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EXISTENCE OF SOLUTIONS FOR RIEMANN-LIOUVILLLE TYPE COUPLED SYSTEMS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

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Abstract. In this article, we study a boundary value problem of coupled systems of nonlinear Riemann-Liouvillle fractional integro-differential equations supplemented with nonlocal Riemann-Liouvillle fractional integro-differential boundary conditions. Our results rely on some standard tools of the fixed point theory. An illustrative example is also discussed.

1. Introduction

Fractional calculus is regarded as an important mathematical modelling tool for describing dynamical systems involving phenomena such as fractal and chaos. The subject started with the speculations of Leibniz (1697) and Euler (1730) about fractional-order derivatives and developed into an important branch of mathematical analysis with the passage of time. It deals with differential and integral operators of arbitrary (non-integer) order. An important and useful feature characterizing fractional-order differential and integral operators (in contrast to integer-order operators) is their nonlocal nature that accounts for the past and hereditary behavior of materials and processes involved in the real world problems. In addition to the extensive applications of fractional-order differential equations in various disciplines of technical and applied sciences, there has been a great focus on developing the theoretical aspects, and analytic and numerical methods for solving fractional order differential equations. For applications of fractional calculus in engineering and physics, we refer the reader to the texts [22, 24, 30], while some recent results on fractional differential equations can be found in [1, 2, 3, 8, 11, 14, 16, 19, 23, 29, 35, 37, 39].

Coupled systems of fractional-order differential equations appear in the mathematical formulation of several real world phenomena and processes. Examples of the occurrence of fractional systems include disease models [7, 9, 10, 26, 27], anomalous diffusion [25, 32], ecological models [18], synchronization of chaotic systems [12, 13, 38], nonlocal thermoelasticity [28], etc. For details concerning the

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theoretical development of coupled systems of fractional-order differential equations supplemented with a variety of boundary conditions, for instance, see [4, 5, 6, 17, 21, 31, 33, 34, 36].

In this article, we study the existence of solutions for a Riemann-Liouville coupled system of nonlinear fractional integro-differential equations given by

$$\begin{gathered} D^{\alpha}u(t) = f(t, u(t), v(t), (\phi_1 u)(t), (\psi_1 v)(t)), \quad t \in [0, T], \\ D^{\beta}v(t) = g(t, u(t), v(t), (\phi_2 u)(t), (\psi_2 v)(t)), \quad 1 < \alpha, \beta \le 2. \end{gathered} \tag{1.1}$$

subject to coupled Riemann-Liouville integro-differential boundary conditions:

$$\begin{gathered} D^{\alpha-2}u(0^+) = 0, \quad D^{\alpha-1}u(0^+) = \nu I^{\alpha-1}v(\eta), \quad 0 < \eta < T, \\ D^{\beta-2}v(0^+) = 0, \quad D^{\beta-1}v(0^+) = \mu I^{\beta-1}u(\sigma), \quad 0 < \sigma < T, \end{gathered} \tag{1.2}$$

where $D^{(\cdot)}, I^{(\cdot)}$ denote the Riemann-Liouville derivatives and integral of fractional order $(\cdot)$, respectively, $f, g : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ are given continuous functions, ν, μ are real constants, and

$$\begin{gathered} (\phi_1 u)(t) = \int_0^t \gamma_1(t, s)u(s)ds, \quad (\phi_2 u)(t) = \int_0^t \gamma_2(t, s)u(s)ds, \\ (\psi_1 v)(t) = \int_0^t \delta_1(t, s)v(s)ds, \quad (\psi_2 v)(t) = \int_0^t \delta_2(t, s)v(s)ds, \end{gathered}$$

with γ_i and δ_i $(i = 1, 2)$ being continuous function on $[0, T] \times [0, T]$.

The rest of the article is organized as follows. In Section 2, we recall some preliminary concepts of Riemann-Liouville calculus and prove an auxiliary lemma. Section 3 contains the existence and uniqueness results. Though we use the standard methodology (Leray-Schauder alternative to prove the existence of solutions and Contraction mapping principle to obtain the uniqueness result), yet its exposition to the given problem is new. Indeed our results are new and contribute to the existing literature on fully Riemann-Liouville type nonlinear nonlocal coupled systems of fractional integro-differential equations and boundary conditions.

