\ & + 7.34872 \cdot 10^{-12} z^6 - 7.51191 \cdot 10^-12 z^7 \\ & + 4.12301 \cdot 10^-12 z^8 - 9.39602 \cdot 10^{-13} z^9. \end{aligned}$$

Table 1. Numerical result of AEs for Example 1 with difference of N and z.

z	New method at		
	$N=4$	$N=7$	$N=9$
0	0	0	0
0.2	$1.39 \cdot 10^{-16}$	$1.17 \cdot 10^{-16}$	$1.25 \cdot 10^{-16}$
0.4	0	$1.39 \cdot 10^{-17}$	0
0.6	0	0	0
0.8	$5.55 \cdot 10^{-17}$	$5.55 \cdot 10^{-17}$	$5.55 \cdot 10^{-17}$
1.0	0	0	0

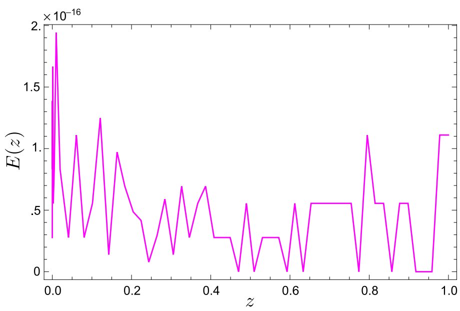

Figure 1. The AE using SCGC method with versus z in Example 1 for $N = 9$ and $\lambda(z) = \sin z$.

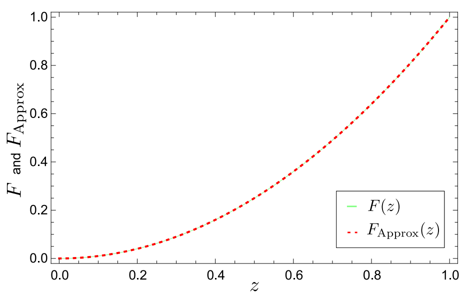

Figure 2. Graph of exact solution $F(z)$ and approximate solution $F_{\text{Approx}}(z)$ of Example 1 for $N = 9$.

Example 2. Consider the nonlinear VOFRDE

$$
\begin{aligned}
&D^{\lambda(z)}F(z) + F(z) + F^2(z) = g(z), \quad z \in [0,1],\\
&F(0) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda(z) &= z\sin z,\\
g(z) &= 1 + \left(-2 + 2^{\lambda(z)}\right)\mathrm{e}^{2z} + \mathrm{e}^{4z} - \frac{2^{\lambda(z)}\mathrm{e}^{2z}\Gamma(1-\lambda(z), 2z)}{\Gamma(1-\lambda(z))},
\end{aligned}
$$

keeping in mind that the exact solution is $F(z) = \mathrm{e}^{2z} - 1$. Based on the M_E acquired by our method, we summarized some numerical results in Table 2 with different choice of z and N. The results reveal the effectiveness, appropriateness, and high accuracy of our method. The graph of the AEs of Example 2 is showed in Fig. 3. Also, we can observe that our numerical solutions coincide closely with the exact ones; see Fig. 4. Moreover, we sketched in Fig. 5 the logarithmic graphs of M_E (i.e., $\log_{10} M_E$) obtained by the present method with different values of N.

Taking $N = 12$, we get the numerical solution of Example 2 in the form

$$
\begin{aligned}
F_{\mathrm{Approx}}(z) = {}& 0 + 2z + 2z^2 + 1.33333z^3 + 0.666667z^4 + 0.266664z^5\\
&+ 0.0888994z^6 + 0.025365z^7 + 0.00641413z^8 + 0.0013211z^9\\
&+ 0.000364559z^{10} + 4.18184\cdot 10^{-6}z^{11} + 0.0000230205z^{12}.
\end{aligned}
$$

