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M-fractional derivative under interval uncertainty: Theory, properties and applications

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ABSTRACT

In the recent years some efforts were made to propose simple and well-behaved fractional derivatives that inherit the classical properties from the first order derivative. In this regards, the truncated *M*-fractional derivative for α-differentiable function was recently introduced that is a generalization of four fractional derivatives presented in the literature and has their important features. In this research, we aim to generalize this novel and effective derivative under interval uncertainty. The concept of interval truncated *M*-fractional derivative is introduced and some of the distinguished properties of this interesting fractional derivative such as Rolle's and mean value theorems, are developed for the interval functions. In addition, the existence and uniqueness conditions of the solution for the interval fractional differential equations (IFDEs) based on this new derivative are also investigated. Finally, we present the applicability of this novel interval fractional derivative for IFDEs based on the notion of w-increasing (w-decreasing) by solving a number of test problems.

1. Introduction

The fractional calculus is a standout amongst the most precise devices to redefine the natural phenomena. In the recent few decades a lot of researchers have shown their interest due to its frequent appearances in many fields of science and engineering. It enhances the accuracy of models due to various types of fractional derivatives and nature [1–11]. In fact, the use of fractional order differential equation for the modelling is more effective than integer order derivatives which can easily explain the hereditary properties and memory [12–21].

While these studies have been carried out, scientists used different definitions of fractional derivative and integral such as Grünwald-Letnikov, Riesz-Fischer, Caputo, Riemann-Liouville, and modified Riemann-Liouville. But almost all of these derivatives have some kind of flaws. For instance, the Riemann-Liouville fractional derivative of a constant is not zero, the Riemann-Liouville derivative and Caputo derivative do not obey the Leibnitz rule and chain rule. The Riemann-Liouville derivative and Caputo do not satisfy the known formula of the derivative of the quotient of two functions [22].

To overcome the above mentioned shortcomings, recently a new type of fractional derivative called conformable fractional derivative has been introduced by Khalil et al. [22]. The applicability, and effectiveness of the conformable derivative have been studied in various fields [23–25]. Hence, it was found worthwhile to work on this new area and develop it with some new features. In this regards, Sousa et al. [26] introduced a generalization of four fractional derivatives including conformable fractional derivative, alternative fractional derivative, generalized alternative fractional derivative [27] and *M*-fractional derivative [28]. It is called truncated *M*-fractional derivative for α-differentiable functions. This new fractional derivative behaves well with respect to the classical properties of the integer-order derivative.

On the other hand, the interval arithmetic and interval differential equations (IDEs) have not been considered enough from the time that were introduced by Markove [29]. In fact, he was the pioneer in this field to propose the interval-valued function, interval derivatives and etc. However, in the recent years, the scientists found the applicability of this significant notion that measures uncertainties in the mathematical modeling with uncertain parameters. Therefore, a number of researches have been done in this re-

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gards to analyze the mathematical systems based on the interval parameters and study the existence and uniqueness of the interval solutions of the IDEs [30–35]. As a matter of fact, interval arithmetic is a branch of fuzzy sets that deals with the intervals from the first step of modeling or numerical algorithm that can reduce the complexity and computational difficulties compared with fuzzy systems.

Considering the above circumstances, we motivated to generalize truncated M-fractional derivative for interval fractional differential equations (IFDEs). Similar to the integer order differential equations, a limited number of researches devoted to IFDEs, even a few studies have been done for fuzzy fractional calculus [36–48]. Because of the vast applications of interval and fuzzy arithmetic arising day to day in the engineering problems [49,50], it is crucial to take these important fields into the consideration. In this regards, we study also in the current proposal the existence and uniqueness of the interval solutions of IFDEs based on the interval truncated M-fractional derivative. The significant and applicable classical properties of this novel fractional derivative are developed under interval uncertainty. To the best of our knowledge, this research has enough potential to open a new window to the interval uncertainty since it proposes a new model for a real-world system (Kelvin-Voigt system) based on this fractional derivative and the behavior of the exact solution is analyzed by considering the concept of w-increasing (decreasing) presented in [29,30].

The paper is organized as follows: in Section 2, some important concept of interval arithmetic are revisited. Besides, the definition and properties of truncated M-fractional derivative are recalled in this section. Section 3 includes the main results of this report. We propose in this section the notion of interval truncated M-fractional derivative and discuss some of important classical properties of this derivative for interval-valued functions. A number of problems including the Kelvin-Voigt models are developed in Section 4 based on this new interval fractional derivative and the procedure to achieve the exact solution is explained complexity. The results demonstrate the applicability and validity of this new concept. To summarize, some conclusions and suggestions for the future works are drawn.

2. Preliminaries and notation

In the current section, we revisit some important features of the interval arithmetic presented in [35]. Also, the definition and some necessary properties of the truncated M-fractional derivative are recalled [28]. Interested reader are referred for a comprehensive study of the fractional calculus to [1–3].

2.1. Interval arithmetic

Let $\mathcal{F}$ indicate the family of all nonempty, compact and convex intervals of the real line $\mathbb{R}$. The addition and scalar multiplication in $\mathcal{F}$, we define as usual, i.e. for $\mathrm{P}, \mathrm{Q} \in \mathcal{F}, \mathrm{P} = [\underline{\mathrm{p}}, \overline{\mathrm{p}}], \mathrm{Q} = [\underline{\mathrm{q}}, \overline{\mathrm{q}}], \underline{\mathrm{p}} \leq \overline{\mathrm{p}}, \underline{\mathrm{q}} \leq \overline{\mathrm{q}}$, and $\varpi \geq 0$ we have

$$\mathrm{P} + \mathrm{Q} = [\underline{\mathrm{p}} + \underline{\mathrm{q}}, \overline{\mathrm{p}} + \overline{\mathrm{q}}],$$
$$\varpi \mathrm{P} = [\varpi \underline{\mathrm{p}}, \varpi \overline{\mathrm{p}}] \quad (-\varpi)\mathrm{P} = [-\varpi \overline{\mathrm{p}}, -\varpi \underline{\mathrm{p}}].$$

The Hausdorff metric $\mathcal{H}$ in $\mathcal{F}$ is defined as follows:

$$\mathcal{H}(\mathrm{P}, \mathrm{Q}) = \max\{|\underline{\mathrm{p}} - \underline{\mathrm{q}}|, |\overline{\mathrm{p}} - \overline{\mathrm{q}}|\},$$

for $\mathrm{P} = [\underline{\mathrm{p}}, \overline{\mathrm{p}}]$, $\mathrm{Q} = [\underline{\mathrm{q}}, \overline{\mathrm{q}}]$. It is known (see e.g. [35]) that $(\mathcal{F}, \mathcal{H})$ is a complete, separable and locally compact metric space.

Let $\mathrm{P}, \mathrm{Q} \in \mathcal{F}$. If there exists an interval $\mathrm{T} \in \mathcal{F}$ such that $\mathrm{P} = \mathrm{Q} + \mathrm{T}$, then we call T the *Hukuhara difference* (H-difference for short) of P and Q. We denote the interval T by $\mathrm{P} \ominus \mathrm{Q}$. Note that $\mathrm{P} \ominus \mathrm{Q} \neq \mathrm{P} + (-1)\mathrm{Q}$.

As it was stated above, the H-difference is unique, but it does not always exist. A generalization of the Hukuharra difference is proposed in [51] to overcome this shortcoming.

Definition 2.1. The generalized Hukuhara difference of two fuzzy numbers $u_1, u_2 \in \mathcal{F}$ (gH-difference for short) is defined as follows

$$u_1 \ominus_g u_2 = u_3 \Leftrightarrow \begin{cases} (i)\ u_1 = u_2 + u_3, \\ \text{or} \\ (ii)\ u_2 = u_1 + (-1)u_3, \end{cases} \tag{2.1}$$

in which $u_3 \in \mathcal{F}$.