2. Preliminaries

This section is devoted to some basic concepts of fractional calculus concerning Riemann-Liouville derivatives and integrals [20]. We also present an auxiliary lemma related to linear variant of the given problem.

Definition 2.1. The Riemann-Liouville fractional integral of order $\rho > 0$ for a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined as

$$I^{\rho}u(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t - s)^{\rho-1} u(s)ds,$$

provided the integral exists.

Definition 2.2. For a continuous function $u : (0, \infty) \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order ρ, $n = [\rho] + 1$ ($[\rho]$denotes the integer part of the real number ρ) is defined as

$$D^{\rho}u(t) = \frac{1}{\Gamma(n - \rho)} \big(\frac{d}{dt}\big)^n \int_0^t (t - s)^{n-\rho-1} u(s)ds = \big(\frac{d}{dt}\big)^n I^{n-\rho}u(t),$$

provided it exists.

For $\rho < 0$, we use the convention that $D^\rho u = I^\rho u$. Also, for $\beta \in [0,\rho]$, we have $D^\beta I^\rho u = I^{\rho-\beta}$. Note that for $\lambda > -1$, $\lambda \neq \rho-1, \rho-2, \dots, \rho-n$, we have

$$D^\rho t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\rho+1)} t^{\lambda-\rho} \text{ and } D^\rho t^{\rho-i} = 0,\ i = 1,2,\dots,n.$$

In particular, for the constant function $u(t) = 1$, we obtain

$$D^\rho 1 = \frac{1}{\Gamma(1-\rho)} t^{-\rho}, \quad \rho \notin \mathbb{N}.$$

For $\rho \in \mathbb{N}$, we obtain, of course, $D^\rho 1 = 0$ because of the poles of the gamma function at the points $0, -1, -2, \dots$. For $\rho > 0$, the general solution of the homogeneous equation $D^\rho u(t) = 0$ in $C(0,T) \cap L(0,T)$ is

$$u(t) = c_0 t^{\rho-n} + c_1 t^{\rho-n-1} + \dots + c_{n-2} t^{\rho-2} + c_{n-1} t^{\rho-1},$$

where c_i, $i = 1,2,\dots,n-1$, are arbitrary real constants. Further, we always have $D^\rho I^\rho u = u$, and

$$I^\rho D^\rho u(t) = u(t) + c_0 t^{\rho-n} + c_1 t^{\rho-n-1} + \dots + c_{n-2} t^{\rho-2} + c_{n-1} t^{\rho-1}. \tag{2.1}$$

To define the solution for problem (1.1)-(1.2), we consider the following lemma.

Lemma 2.3. *Let $h_1, h_2 \in C[0,T] \cap L[0,T]$. Then the integral solution for the linear system of fractional differential equations:*

$$D^\alpha u(t) = h_1(t), \quad D^\beta v(t) = h_2(t), \tag{2.2}$$

supplemented with the boundary conditions (1.2) *is given by*

$$\begin{aligned} u(t) &= \frac{\nu\Gamma(\beta) t^{\alpha-1}}{\Delta} \Big\{ \int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h_2(\tau) d\tau \Big) ds \\ &\quad + \frac{\mu \eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)} \Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h_1(\tau) d\tau \Big) ds \Big\} \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s) ds, \end{aligned} \tag{2.3}$$

$$\begin{aligned} v(t) &= \frac{\mu\Gamma(\alpha) t^{\beta-1}}{\Delta} \Big\{ \int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)} \Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h_1(\tau) d\tau \Big) ds \\ &\quad + \frac{\nu \sigma^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h_2(\tau) d\tau \Big) ds \Big\} \\ &\quad + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h_2(s) ds, \end{aligned} \tag{2.4}$$

where

$$\Delta = \Gamma(\alpha)\Gamma(\beta) - \frac{\nu\mu\Gamma(\alpha)\Gamma(\beta)(\eta\sigma)^{\alpha+\beta-2}}{(\Gamma(\alpha+\beta-1))^2} \neq 0. \tag{2.5}$$