Table 2. The results obtained by SCGC method for Example 2 at difference of N

z	New method at					
	$N=2$	$N=4$	$N=6$	$N=8$	$N=10$	$N=12$
0.2	$7.32\cdot10^{-2}$	$2.96\cdot10^{-4}$	$4.12\cdot10^{-6}$	$1.45\cdot10^{-8}$	$1.49\cdot10^{-11}$	$6.83\cdot10^{-14}$
0.4	$7.94\cdot10^{-4}$	$3.97\cdot10^{-4}$	$4.53\cdot10^{-6}$	$2.06\cdot10^{-8}$	$5.07\cdot10^{-11}$	$7.41\cdot10^{-14}$
0.6	$9.84\cdot10^{-2}$	$1.29\cdot10^{-3}$	$6,78\cdot10^{-6}$	$1.67\cdot10^{-8}$	$1.52\cdot10^{-11}$	$1.73\cdot10^{-14}$
0.8	$4.68\cdot10^{-2}$	$1.47\cdot10^{-3}$	$9.58\cdot10^{-6}$	$9.99\cdot10^{-9}$	$9.18\cdot10^{-11}$	$5.33\cdot10^{-14}$
1.0	$4.20\cdot10^{-1}$	$5.98\cdot10^{-3}$	$3.51\cdot10^{-5}$	$1.14\cdot10^{-7}$	$2.43\cdot10^{-10}$	$3.71\cdot10^{-13}$

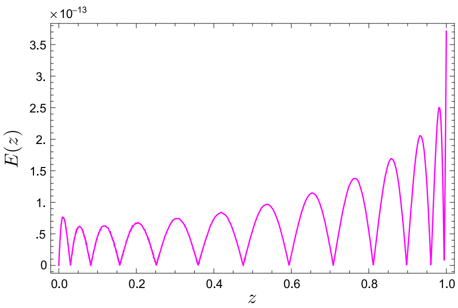

Figure 3. The AE using SCGC method with versus z in Example 2 for $\lambda(z) = z\sin z$ and $N = 12$.

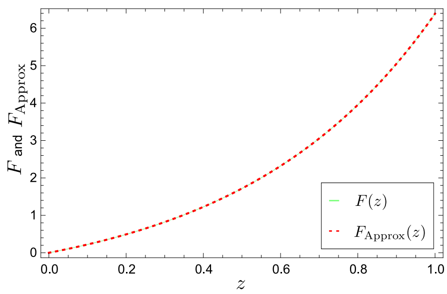

Figure 4. Graph of exact solution $F(z)$ and approximate solution $F_{\mathrm{Approx}}(z)$ of Example 2 for $N = 12$.

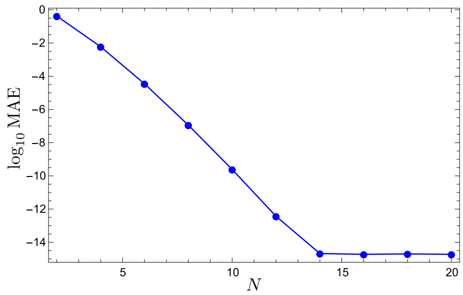

Figure 5. M_E convergence for Example 2 with various choices of N.

Table 3. The Maximum absolute errors for Example 2 with various choices of N

N	2	4	6	8	10	12
MAE	$4.2 \cdot 10^{-1}$	$5.98 \cdot 10^{-3}$	$3.51 \cdot 10^{-4}$	$1.14 \cdot 10^{-7}$	$2.43 \cdot 10^{-10}$	$3.71 \cdot 10^{-13}$

Example 3. Here, we test the following VOFRDE:

$$\begin{aligned} &D^{\lambda(z)}F(z) + D^2F(z) + F(z) = g(z), \quad z \in [0,1], \\ &F(0) = 1, \qquad F'(0) = 1, \end{aligned}$$

where

$$\lambda(z) = \mathrm{e}^z, \qquad g(z) = 1 + z + \frac{z^{-\lambda(z)}(1 + z - \lambda(z))}{\Gamma(2 - \lambda(z))},$$

keeping in mind that the exact solution is $F(z) = z + 1$. Applying the method mentioned in Section 3 with different choice of N, we obtain results given in Table 4. The curve of the AE of Example 3 for $N = 10$ is displayed in Fig. 6.