The most important definition of the interval derivative, based on the fuzzy differentiability concept introduced in [51], is defined as:

Definition 2.2. Let $\mathcal{Y} : (a, b) \to \mathcal{F}$ and $\omega \in (a, b)$. We say that $\mathcal{Y}$ is strongly generalized (Hukuhara) differentiable at ω, if there exists an element $\mathcal{Y}'(\omega) \in \mathcal{F}$, such that $\mathcal{Y}'(\omega)$ satisfies in one of the following cases:

(i) for all $h > 0$ sufficiently small, $\exists \mathcal{Y}(\omega + h) \ominus \mathcal{Y}(\omega)$, $\exists \mathcal{Y}(\omega) \ominus \mathcal{Y}(\omega - h)$ and
$$\lim_{h \searrow 0} \frac{\mathcal{Y}(\omega+h) \ominus \mathcal{Y}(\omega)}{h} = \lim_{h \searrow 0} \frac{\mathcal{Y}(\omega) \ominus \mathcal{Y}(\omega-h)}{h}, = \mathcal{Y}'(\omega)$$

(ii) for all $h > 0$ sufficiently small, $\exists \mathcal{Y}(\omega) \ominus \mathcal{Y}(\omega + h)$, $\exists \mathcal{Y}(\omega - h) \ominus \mathcal{Y}(\omega)$ and
$$\lim_{h \searrow 0} \frac{\mathcal{Y}(\omega) \ominus \mathcal{Y}(\omega+h)}{-h} = \lim_{h \searrow 0} \frac{\mathcal{Y}(\omega-h) \ominus \mathcal{Y}(\omega)}{-h}, = \mathcal{Y}'(\omega)$$

(iii) for all $h > 0$ sufficiently small, $\exists \mathcal{Y}(\omega + h) \ominus \mathcal{Y}(\omega)$, $\exists \mathcal{Y}(\omega - h) \ominus \mathcal{Y}(\omega)$ and
$$\lim_{h \searrow 0} \frac{\mathcal{Y}(\omega+h) \ominus \mathcal{Y}(\omega)}{h} = \lim_{h \searrow 0} \frac{\mathcal{Y}(\omega-h) \ominus \mathcal{Y}(\omega)}{-h}, = \mathcal{Y}'(\omega)$$

(iv) for all $h > 0$ sufficiently small, $\exists \mathcal{Y}(\omega) \ominus \mathcal{Y}(\omega + h)$, $\exists \mathcal{Y}(\omega) \ominus \mathcal{Y}(\omega - h)$ and
$$\lim_{h \searrow 0} \frac{\mathcal{Y}(\omega) \ominus \mathcal{Y}(\omega+h)}{-h} = \lim_{h \searrow 0} \frac{\mathcal{Y}(\omega) \ominus \mathcal{Y}(\omega-h)}{h}, = \mathcal{Y}'(\omega)$$

Next is definition of the generalized Hukuhara differentiability (gH-differentiable for short) which was introduced in [35] based on the gH-difference.

Definition 2.3. Let $t \in (a, b)$ and h be such that $t + h \in (a, b)$, then the generalized Hukuhara derivative of a fuzzy-valued function $x : (a, b) \to \mathcal{F}$ is defined as

$$x'_{gH}(t) = \lim_{h \to 0} \frac{x(t+h) \ominus_g x(t)}{h}. \tag{2.2}$$

If $x'_{gH}(t) \in \mathcal{F}$ satisfying Eq. (2.2) exists, we say that x is gH-differentiable at t. Also, we say that x is [(i)-gH]-differentiable at t, if x satisfies in Definition (2.2)-(i), then we have $x'_{gH}(t) = [\underline{x}'(t), \overline{x}'(t)]$, similarly, x is [(ii)-gH]-differentiable at t, if x satisfies in Definition (2.2)-(ii), then we have $x'_{gH}(t) = [\overline{x}'(t), \underline{x}'(t)]$.

We say that an interval-valued function $F : [a, b] \to \mathcal{F}$ is *w-increasing* (w-decreasing) on $[a, b]$ if the real function $t \to w_F(t) := w(F(t))$ is increasing (decreasing) on $[a, b]$. If F is w-increasing or w-decreasing on $[a, b]$, then we say that F is w-monotone on $[a, b]$ (see, [30]).

Proposition 2.1. *(see, [29]). Let $F : [a, b] \to \mathcal{F}$ be such that $F(t) = [f^-(t), f^+(t)], t \in [a, b]$. If F is w-monotone and gH-differentiable on*

$[a, b]$, then $\frac{d}{dt}f^-(t)$ and $\frac{d}{dt}f^+(t)$ exist for all $t \in [a, b]$. Moreover, we have that:

(i) $F'(t) = [\frac{d}{dt}f^-(t), \frac{d}{dt}f^+(t)]$ for all $t \in [a, b]$, if F is w-increasing,

(ii) $F'(t) = [\frac{d}{dt}f^+(t), \frac{d}{dt}f^-(t)]$ for all $t \in [a, b]$, if F is w-decreasing.

2.2. Truncated M-fractional derivative

Definition 2.4. The truncated Mittag-Leffler function with one parameter is defined as follows [26]:

$$_iE_\beta(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(\beta k + 1)}, \tag{2.3}$$

in which $\beta > 0$ and $z \in \mathbb{C}$.

The truncated M-fractional derivative in the sense of non-fuzzy concept is stated in the following definition [28].

Definition 2.5. Let $f : [0, \infty) \to \mathbb{R}$. For $0 < \alpha < 1$ the truncated M-fractional derivative of f of order α, indicated by

$$_if_\beta^{(\alpha)}(x) := \lim_{\varepsilon \to 0} \frac{f(x_iE_\beta(\varepsilon x^{-\alpha})) - f(x)}{\varepsilon} \tag{2.4}$$

$\forall t > 0$ and $_iE_\beta(.)$, $\beta > 0$ is a truncated Mittag-Leffler function of one parameter, prescribed in Definition 2.4.

Some classical properties of the new fractional derivative are presented in the next theorem.

Theorem 2.1. *Assume that $0 < \alpha \le 1$, $\beta > 0$, $a, b \in \mathbb{R}$ and f, g are α-differentiable at point $x > 0$. Therefore:*

(1) $_i(af + bg)_\beta^{(\alpha)}(x) = a_if_\beta^{(\alpha)}(x) + b_ig_\beta^{(\alpha)}(x)$

(2) $_i(f.g)_\beta^{(\alpha)}(x) = f(x)_ig_\beta^{(\alpha)}(x) + g(x)_if_\beta^{(\alpha)}(x)$

(3) $_i(\frac{f}{g})_\beta^{(\alpha)}(x) = \frac{g(x)_if_\beta^{(\alpha)}(x) - f(x)_ig_\beta^{(\alpha)}(x)}{[g(t)]^2}$

(4) $_i(c)_\beta^{(\alpha)} = 0$, where $f(t) = c$ is a constant.

(5) *(Chain rule) If f is differentiable, then* $_if_\beta^{(\alpha)}(x) = \frac{x^{1-\alpha}}{\Gamma(\beta+1)} \frac{df(x)}{dx}$.

The M-fractional integral of order α is defined in the next definition.

Definition 2.6. Let $a \ge 0$ and $t \ge a$. Also, let f be a function defined in $(a, t]$ and $0 < \alpha < 1$. Then, the M-fractional integral of order α of function f is defined by

$$\left(I_a^{\alpha,\beta} f\right)(t) = \Gamma(\beta + 1) \int_a^t \frac{f(x)}{x^{1-\alpha}} dx, \tag{2.5}$$

in which $b > 0$.

3. Main results

In this section, the interval truncated M-fractional derivative is presented and its important relevant classical properties are provided in details. Furthermore, the existence and uniqueness conditions of the interval solution of the IFDEs based on this new interval fractional derivative are analyzed.