Proof. Using the formula (2.1), the general solution of the system (2.2) can be written as

$$u(t) = a_0 t^{\alpha-2} + a_1 t^{\alpha-1} + I^\alpha h_1(t), \tag{2.6}$$

$$v(t) = b_0 t^{\beta-2} + b_1 t^{\beta-1} + I^\beta h_2(t), \tag{2.7}$$

where a_i, b_i, $(i = 0, 1)$ are unknown arbitrary constants. From (2.6) and (2.7), we have

$$D^{\alpha-1}u(t) = a_1\Gamma(\alpha) + Ih_1(t), \tag{2.8}$$

$$D^{\beta-1}v(t) = b_1\Gamma(\beta) + Ih_2(t), \tag{2.9}$$

$$D^{\alpha-2}u(t) = a_0\Gamma(\alpha-1) + a_1\Gamma(\alpha)t + I^2h_1(t), \tag{2.10}$$

$$D^{\beta-2}v(t) = b_0\Gamma(\beta-1) + b_1\Gamma(\beta)t + I^2h_2(t). \tag{2.11}$$

Using the given conditions: $D^{\alpha-2}u(0^+) = 0 = D^{\beta-2}v(0^+)$ in (2.10)-(2.11), we find that $a_0 = 0$, $b_0 = 0$. Thus (2.6) and (2.7) take the form

$$u(t) = a_1t^{\alpha-1} + I^\alpha h_1(t), \tag{2.12}$$

$$v(t) = b_1t^{\beta-1} + I^\beta h_2(t). \tag{2.13}$$

Using the coupled integral boundary conditions: $D^{\alpha-1}u(0^+) = \nu I^{\alpha-1}v(\eta)$ and $D^{\beta-1}v(0^+) = \mu I^{\beta-1}u(\sigma)$ in (2.8) and (2.9), we obtain

$$\begin{gathered}
\Gamma(\alpha)a_1 - \frac{\nu\Gamma(\beta)\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}b_1 = \nu I^{\alpha+\beta-1}h_2(\eta), \\
\frac{\mu\Gamma(\alpha)\sigma^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}a_1 - \Gamma(\beta)b_2 = -\mu I^{\alpha+\beta-1}h_1(\sigma).
\end{gathered} \tag{2.14}$$

Solving the system (2.14), we find that

$$a_1 = \frac{\nu}{\Delta}\Big\{\Gamma(\beta)I^{\alpha+\beta-1}h_2(\eta) + \frac{\mu\Gamma(\beta)\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}I^{\alpha+\beta-1}h_1(\sigma)\Big\}, \tag{2.15}$$

$$b_1 = \frac{\mu}{\Delta}\Big\{\Gamma(\alpha)I^{\alpha+\beta-1}h_1(\sigma) + \frac{\nu\Gamma(\alpha)\sigma^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}I^{\alpha+\beta-1}h_2(\eta)\Big\}, \tag{2.16}$$

where Δ is given by (2.5). Substituting the values of a_1 and b_1 (from (2.15) and (2.16)) in (2.12) and (2.13), we obtain the solution (2.3)-(2.4). Note that the converse follows by direct computation. This completes the proof. □

The following lemma contains certain estimates that we need in the sequel. We do not provide the proof as it is based on simple computation.

Lemma 2.4. *For $h_1, h_2 \in C[0,T] \cap L[0,T]$ with $\|h_1\| = \sup_{t\in[0,T]}|h_1(t)|$ and $\|h_2\| = \sup_{t\in[0,T]}|h_2(t)|$, we have*

(i) $$\Big|\int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)}\Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}h_1(\tau)d\tau\Big)ds\Big| \le \frac{\sigma^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\|h_1\|.$$

(ii) $$\Big|\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h_2(\tau)d\tau\Big)ds\Big| \le \frac{\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\|h_2\|.$$

(iii) $$\Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds\Big| \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|h_1\|.$$