Taking $N = 10$, we get the numerical solution of Example 3 in the form

$$\begin{aligned} F_{\mathrm{Approx}}(z) = 1 &+ z - 2.82369 \cdot 10^{-15}z^2 - 1.77001 \cdot 10^{-14}z^3 \\ &+ 1.02814 \cdot 10^{-13}z^4 - 3.34339 \cdot 10^{-13}z^5 \\ &+ 6.8861 \cdot 10^{-13}z^6 - 8.86254 \cdot 10^{-13}z^7 \\ &+ 6.88274 \cdot 10^{-13}z^8 - 2.95128 \cdot 10^{-13}z^9 \\ &+ 5.37018 \cdot 10^{-14}z^{10}. \end{aligned}$$

Table 4. Numerical solution of AEs for Example 3 with $N = 8, 10$

z	New method at		z	New method at	
	$N = 8$	$N = 10$		$N = 8$	$N = 10$
0	$5.55 \cdot 10^{-17}$	$1.67 \cdot 10^{-16}$	0.6	$9.44 \cdot 10^{-15}$	$3.11 \cdot 10^{-15}$
0.1	$2.08 \cdot 10^{-15}$	$5.55 \cdot 10^{-16}$	0.7	$1.07 \cdot 10^{-14}$	$3.50 \cdot 10^{-15}$
0.2	$3.91 \cdot 10^{-15}$	$1.17 \cdot 10^{-15}$	0.8	$1.18 \cdot 10^{-14}$	$3.89 \cdot 10^{-15}$
0.3	$5.53 \cdot 10^{-15}$	$1.78 \cdot 10^{-15}$	0.9	$1.28 \cdot 10^{-14}$	$4.33 \cdot 10^{-15}$
0.4	$6.94 \cdot 10^{-15}$	$2.30 \cdot 10^{-15}$	1.0	$1.38 \cdot 10^{-14}$	$4.55 \cdot 10^{-15}$
0.5	$8.27 \cdot 10^{-15}$	$2.72 \cdot 10^{-15}$			

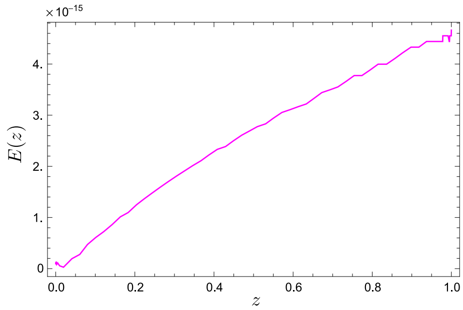

Figure 6. The AE using SCGC method with versus z in Example 3 for $N = 10$ and $\lambda(z) = e^z$.

Example 4. Finally, we consider the nonlinear VOFRDE [29]

$$D^{\lambda(z)}F(z) + F(z) + \sqrt{z}F^2(z) = g(z), \quad z \in [0, 1],$$
$$F(0) = 0,$$

where

$$\lambda(z) = z\cos z,$$
$$g(z) = x^q\left(-1 - x^{1/2+q} + \frac{x^{-\lambda(z)}\Gamma(1+q)}{\Gamma(1+q-\lambda(z))}\right),$$

keeping in mind that the exact solution is $F(z) = z^q$. We compare our method with that proposed in [29]. For various values of q, the MAE at difference N are listed in Table 5. For all choices of N and q, our method is more accurate than the method in [29]. The graph of the AEs of Example 4 is showed in Fig. 7. Moreover, we sketched in Fig. 8 the logarithmic graphs of M_E (i.e., $\log_{10} M_E$) obtained by the present method with different values of N.