Considering the Definition 2.4, we introduce the interval truncated M-fractional derivative as follows:

Definition 3.1. Let $f \in \mathcal{F}$ and $0 < \alpha \le 1$. The interval truncated M-fractional derivative is defined as:

$$_if_\beta^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f(t_iE_\beta(\varepsilon t^{-\alpha})) \ominus_g f(t)}{\varepsilon}, \tag{3.1}$$

for all β, $t > 0$. Also, we have

$$_if_\beta^{(\alpha)}(0) = \lim_{t \to 0^+} {}_if_\beta^{(\alpha)}(t).$$

Note that for the simplicity, if f satisfies in Definition 3.1, then we say that f is α-differentiable.

The following theorem discusses the continuity of an interval function f that is α-differentiable under specified conditions.

Theorem 3.1. *Assume that $f \in \mathcal{F}$ is α-differentiable at t_0 and $0 < \alpha \le 1$, $\beta > 0$. Then, f is a continuous function.*

Proof. In fact we have:

$$f(t_{0i}E_\beta(\varepsilon t_0^{-\alpha})) \ominus_g f(t_0) = \frac{f(t_{0i}E_\beta(\varepsilon t_0^{-\alpha})) \ominus_g f(t_0)}{\varepsilon} \times \varepsilon.$$

Then

$$\begin{aligned} \lim_{\varepsilon \to 0} \left[f(t_{0i}E_\beta(\varepsilon t_0^{-\alpha})) \ominus_g f(t_0)\right] &= \lim_{\varepsilon \to 0} \frac{f(t_{0i}E_\beta(\varepsilon t_0^{-\alpha})) \ominus_g f(t_0)}{\varepsilon} \times \lim_{\varepsilon \to 0} \varepsilon \\ &= {}_if_\beta^{(\alpha)}(t_0) \times \lim_{\varepsilon \to 0} \varepsilon = 0. \end{aligned}$$

Hence, f is continuous at t_0. □

In the next definition, the concept of gH-differentiability, stated in Definition 2.3, is generalized for the interval truncated M-fractional derivative.

Definition 3.2. Let $f \in \mathcal{F}$ then:

(i) We say that f is $(\alpha, 2)_\beta$-differentiable, if there exists $_if_\beta^{(\alpha)}(x) \in \mathcal{F}$ such that for all $\varepsilon > 0$ sufficiently small, $f(x) \ominus f(x_iE_\beta(\varepsilon x^{-\alpha}))$ exist and the limits

$$\lim_{\varepsilon \to 0} \frac{f(x) \ominus f(x_iE_\beta(\varepsilon x^{-\alpha}))}{\varepsilon} =_i f_\beta^{(\alpha)}(x).$$

(ii) We say that f is $(\alpha, 1)_\beta$-differentiable, if there exists $_if_\beta^{(\alpha)}(x) \in \mathcal{F}$ such that for all $\varepsilon > 0$ sufficiently small, $f(x_iE_\beta(\varepsilon x^{-\alpha})) \ominus f(x)$ exist and the limits

$$\lim_{\varepsilon \to 0} \frac{f(x_iE_\beta(\varepsilon x^{-\alpha})) \ominus f(x)}{\varepsilon} =_i f_\beta^{(\alpha)}(x).$$

Theorem 3.2. *Let $f(x) = [f_1(x), f_2(x)]$ is α-dfferentiable and w-monotone on (a, b), then for every $x \in (a, b)$, the derivative $f_1^{(\alpha)}(x)$ and $f_2^{(\alpha)}(x)$ exist and $f^{(\alpha)}(x) = [f_1^{(\alpha)}(x), f_2^{(\alpha)}(x)]$, if f is w-increasing and $f^{(\alpha)}(x) = [f_2^{(\alpha)}(x), f_1^{(\alpha)}(x)]$, if f is w-decreasing.*

Proof. Markov [29] proposed the concept of interval differentiability for interval differential equations of the first order. In fact, considering Definition (3.1), we generalize aforesaid concept for IFDEs under the assumption of w-increasing:

$$\begin{aligned} _if_\beta^{(\alpha)}(x) &= \lim_{\varepsilon \to 0} \frac{f(x_iE_\beta(\varepsilon x^{-\alpha})) \ominus f(x)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\left[f_1(x_iE_\beta(\varepsilon x^{-\alpha})), f_2(x_iE_\beta(\varepsilon x^{-\alpha}))\right] \ominus [f_1(x), f_2(x)]}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\left[f_1(x_iE_\beta(\varepsilon x^{-\alpha})) - f_1(x), f_2(x_iE_\beta(\varepsilon x^{-\alpha})) - f_2(x)\right]}{\varepsilon} \\ &= \left[{}_i\left(f_{\beta,1}^{(\alpha)}\right)(x), {}_i\left(f_{\beta,2}^{(\alpha)}\right)(x).\right] \end{aligned}$$

For the case under the assumption of w-decreasing, the procedure is similar to prove the results, hence we omit it.

□

Proposition 3.1. *In [26], the authors have shown that*

$$\lim_{\varepsilon \to 0} f(t_iE_\beta(\varepsilon t^{-\alpha})) = f\left(t \lim_{\varepsilon \to 0} \sum_{k=0}^{i} \frac{(\varepsilon t^{-\alpha})^k}{\Gamma(\beta k + 1)}\right).$$

Using the fact that $\lim_{\varepsilon\to 0}\sum_{k=0}^{i}\frac{(\varepsilon t^{-\alpha})^k}{\Gamma(\beta k+1)}=1$, *we have:*

$$\lim_{\varepsilon\to 0} f(t_iE_\beta(\varepsilon t^{-\alpha}))=f(t),$$

in which f is a continuous function.

Now, we present some classical properties of the interval α-derivative in the following theorem.

Theorem 3.3. *Suppose that* $0<\alpha\le 1$, $\beta>0$, $a,b\in\mathbb{R}$ *and f, g are* α*-differentiable. The, the following properties hold:*

(i) ${}_i(af(t)+bg(t))_\beta^{(\alpha)}=a_if_\beta^{(\alpha)}(t)+b_ig_\beta^{(\alpha)}(t).$

(ii) ${}_if_\beta^{(\alpha)}(t)=0,\quad$ where $\quad f(t)=c\in\mathcal{F}.$

(iii) ${}_if_\beta^{(\alpha)}(t)=\frac{t^{1-\alpha}}{\Gamma(\beta+1)}.\,f'(t),\quad$ *if f is gH-differentiable.*

Proof. We choose to prove (iii), and the others can be proved directly from the definition. From the Eq. (2.6) in [26] we have:

$$\begin{aligned}t_iE_\beta(\varepsilon t_0^{-\alpha})&=t\sum_{k=0}^{i}\frac{(\varepsilon t^{-\alpha})^k}{\Gamma(\beta k+1)}\\&=t+\frac{\varepsilon t^{1-\alpha}}{\Gamma(\beta+1)}+\frac{t(\varepsilon t^{-\alpha})^2}{\Gamma(2\beta+1)}+\frac{t(\varepsilon t^{-\alpha})^3}{\Gamma(3\beta+1)}\\&\quad+\ldots+\frac{t(\varepsilon t^{-\alpha})^i}{\Gamma(i\beta+1)}.\end{aligned}$$

Therefore, we can truncate the above series and state it as follows:

$$t_iE_\beta(\varepsilon t_0^{-\alpha})=t+\frac{\varepsilon t^{1-\alpha}}{\Gamma(\beta+1)}+O(\varepsilon^2).$$

If $h=\varepsilon t^{1-\alpha}\left(\frac{1}{\Gamma(\beta+1)}+O(\varepsilon)\right)$, then

$$\varepsilon=\frac{h}{t^{1-\alpha}\left(\frac{1}{\Gamma(\beta+1)}+O(\varepsilon)\right)},$$

and we have:

$$\begin{aligned}{}_if_\beta^{(\alpha)}(t)&=\frac{t^{1-\alpha}}{\Gamma(\beta+1)}\lim_{h\to 0}\frac{\frac{f(t+h)\ominus_g f(t)}{h}}{1+\Gamma(\beta+1)O(\varepsilon)}\\&=\frac{t^{1-\alpha}}{\Gamma(\beta+1)}.f'(t),\end{aligned}$$

where $\beta>0$.