(iv) $$\Big|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h_2(s)ds\Big| \le \frac{T^\beta}{\Gamma(\beta+1)}\|h_2\|.$$

3. Existence and uniqueness of solutions

Denote by $X = \{x : x \in C([0,T],\mathbb{R})\}$ and $Y = \{y : y \in C([0,T],\mathbb{R})\}$ the spaces equipped respectively with the norms $\|x\|_X = \sup_{t\in[0,T]} |x(t)|$ and $\|y\|_Y = \sup_{t\in[0,T]} |y(t)|$. Observe that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces. In consequence, the product space $(X\times Y, \|\cdot\|_{X\times Y})$ is a Banach space endowed with the norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ for $(x,y)\in X\times Y$.

By Lemma 2.3, we define an operator $F : X\times Y \to X\times Y$ associated with the problem (1.1)-(1.2) as follows:

$$F(u,v)(t) := (F_1(u,v)(t), F_2(u,v)(t)), \tag{3.1}$$

where

$$\begin{aligned} F_1(u,v)(t) = \frac{\nu\Gamma(\beta)t^{\alpha-1}}{\Delta}\Big\{ &\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\times \Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau,u(\tau),v(\tau),(\phi_2 u)(\tau),(\psi_2 v)(\tau))d\tau\Big)ds \\ &+ \frac{\mu\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)} \\ &\times \Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau),v(\tau),(\phi_1 u)(\tau),(\psi_1 v)(\tau))d\tau\Big)ds\Big\} \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s),(\phi_1 u)(s),(\psi_1 v)(s))ds, \end{aligned} \tag{3.2}$$

$$\begin{aligned} F_2(u,v)(t) = \frac{\mu\Gamma(\alpha)t^{\beta-1}}{\Delta}\Big\{ &\int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)} \\ &\times \Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau),v(\tau),(\phi_1 u)(\tau),(\psi_1 v)(\tau))d\tau\Big)ds \\ &+ \frac{\nu\sigma^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\times \Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau,u(\tau),v(\tau),(\phi_2 u)(\tau),(\psi_2 v)(\tau))d\tau\Big)ds\Big\} \\ &+ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s),(\phi_2 u)(s),(\psi_2 v)(s))ds. \end{aligned} \tag{3.3}$$

For computational convenience, we set

$$\lambda = 1+\gamma_0, \quad \bar\lambda = 1+\bar\gamma_0, \tag{3.4}$$

$$\theta = 1+\delta_0, \quad \bar\theta = 1+\bar\delta_0, \tag{3.5}$$

$$\varepsilon_1 = T^{\alpha-1}\Big\{\frac{|\nu\mu|\Gamma(\beta)(\eta\sigma)^{\alpha+\beta-1}}{\eta|\Delta|\Gamma(\alpha+\beta-1)\Gamma(\alpha+\beta)} + \frac{T}{\Gamma(\alpha+1)}\Big\}, \tag{3.6}$$

$$\varepsilon_2 = \frac{|\nu|\Gamma(\beta)\eta^{\alpha+\beta-1}T^{\alpha-1}}{|\Delta|\Gamma(\alpha+\beta)}, \tag{3.7}$$

$$\bar\varepsilon_1 = \frac{|\mu|\Gamma(\alpha)\sigma^{\alpha+\beta-1}T^{\beta-1}}{|\Delta|\Gamma(\alpha+\beta)}, \tag{3.8}$$

$$\bar\varepsilon_2 = T^{\beta-1}\Big\{\frac{|\nu\mu|(\eta\sigma)^{\alpha+\beta-1}\Gamma(\alpha)}{\sigma|\Delta|\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-1)} + \frac{T}{\Gamma(\beta+1)}\Big\}, \tag{3.9}$$

$$\Omega_1 = \varpi_0(\varepsilon_1 + \bar{\varepsilon}_1) + \kappa_0(\varepsilon_2 + \bar{\varepsilon}_2), \tag{3.10}$$

$$\Omega_2 = \lambda \max\{\varpi_1, \varpi_3\}(\varepsilon_1 + \bar{\varepsilon}_1) + \bar{\lambda} \max\{\kappa_1, \kappa_3\}(\varepsilon_2 + \bar{\varepsilon}_2), \tag{3.11}$$

$$\Omega_3 = \theta \max\{\varpi_2, \varpi_4\}(\varepsilon_1 + \bar{\varepsilon}_1) + \bar{\theta} \max\{\kappa_2, \kappa_4\}(\varepsilon_2 + \bar{\varepsilon}_2). \tag{3.12}$$