Table 5. Comparison of the MAE with the other method Example 4 with various choices of N and q

N	Chebyshev bases [29]	New method	Chebyshev bases [29]	New method
	$q = 1.2$		$q = 1.4$	
5	$5.28 \cdot 10^{-2}$	$4.17 \cdot 10^{-2}$	$3.53 \cdot 10^{-2}$	$2.56 \cdot 10^{-2}$
8	$8.06 \cdot 10^{-3}$	$7.33 \cdot 10^{-3}$	$4.46 \cdot 10^{-3}$	$3.99 \cdot 10^{-3}$
11	$2.68 \cdot 10^{-3}$	$2.28 \cdot 10^{-3}$	$1.29 \cdot 10^{-3}$	$1.10 \cdot 10^{-3}$
14	$1.17 \cdot 10^{-3}$	$9.32 \cdot 10^{-4}$	$5.11 \cdot 10^{-4}$	$4.14 \cdot 10^{-4}$
17	$6.01 \cdot 10^{-4}$	$4.40 \cdot 10^{-4}$	$2.43 \cdot 10^{-4}$	$1.80 \cdot 10^{-4}$
20	$3.44 \cdot 10^{-4}$	$2.37 \cdot 10^{-4}$	$1.31 \cdot 10^{-4}$	$9.21 \cdot 10^{-5}$
N	Chebyshev bases [29]	New method	Chebyshev bases [29]	New method
	$q = 1.2$		$q = 1.4$	
5	$1.57 \cdot 10^{-2}$	$1.12 \cdot 10^{-2}$	$4.68 \cdot 10^{-3}$	$3.41 \cdot 10^{-3}$
8	$1.66 \cdot 10^{-3}$	$1.52 \cdot 10^{-3}$	$4.07 \cdot 10^{-4}$	$4.0 \cdot 10^{-4}$
11	$4.21 \cdot 10^{-4}$	$3.72 \cdot 10^{-4}$	$9.01 \cdot 10^{-5}$	$8.57 \cdot 10^{-5}$
14	$1.49 \cdot 10^{-4}$	$1.29 \cdot 10^{-4}$	$2.91 \cdot 10^{-5}$	$2.74 \cdot 10^{-5}$
17	$6.57 \cdot 10^{-5}$	$5.18 \cdot 10^{-5}$	$1.17 \cdot 10^{-5}$	$1.01 \cdot 10^{-5}$
20	$3.29 \cdot 10^{-5}$	$2.51 \cdot 10^{-5}$	$5.51 \cdot 10^{-6}$	$4.68 \cdot 10^{-6}$

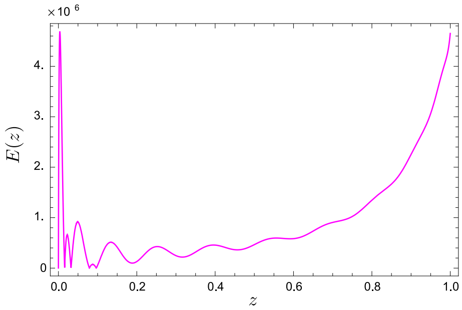

Figure 7. The AE using SCGC method with versus z in Example 4 for $N = 20$ and $\lambda(z) = z\cos z$.

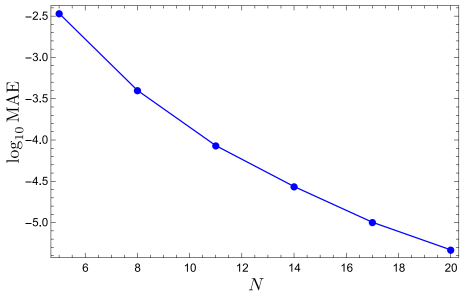

Figure 8. M_E convergence for Example 4 with various choices of N.

5 Conclusion

Our main goal is to provide and improve spectral algorithms to solve VOFRDEs, and it is acquired by means of the SJGC method. The novel algorithms are based upon decrease the aforementioned problems into a system of algebraic equations. We listed illustrative examples to examine the legality and applicability of the current algorithms. The given comparisons demonstrated the effectiveness and accuracy of the spectral collection method.

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