□

Theorem 3.4. $f,g\in\mathcal{F}$ *and differentiable of the same type (both of them are* $(\alpha,\,1)_\beta$*-differentiable or* $(\alpha,\,2)_\beta$*-differentiable). Then,* $(f\ominus_g g)(x)$ *is differentiable and*

$${}_i(f\ominus_g g)_\beta^{(\alpha)}(x)={}_if_\beta^{(\alpha)}(x)\ominus_{gi}g_\beta^{(\alpha)}(x),$$

where $0<\alpha<1$, $\beta>0$, $x\in(a,\,b)$.

Proof. Let f, g are $(\alpha,\,1)_\beta$-differentiable. Then,

$$f(x)\ominus_g g(x)=k(x)\Leftrightarrow\begin{cases}(i)\;f(x)=g(x)+k(x)\\(ii)f(x)=g(x)-k(x)\end{cases}$$

under case (i), we have:

$${}_if_\beta^{(\alpha)}(x)={}_ig_\beta^{(\alpha)}(x)+{}_ik_\beta^{(\alpha)}(x),$$

then,

$${}_ik_\beta^{(\alpha)}(x)={}_if_\beta^{(\alpha)}(x)\ominus_{gi}g_\beta^{(\alpha)}(x).$$

Similarly, for case (ii),

$${}_ik_\beta^{(\alpha)}(x)=-{}_ig_\beta^{(\alpha)}(x)\ominus(-{}_if_\beta^{(\alpha)}(x)).$$

Hence, we conclude that

$${}_i(f\ominus_g g)_\beta^{(\alpha)}(x)={}_if_\beta^{(\alpha)}(x)\ominus_{gi}g_\beta^{(\alpha)}(x).$$

□

Allahviranloo et al. [53] have used the following partial ordering in $\mathcal{F}$ proposed by Kaleva [52]:

For $u,v\in\mathcal{F}, u\le v$ if and only if $\underline{u}(\alpha)\le\underline{v}(\alpha)$ and $\bar{u}(\alpha)\le\bar{v}(\alpha)$,

Using the above partial ordering, they obtained Rolle's theorem and fuzzy Cauchy mean value theorem under fuzzy notion. Similarly, we aim to develop some new and interesting results such as interval Rolle's theorem and interval mean value theorem for interval truncated M-fractional derivative as follows.

Theorem 3.5. *Let* $c\in(a,\,b)$, *and* $f:[a,b]\to\mathcal{F}$. *Suppose that f is* $(\alpha,\,\dagger)_\beta$*-differentiable at c,* $\dagger\in\{1,\,2\}$. *If f has a local maximum or minimum at c, then* ${}_if_\beta^{(\alpha)}(c)=0$

Proof. Without loss of generality, let c is a local minimum. For $x\in(a,\,c)$ we have $f(x)\ominus_g f(c)\le 0$ and $x-c<0$, then set $x=t_iE_\beta(\varepsilon t^{-\alpha})$, we have

$$\frac{f(t_iE_\beta(\varepsilon t^{-\alpha}))\ominus_g f(c)}{t_iE_\beta(\varepsilon t^{\alpha})-c}\ge 0.$$

It follows that ${}_if_\beta^{(\alpha)}(c)\geqslant$. Similarly, for $x\in(c,\,b)$, we have

$$\frac{f(t_iE_\beta(\varepsilon t^{-\alpha}))\ominus_g f(c)}{t_iE_\beta(\varepsilon t^{\alpha})-c}\le 0,$$

hence, we have ${}_if_\beta^{(\alpha)}(c)\le 0$. Finally, we deduce that ${}_if_\beta^{(\alpha)}(c)=0$ □

Theorem 3.6. *(Interval Rolle's theorem). Assume that* $f\in\mathcal{F}$. *It is differentiable on* $(a,\,b)$ *and continuous on* $[a,\,b]$. *If* $f(a)=f(b)$, *then there exist some* $c\in(a,\,b)$ *such that*

$${}_if_\beta^{(\alpha)}(c)=0,\quad 0<\alpha<1,\;\beta>0.$$

Proof. The proof is analogously to the demonstration of the proof of Lemma 2.4 in [53]. In fact, we have $f(x_{\min})\le f(x)\le f(x_{\max})$ for all $x\in[a,\,b]$. If $f(x_{\min})=f(x_{\max})$, then f should be a constant function and we have:

$${}_if_\beta^{(\alpha)}(x)=0,\;\text{for all } x\in(a,b).$$

If $f(x_{\min})\le f(x_{\max})$, at least $f(x_{\min})$ or $f(x_{\max})$ does not equal to $f(a)$ (or $f(b)$). Suppose that

$$f(x_{\max})\neq f(a).$$

Let $c=x_{\max}$, then $c\in(a,\,b)$ and by using Theorem 3.5, we have ${}_if_\beta^{(\alpha)}(c)=0$ that completes the proof. □

Theorem 3.7. *(Interval mean value theorem). Suppose that* $f\in\mathcal{F}$. *It is differentiable on* $(a,\,b)$ *and continuous on* $[a,\,b]$ *then, there exists some* $c\in(a,\,b)$ *such that*

$${}_if_\beta^{(\alpha)}(c)=\left(\frac{f(b)\ominus_g f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}\right)\left(\frac{1}{\Gamma(\beta+1)}\right),$$

in which $0<\alpha<1,\beta>0$.

Proof. Suppose that

$$\Upsilon(x)=(f(x)\ominus_g f(a))\ominus_g\left(\frac{f(b)\ominus_g f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}\right)\left(\frac{x^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right).$$

Then, it is easy to verify that

$$\Upsilon(a)=(f(a)\ominus_g f(a))\ominus_g\left(\frac{f(b)\ominus_g f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}\right)\left(\frac{a^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right)=0,$$

and

$$\Upsilon(b) = (f(b)\ominus_g f(a))\ominus_g\left(\frac{f(b)\ominus_g f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}\right)\left(\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right)$$
$$= (f(b)\ominus_g f(a))\ominus_g(f(b)\ominus_g f(a)) = 0.$$

Then using interval Rolle's theorem, we have:

$$\exists\, c\in(a,b)\quad s.t.\ \ {}_i\Upsilon_\beta^{(\alpha)}(c) = 0$$

Hence,

$${}_i(f(x)\ominus_g f(a))_\beta^{(\alpha)}\ominus_g\,(f(b)\ominus_g f(a))\ominus_g\left(\frac{f(b)\ominus_g f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}\right)_i$$
$$\left(\frac{x^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right)_\beta^{(\alpha)} = 0.$$

Then, we have:

$${}_if_\beta^{(\alpha)}(x)\ominus_g\left(\frac{f(b)\ominus_g f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}\right).\frac{1}{\Gamma(\beta+1)} = 0$$

and finally we obtain:

$${}_if_\beta^{(\alpha)}(c) = \frac{f(b)\ominus_g f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}.\frac{1}{\Gamma(\beta+1)}.$$

In fact, we used the results ${}_i\left(\frac{x^\alpha}{\alpha}\right)_\beta^{(\alpha)} = \frac{1}{\Gamma(\beta+1)}$ and ${}_i\left(\frac{a^\alpha}{\alpha}\right)_\beta^{(\alpha)} = 0$ that completes the proof. □

Remark 3.1. Clearly, once β becomes 0, the results reduce to the Rolle' theorem under conformable derivative with interval uncertainty.