Observe that problem (1.1)-(1.2) has solutions if and only if the operator equation $F(u, v) = (u, v)$ has a fixed point.

Now we are ready to present our first existence result, which is based on Leray-Schauder alternative.

Lemma 3.1 (Leray-Schauder alternative [15]). *Let $F : E \to E$ be a completely continuous operator. Let $V(F) = \{x \in E : x = \lambda F(x)$ for some$0 < \lambda < 1\}$. Then either the set $V(F)$ is unbounded or F has at least one fixed point.*

Theorem 3.2. *Let $f, g : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous functions and there exist real constants $\varpi_i, \kappa_i \geq 0$ $(i = 1, \dots, 4)$ and $\varpi_0, \kappa_0 > 0$ such that*

(H1) $|f(t, u(t), v(t), (\phi_1 u)(t), (\psi_1 v)(t))| \leq \varpi_0 + \varpi_1|u| + \varpi_2|v| + \varpi_3|\phi_1 u| + \varpi_4|\psi_1 v|$,
$|g(t, u(t), v(t), (\phi_2 u)(t), (\psi_2 v)(t))| \leq \kappa_0 + \kappa_1|u| + \kappa_2|v| + \kappa_3|\phi_2 u| + \kappa_4|\psi_2 v|$,
for all $(u, v) \in X \times Y$.

Further it is assumed that $\max\{\Omega_2, \Omega_3\} < 1$, *where* Ω_2 *and* Ω_3 *are given by* (3.11) *and* (3.12) *respectively. Then problem* (1.1)-(1.2) *has at least one solution on* $[0, T]$.

Proof. In the first step, we show that the operator $F : X \times Y \to X \times Y$ defined by (3.1) is completely continuous. By continuity of the functions f and g, we deduce that the operators F_1 and F_2 respectively given by (3.2) and (3.3) are continuous. In consequence, the operator F is continuous. Next we show that the operator F is uniformly bounded. For that, let $\mathcal{A} \subset X \times Y$ be a bounded set. Then, for any $(u, v) \in \mathcal{A}$, there exist positive constants L_1 and L_2 such that $|f(t, u(t), v(t), (\phi_1 u)(t), (\psi_1 v)(t))| \leq L_1$, $|g(t, u(t), v(t), (\phi_2 u)(t), (\psi_2 v)(t))| \leq L_2$, for all $(u, v) \in \mathcal{A}$. Then, for any $(u, v) \in \mathcal{A}$, we have

$$\begin{aligned}
&|F_1(u, v)(t)| \\
&\leq \frac{\Gamma(\beta)|\nu|t^{\alpha-1}}{|\Delta|}\Big\{ \int_0^\eta \frac{(\eta - s)^{\alpha-2}}{\Gamma(\alpha - 1)} \\
&\quad \times \Big(\int_0^s \frac{(s - \tau)^{\beta-1}}{\Gamma(\beta)} |g(\tau, u(\tau), v(\tau), (\phi_2 u)(\tau), (\psi_2 v)(\tau))| d\tau \Big) ds \\
&\quad + \frac{|\mu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha + \beta - 1)} \int_0^\sigma \frac{(\sigma - s)^{\beta-2}}{\Gamma(\beta - 1)} \\
&\quad \times \Big(\int_0^s \frac{(s - \tau)^{\alpha-1}}{\Gamma(\alpha)} |f(\tau, u(\tau), v(\tau), (\phi_1 u)(\tau), (\psi_1 v)(\tau))| d\tau \Big) ds \Big\} \\
&\quad + \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), v(s), (\phi_1 u)(s), (\psi_1 v)(s))| ds, \\
&\leq \frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big\{ L_2 \int_0^\eta \frac{(\eta - s)^{\alpha-2}}{\Gamma(\alpha - 1)} \Big(\int_0^s \frac{(s - \tau)^{\beta-1}}{\Gamma(\beta)} d\tau \Big) ds \\
&\quad + \frac{|\mu|\eta^{\alpha+\beta-2}L_1}{\Gamma(\alpha + \beta - 1)} \int_0^\sigma \frac{(\sigma - s)^{\beta-2}}{\Gamma(\beta - 1)} \Big(\int_0^s \frac{(s - \tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \Big) ds \Big\} + L_1 \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} ds, \\
&\leq T^{\alpha-1} L_1 \Big\{ \frac{|\nu\mu|(\eta\sigma)^{\alpha+\beta-1}\Gamma(\beta)}{\eta|\Delta|\Gamma(\alpha + \beta - 1)\Gamma(\alpha + \beta)} + \frac{T}{\Gamma(\alpha + 1)} \Big\} + \frac{|\nu|\Gamma(\beta)\eta^{\alpha+\beta-1}T^{\alpha-1}L_2}{|\Delta|\Gamma(\alpha + \beta)},
\end{aligned}$$

which, on taking the norm for $t \in [0,T]$ and using the notation (3.7), yields

$$\|F_1(u,v)\| \le \varepsilon_1 L_1 + \varepsilon_2 L_2. \tag{3.13}$$

In a similar manner, we can find that

$$|F_2(u,v)(t)| \le \frac{|\mu|\Gamma(\alpha)T^{\beta-1}}{|\Delta|}\Big\{\frac{L_1\sigma^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{|\nu|L_2(\eta\sigma)^{\alpha+\beta-1}}{\sigma\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-1)}\Big\} + \frac{L_2T^\beta}{\Gamma(\beta+1)},$$

which together with (3.8) and (3.9) implies

$$\|F_2(u,v)\| \le \bar\varepsilon_1 L_1 + \bar\varepsilon_2 L_2. \tag{3.14}$$

From the inequalities (3.13) and (3.14), we infer that F_1 and F_2 are uniformly bounded, and hence the operator F is uniformly bounded.

Next, we show that F is equicontinuous. Let $t_1,\ t_2 \in [0,T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|F_1(u,v)(t_2) - F_1(u,v)(t_1)|\\
&\le \frac{|\nu|\Gamma(\beta)|t_2^{\alpha-1} - t_1^{\alpha-1}|}{|\Delta|}\Big\{\frac{L_2\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{|\mu|(\eta\sigma)^{\alpha+\beta-1}L_1}{\eta\Gamma(\alpha+\beta-1)\Gamma(\alpha+\beta)}\Big\}\\
&\quad + \frac{L_1}{\Gamma(\alpha+1)}\big(2(t_2-t_1)^\alpha + |t_2^\alpha - t_1^\alpha|\big).
\end{aligned}$$

Obviously $|F_1(u,v)(t_2) - F_1(u,v)(t_1)| \to 0$ as $t_2 \to t_1$.

In a similar manner, one can show that $|F_2(u,v)(t_2) - F_2(u,v)(t_1)| \to 0$ as $t_2 \to t_1$. Thus the operator F is equicontinuous in view of equicontinuity of F_1 and F_2. Therefore, by Arzela-Ascoli's theorem, we deduce that the operator F is compact (completely continuous).