In the next definition, the interval fractional integral of the interval function f is defined.

Definition 3.3. Let $0\le a\le t$ and $f\in\mathcal{F}$ is integrable over $(a, t]$. Then, the fractional integral of order α for $0<\alpha\le 1$ is defined by

$$\left(I_a^{\alpha,\beta}f\right)(t) = \Gamma(\beta+1)\int_a^t\frac{f(x)dx}{x^{1-\alpha}}. \tag{3.2}$$

Theorem 3.8. *Let $a\ge 0$ and $0<\alpha\le 1$. Then for a given continuous interval-valued function f we have:*

$${}_i\left(I_a^{\alpha,\beta}f\right)_\beta^{(\alpha)}(t) = f(t). \tag{3.3}$$

Proof

$${}_i\left(I_a^{\alpha,\beta}f\right)_\beta^{(\alpha)}(t) = \frac{t^{1-\alpha}}{\Gamma(\beta+1)}\frac{d}{dt}(I_a^{\alpha,\beta}f)(t)$$
$$= \frac{t^{1-\alpha}}{\Gamma(\beta+1)}.\Gamma(\beta+1)\frac{d}{dt}\int_a^t\frac{f(x)dx}{x^{1-\alpha}} = f(t).$$

Therefore, the proof is complete. □

Theorem 3.9. *Let $f\in\mathcal{F}$ be a α-differentiable interval-valued function for $0<\alpha\le 1$. Then*

$$I_a^{\alpha,\beta}\left({}_if_\beta^{(\alpha)}\right)(t) = f(t)\ominus_g f(a).$$

Proof. Let f is α-differentiable under $(\alpha,\,1)_\beta$-differentiability, then

$$I_a^{\alpha,\beta}\left({}_if_\beta^{(\alpha)}\right)(t) = \Gamma(\beta+1)\int_a^t\frac{{}_if_\beta^{(\alpha)}(x)dx}{x^{1-\alpha}}$$
$$= \Gamma(\beta+1)\int_a^t\frac{x^{1-\alpha}}{\Gamma(\beta+1)}.\frac{1}{x^{1-\alpha}}f'(x)dx$$
$$= \left[\int_a^t f_1'(x)dx,\int_a^t f_2'(x)dx\right]$$
$$= [f_1(t)-f_1(a),\, f_2(t)-f_2(a)]$$
$$= f(t)\ominus f(a).$$

Similarly, if f is α-differentiable under $(\alpha,\,2)_\beta$-differentiability, then

$$I_a^{\alpha,\beta}\left({}_if_\beta^{(\alpha)}\right)(t) = \Gamma(\beta+1)\int_a^t\frac{{}_if_\beta^{(\alpha)}(x)dx}{x^{1-\alpha}}$$
$$= \int_a^t f'(x)dx$$
$$= \left[\int_a^t f_2'(x)dx,\int_a^t f_1'(x)dx\right]$$
$$= [f_2(t)-f_2(a),\, f_1(t)-f_1(a)]$$
$$= -f(a)\ominus(-f(t)).$$

□

Theorem 3.10. *Let $f\in\mathcal{F}$ be a continuous interval-valued function, then*

$$\mathcal{H}\left((I_a^{\alpha,\beta}f)(t),0\right)\leqslant I_a^{\alpha,\beta}\mathcal{H}(f(t),0), \tag{3.4}$$

also, if $K=\sup_{t\in[a,b]}\mathcal{H}(f(t),0)$, then

$$\mathcal{H}\left((I_a^{\alpha,\beta}f)(t),0\right)\leqslant\Gamma(\beta+1)K\left(\frac{t^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right).$$

Proof. The proofs are straightforward. □

Theorem 3.11. *Let $f\in\mathbb{R}$ and $g\in\mathcal{F}$ are two differentiable functions, then*

(1) *If $f(x).{}_if_\beta^{(\alpha)}(x)>0$ and g is $(\alpha,\,1)_\beta$-differentiable. Then, f.g is $(\alpha,\,1)_\beta$-differentiable and*

$${}_i(f.g)_\beta^{(\alpha)}(x) = {}_if_\beta^{(\alpha)}(x).g(x)+f(x).{}_ig_\beta^{(\alpha)}(x). \tag{3.5}$$

(2) *If $f(x).{}_if_\beta^{(\alpha)}(x)<0$ and g is $(\alpha,\,2)_\beta$-differentiable. Then, f.g is $(\alpha,\,2)_\beta$-differentiable and*

$${}_i(f.g)_\beta^{(\alpha)}(x) = {}_if_\beta^{(\alpha)}(x).g(x)+f(x).{}_ig_\beta^{(\alpha)}(x). \tag{3.6}$$

(3) *If $f(x).{}_if_\beta^{(\alpha)}(x)>0$, g is $(\alpha,\,2)_\beta$-differentiable and f.g satisfies S_1 at x. Then, f.g is $(\alpha,\,1)_\beta$-differentiable and*

$${}_i(f.g)_\beta^{(\alpha)}(x) = {}_if_\beta^{(\alpha)}(x).g(x)\ominus(-1)f(x).{}_ig_\beta^{(\alpha)}(x). \tag{3.7}$$

(4) *If $f(x).{}_if_\beta^{(\alpha)}(x)>0$, g is $(\alpha,\,2)_\beta$-differentiable and f.g satisfies S_2 at x. Then, f.g is $(\alpha,\,2)_\beta$-differentiable and*

$${}_i(f.g)_\beta^{(\alpha)}(x) = f(x).{}_ig_\beta^{(\alpha)}(x)\ominus(-1){}_if_\beta^{(\alpha)}(x).g(x). \tag{3.8}$$

(5) *If $f(x).{}_if_\beta^{(\alpha)}(x)<0$, g is $(\alpha,\,1)_\beta$-differentiable and f.g satisfies S_1 at x. Then, f.g is $(\alpha,\,1)_\beta$-differentiable and*

$${}_i(f.g)_\beta^{(\alpha)}(x) = f(x).{}_ig_\beta^{(\alpha)}(x)\ominus(-1){}_if_\beta^{(\alpha)}(x).g(x). \tag{3.9}$$

(6) *If $f(x).{}_if_\beta^{(\alpha)}(x)<0$, g is $(\alpha,\,1)_\beta$-differentiable and f.g satisfies S_2 at x. Then, f.g is $(\alpha,\,2)_\beta$-differentiable and*

$${}_i(f.g)_\beta^{(\alpha)}(x) = {}_if_\beta^{(\alpha)}(x).g(x)\ominus(-1)f(x).{}_ig_\beta^{(\alpha)}(x). \tag{3.10}$$

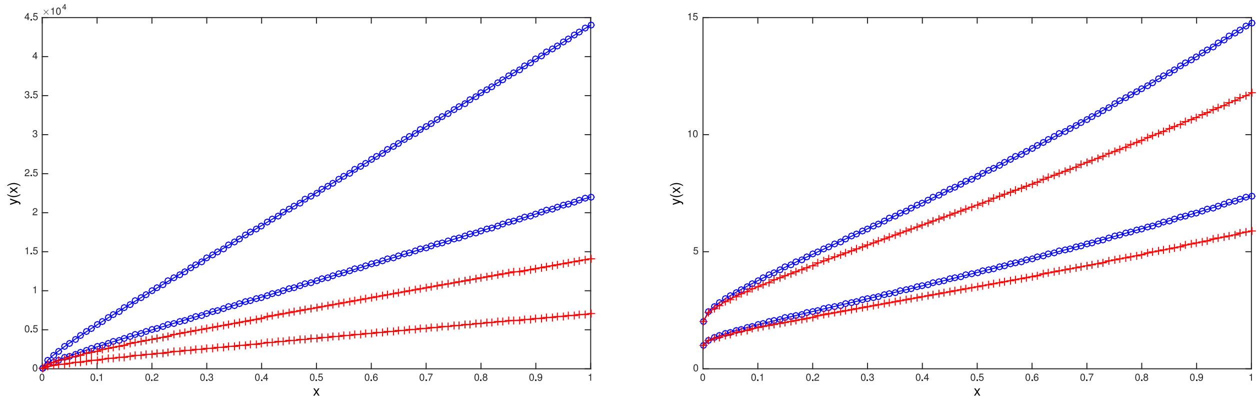
where S_1 and S_2 are defined as follows:

(S_1) For $\varepsilon>0$, sufficiently small, there exists:

$g(xE_\beta(\varepsilon x^{-\alpha}))\ominus g(x).$

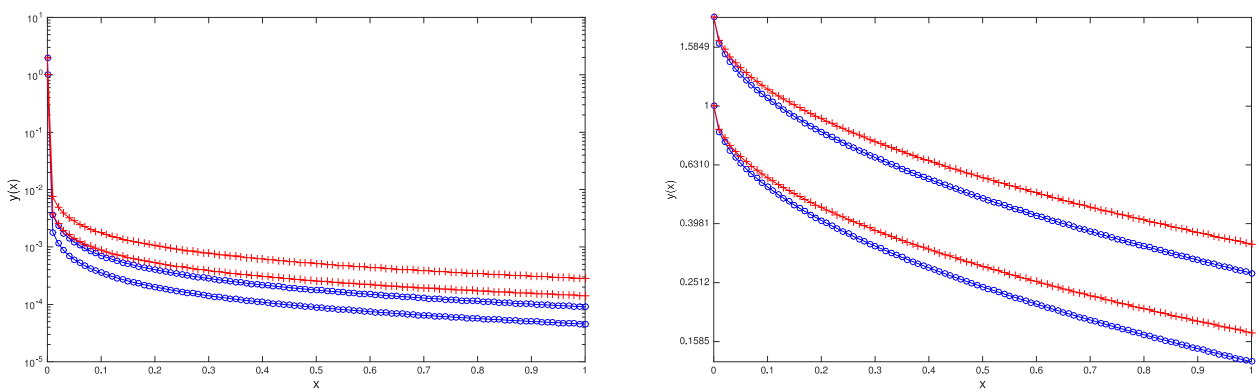
(S_2) For $\varepsilon>0$, sufficiently small, there exists:

$g(x)\ominus g(xE_\beta(\varepsilon x^{-\alpha})).$



(a) $\alpha = 0.1$ (b) $\alpha = 0.5$

Fig. 1. Comparison of the exact solution for different values of α and $\beta : 0$ (*blue* $-$ *o* $-$); 0.5 (*red* $- + -$), Example 4.1, Case I. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)



(a) $\alpha = 0.1$ (b) $\alpha = 0.5$

Fig. 2. Comparison of the exact solution for different values of α and $\beta : 0$ (*blue* $-$ *o* $-$); 0.5 (*red* $- + -$), Example 4.1, Case II. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Proof. We prove Case (2) and the procedure to prove the other statements are similar. Under the assumptions of Case (2), we have for sufficiently small $\varepsilon > 0$:

$$\begin{cases} g(x) = g(xE_\beta(\varepsilon x^{-\alpha})) + p(x, \varepsilon, \alpha), \\ f(x) - f(xE_\beta(\varepsilon x^{-\alpha})) = q(x, \varepsilon, \alpha). \end{cases}$$

Then,

$$\begin{aligned} f(x).g(x) = {} & f(xE_\beta(\varepsilon x^{-\alpha})).g(xE_\beta(\varepsilon x^{-\alpha})) \\ & + g(xE_\beta(\varepsilon x^{-\alpha}))p(x, \varepsilon, \alpha) \\ & + q(x, \varepsilon, \alpha).g(xE_\beta(\varepsilon x^{-\alpha})) \\ & + q(x, \varepsilon, \alpha).p(x, \varepsilon, \alpha). \end{aligned} \tag{3.11}$$

Indeed, Eq. (3.11) is H-difference of $f.g$ and $f(xE_\beta(\varepsilon x^{-\alpha})).g(xE_\beta(\varepsilon x^{-\alpha}))$. Hence,

$$\begin{aligned} &(f.g)(x) \ominus (f.g)(xE_\beta(\varepsilon x^{-\alpha})) = f(xE_\beta(\varepsilon x^{-\alpha}))p(x, \varepsilon, \alpha) \\ &\quad + q(x, \varepsilon, \alpha)g(xE_\beta(\varepsilon x^{-\alpha})) + q(x, \varepsilon, \alpha)p(x, \varepsilon, \alpha). \end{aligned}$$

Then by taking $\lim_{\varepsilon\to 0}$ from both sides, we have:

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{(f.g)(x) \ominus (f.g)(xE_\beta(\varepsilon x^{-\alpha}))}{-\varepsilon} = \lim_{\varepsilon\to 0} f(xE_\beta(\varepsilon x^{-\alpha})).\frac{p(x, \varepsilon, \alpha)}{\varepsilon} \\ &+ \lim_{\varepsilon\to 0} \frac{q(x, \varepsilon, \alpha)}{-\varepsilon}.g(xE_\beta(\varepsilon x^{-\alpha})) + \lim_{\varepsilon\to 0} \frac{(q.p)(x, \varepsilon, \alpha)}{-\varepsilon}. \end{aligned}$$

Considering the following assumptions that hold:

(a) $\lim_{\varepsilon\to 0} p(x, \varepsilon, \alpha) = 0, \quad p$ is continous.

(b) $\lim_{\varepsilon\to 0} f(xE_\beta(\varepsilon x^{-\alpha})).\frac{p(x,\varepsilon,\alpha)}{-\varepsilon} = f(x)g_\beta^{(\alpha)}(x)$.

(c) $\lim_{\varepsilon\to 0} \frac{q(x,\varepsilon,\alpha)}{-\varepsilon} g(xE_\beta(\varepsilon x^{-\alpha})) = f_\beta^{(\alpha)}(x)g(x)$.

Now using the above statements, we obtain:

$$\lim_{\varepsilon\to 0} \frac{(f.g)(x) \ominus (f.g)(xE_\beta(\varepsilon x^{-\alpha}))}{-\varepsilon} = f(x).g_\beta^{(\alpha)}(x) + f_\beta^{(\alpha)}(x).g(x).$$

□

3.1. Existence and uniqueness of the solution

In this part, the existence and uniqueness conditions of the interval solution for IFDE based on the interval truncated M-fractional derivative are discussed. Let us to consider the following IFDE

$$\begin{cases} {}_iy_\beta^{(\alpha)}(x) = f(x, y) \\ y(x_0) = y_0 \in \mathcal{F}, \end{cases} \tag{3.12}$$

where $x_0 \in \mathbb{R}$ and $f : \mathbb{R} \times \mathcal{F} \to \mathcal{F}$ is a interval continuous function. The problem (3.12) is equivalent to the following system of equations:

$$\begin{cases} y'(x) = \Gamma(\beta + 1)x^{\alpha-1}f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{3.13}$$

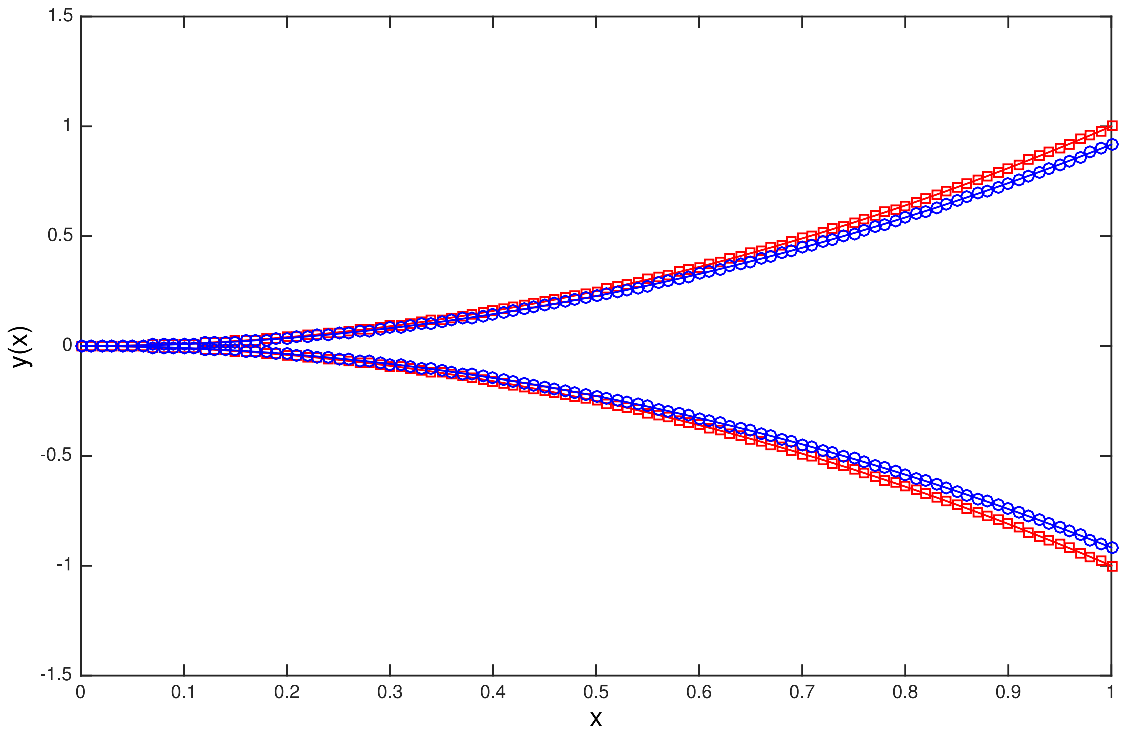


Fig. 3. Exact solution of Example 4.2 for different values of β: $\beta = 0$ $(-\square - red)$, $\beta = 0.5$ $(-o - blue)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

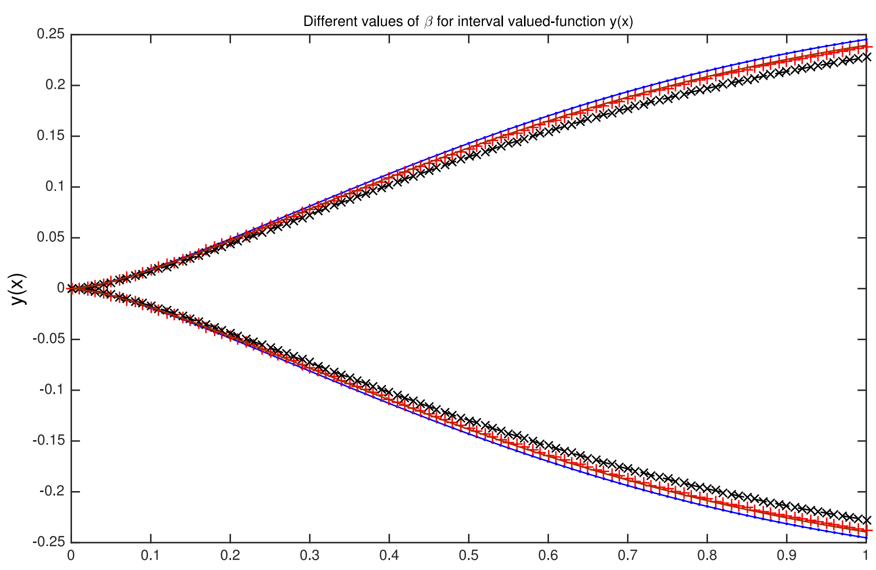


Fig. 4. Exact solution of Example 4.3 for different values of β: $\beta = 0(-o - blue)$, $\beta = 0.1(- + -, red)$, $\beta = 0.5(- * -yellow)$, $\beta = 0.9(- - black)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Set $g(x, y) = \Gamma(\beta + 1)x^{\alpha-1}f(x, y)$, then Eq. (3.13) is converted to:

$$\begin{cases} y'(x) = g(x, y) \\ y(x_0) = y_0 \end{cases} \tag{3.14}$$

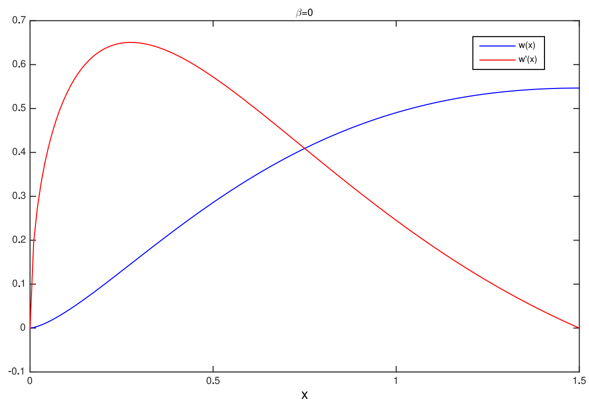
Using Lemma 20 in [51], it is easy to verify that the Eq. (3.14) is equivalent to the one of the following integral equations:

$$y(x) = y_0 + \int_{x_0}^{x} g(t, y(t))dt,$$

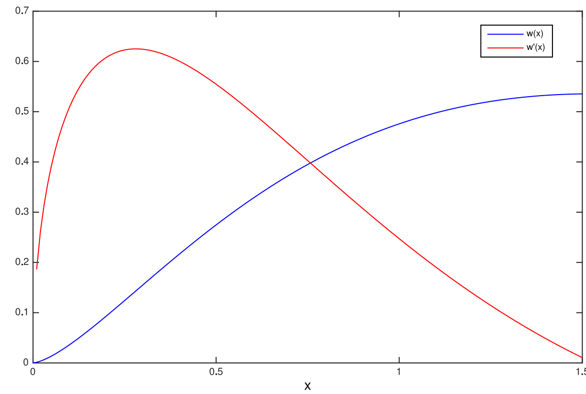
or

$$y(x) = y_0 \ominus (-1)\int_{x_0}^{x} g(t, y(t))dt.$$

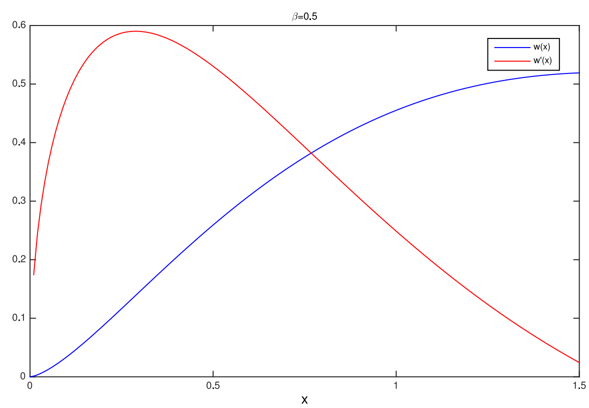
Remark 3.2. Using the equivalent form of the Eq. (3.12), we can easily obtain the existence and uniqueness of the solution. Indeed, by replacing the function f with g in Theorems 22 and 25 in [51], we obtain the existence and uniqueness of the solution of the Eq. (3.12).



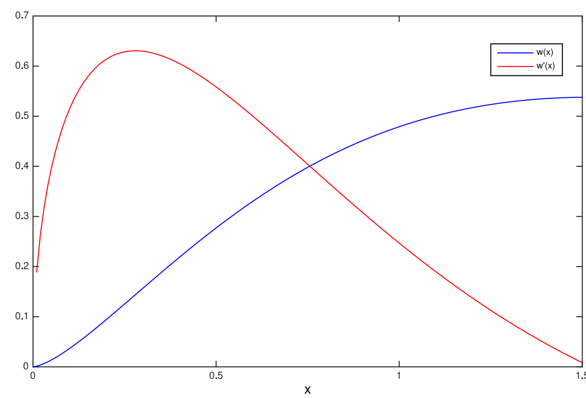
(a) $\beta = 0$



(b) $\beta = 0.1$



(c) $\beta = 0.5$



(d) $\beta = 0.9$

Fig. 5. Comparison of $w(x)$ and its derivative, $w'(x)$, for different values of β, Example 4.3.

4. Examples

In this section, a number of IFDEs based on the interval truncated M-fractional derivative are presented. The problems are solved under $(\alpha,\ 1)_\beta$-and $(\alpha,\ 2)_\beta$-differentiability to demonstrate the validity and applicability of the proposed interval fractional derivative.

In the current section, we propose a new viscoelastic model using IFDEs under interval truncated M-fractional differentiability. In this regards, interval parameters are defined which are corresponding to the frequency-dependent factors in the Kelin-Voigt equations [3] to explain the reality far better.

Example 4.1. Let consider the following IFDE

$$\begin{cases} {}_iy_\beta^{(\alpha)}(x) = \lambda y(x) \\ y(0) = y_0 = [1, 2] \end{cases} \tag{4.15}$$

Case I. Set $\lambda = 1$ and if we assume $y(x)$ is $(\alpha,\ 1)_\beta$-differentiable, then we have:

$$y(x) = y_0 e^{\Gamma(\beta+1)\frac{x^\alpha}{\alpha}}. \tag{4.16}$$

Indeed, when $\begin{cases} \beta \to 0^+ \\ \alpha \to 1 \end{cases}$ we get to the solution for IFDE as follows:

$$y(x) = [1, 2]e^x.$$

Also, only for the case $\beta \to 0^+$, we have

$$y(x) = y_0 e^{\frac{x^\alpha}{\alpha}},$$

that coincides with the solution based on the conformable derivative.

Fig. 1 shows the interval exact solution for Example 4.1, Case I with different values of α and β. As it is obvious, the IFDE experiences an interval solution by changing the values of α and β over $x \in [0,\ 1]$.

Case II. Set $\lambda = -1$ and if we assume $y(x)$ is $(\alpha,\ 2)_\beta$-differentiable, then we have:

$$y(x) = [1, 2]e^{-\Gamma(\beta+1)\frac{x^\alpha}{\alpha}} \tag{4.17}$$

In a similar way, once $\begin{cases} \beta \to 0^+ \\ \alpha \to 1 \end{cases}$ the solution for IFDE is given by:

$$y(x) = [1, 2]e^{-x}.$$

which is similar with the solution of the interval ODE. Also, just for the case that $\beta \to 0^+$, we have

$$y(x) = [1, 2]e^{-\frac{x^\alpha}{\alpha}}.$$

Similar to Case I, Fig. 2 depicts the interval exact solution under the assumption $\lambda = -1$ under $(\alpha,\ 2)_\beta$-differentiability. It is again clear that the IFDE based on the interval truncated M-fractional derivative achieves interval solutions by changing different values of α and β.

Example 4.2. Consider the following Kelin-Voigt model:

$$\begin{cases} {}_iy_\beta^{(1/2)}(x) + y(x) = \bar{c}(x^2 + 2x^{\frac{3}{2}}) \\ y(0) = 0, \quad \bar{c} = [-1, 1] \end{cases} \tag{4.18}$$

Using the fact that ${}_iy_\beta^{(1/2)}(x) = \frac{x^{\frac{1}{2}}}{\Gamma(\beta+1)}y'(x)$, the equation is converted to

$$\begin{cases} \frac{x^{\frac{1}{2}}}{\Gamma(\beta+1)}y'(x) + y(x) = \bar{c}(x^2 + 2x^{\frac{3}{2}}), \\ y(0) = 0, \quad \bar{c} = [-1, 1]. \end{cases} \tag{4.19}$$

Then, the solution is obtained under $(\alpha, 1)_\beta$-differentiablity as follows:

$$y(x) = \frac{\bar{c}(6b^2x - 3b - 6bx^{1/2} - 6b^3x + 6b^2x^{1/2} + 2b^4x^2 - 4b^3x^{3/2} + 4b^4x^{3/2} + 3}{2b^4} + \frac{\bar{c}(3b-3)}{2b^4\exp(2bx^{1/2})}, \tag{4.20}$$

where $b = \Gamma(\beta + 1)$.

For the case that $\beta \to 0^+$, we get

$$y(x) = \bar{c}x^2.$$

that coincides with the solution of Example 4.1 in [54].

Fig. 3 displays the interval exact solution for Example 4.2 with different values of β. Form the figure, it can conclude that by increasing the value of β, the lower and upper bounds of the interval solutions are closing together.

Example 4.3. Let suppose another Kelin-Voigt model based on the truncated M-fractional differential equation under interval uncertainty:

$$\begin{cases} {}_iy_\beta^{(1/2)}(x) + \sqrt{x}y(x) = \bar{c}xe^{-x} \\ y(0) = 0, \quad \bar{c} = [-1, 1], \end{cases} \tag{4.21}$$

The problem under $(\alpha, 1)_\beta$-differentibaility is converted to the following system:

$$\begin{cases} \frac{x^{\frac{1}{2}}}{\Gamma(\beta+1)}y'(x) + \sqrt{x}y(x) = \bar{c}xe^{-x} \\ y(0) = 0, \quad \bar{c} = [-1, 1], \end{cases} \tag{4.22}$$

Then, the solution is obtained as:

$$y(x) = b\bar{c}\left[\frac{\sqrt{x}\exp(x(b-1))}{-1+b} - \frac{1}{2}\frac{\sqrt{\pi}\ erf(\sqrt{1-b})\sqrt{x})}{(-1+b)\sqrt{1-b})}\right] \times e^{-\Gamma(\beta)\beta x}, \tag{4.23}$$

where $b = \Gamma(\beta + 1)$.

For the case that $\beta \to 0^+$, we get

$$y(x) = \frac{2}{3}\bar{c}e^{-x}x^{3/2},$$

that adapts with the solution of Example 4.2 in [54].

Fig. 4 shows the exact solution of Example 4.3 under interval uncertainty for different values of β. Once again, it is obvious that the bounds of the interval solution are approaching to each other by increasing the value of β. Also, Fig. 5 demonstrates the concept of w-increasing (decreasing) for this examples. It is implied from all of the cases in this figure that the value of β affect considerably on the behaviour of the interval solution.

Remark 4.1. From the results, once can conclude that by introducing the new interval fractional derivative, we obtain more flexible interval solution as β experiences different values. It affects on the shape of the interval solutions and the time interval domains.

5. Conclusions

In this paper, we proposed a new type of fractional derivative under interval uncertainty, based on the M-truncated fractional derivative. At the same time, we introduced the related fractional integral for interval-valued functions. For such uncertain fractional differentiable functions, we proved some new results such as Rolle's theorem, Mean value theorem under new type of fractional derivative. Indeed, our new results developed the interval conformable derivative using new reflexive parameter β. In fact, our achievements will be considered as a generalization of the proposed derivative in [27] under interval uncertainty. Moreover, we obtained existence and uniqueness of solutions of IFDEs under both types of interval differentiability, by converting the original problem to the equivalent IDE.

Finally, in order to show the ability and effectiveness of the proposed interval fractional derivative, we solved a number of applicable examples arising in the mathematical modeling of viscoelastic materials.

For a future research, one can consider such uncertainty for the fuzzy cases and develop this new derivative for fuzzy fractional differential equations.

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