Finally, we consider a set $V(F) = \{(u,v) \in X \times Y : (u,v) = \lambda F(u,v)\ ;\ 0 \le \lambda \le 1\}$ and show that it is bounded. Let $(u,v) \in V$. Then $(u,v) = \lambda F(u,v)$. For any $t \in [0,T]$, we have $u(t) = \lambda F_1(u,v)(t),\ v(t) = \lambda F_2(u,v)(t)$. Using the assumption $(H1)$ together with the notation (3.4) and (3.5), we obtain

$$\begin{aligned}
&|u(t)|\\
&\le |\lambda||F_1(u,v)(t)| \le |F_1(u,v)(t)|\\
&\le \frac{\Gamma(\beta)|\nu|t^{\alpha-1}}{|\Delta|}\Big\{\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}[\kappa_0 + \kappa_1|u(\tau)| + \kappa_2|v(\tau)|\\
&\quad + \kappa_3|(\phi_2 u)(\tau)| + \kappa_4|(\psi_2 v)(\tau)|]d\tau\Big)ds\\
&\quad + \frac{|\mu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)}\Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\varpi_0 + \varpi_1|u(\tau)| + \varpi_2|v(\tau)|\\
&\quad + \varpi_3|(\phi_1 u)(\tau)| + \varpi_4|(\psi_1 v)(\tau)|]d\tau\Big)ds\Big\}\\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\varpi_0 + \varpi_1|u(s)| + \varpi_2|v(s)| + \varpi_3|(\phi_1 u)(s)| + \varpi_4|(\psi_1 v)(s)|]ds\\
&\le \frac{\Gamma(\beta)|\nu|t^{\alpha-1}}{|\Delta|}\Big\{\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}[\kappa_0 + (\kappa_1 + \bar\gamma_0\kappa_3)|u(\tau)|\\
&\quad + (\kappa_2 + \bar\delta_0\kappa_4)|v(\tau)|]d\tau\Big)ds\\
&\quad + \frac{|\mu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)}\Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}[\varpi_0 + (\varpi_1 + \gamma_0\varpi_3)|u(\tau)|
\end{aligned}$$

$$
\begin{aligned}
&+ (\varpi_2 + \delta_0 \varpi_4)|v(\tau)|]d\tau\Big)ds\Big\}\\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[\varpi_0 + (\varpi_1 + \gamma_0\varpi_3)|u(s)| + (\varpi_2 + \delta_0\varpi_4)|v(s)|]ds,\\
\le\ & \frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big\{[\kappa_0 + \bar{\lambda}\max\{\kappa_1,\kappa_3\}\|u\|_X\\
&+ \bar{\theta}\max\{\kappa_2,\kappa_4\}\|v\|_Y]\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\Big)ds\\
&+ \frac{|\mu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)}\Big([\varpi_0 + \lambda\max\{\varpi_1,\varpi_3\}\|u\|_X\\
&+ \theta\max\{\varpi_2,\varpi_4\}\|v\|_Y]\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau\Big)ds\Big\}\\
&+ [\varpi_0 + \lambda\max\{\varpi_1,\varpi_3\}\|u\|_X + \theta\max\{\varpi_2,\varpi_4\}\|v\|_Y]\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds,\\
\le\ & T^{\alpha-1}\Big\{\frac{|\nu\mu|(\eta\sigma)^{\alpha+\beta-1}\Gamma(\beta)}{\eta|\Delta|\Gamma(\alpha+\beta-1)\Gamma(\alpha+\beta)} + \frac{T}{\Gamma(\alpha+1)}\Big\}[\varpi_0 + \lambda\max\{\varpi_1,\varpi_3\}\|u\|_X\\
&+ \theta\max\{\varpi_2,\varpi_4\}\|v\|_Y]\\
&+ \frac{|\nu|\Gamma(\beta)\eta^{\alpha+\beta-1}T^{\alpha-1}}{|\Delta|\Gamma(\alpha+\beta)}[\kappa_0 + \bar{\lambda}\max\{\kappa_1,\kappa_3\}\|u\|_X + \bar{\theta}\max\{\kappa_2,\kappa_4\}\|v\|_Y],
\end{aligned}
$$

which, on taking the norm for $t \in [0,T]$ and using (3.7), yields

$$
\begin{aligned}
\|u\|_X \le\ & \varepsilon_1[\varpi_0 + \lambda\max\{\varpi_1,\varpi_3\}\|u\|_X + \theta\max\{\varpi_2,\varpi_4\}\|v\|_Y] + \varepsilon_2[\kappa_0\\
&+ \bar{\lambda}\max\{\kappa_1,\kappa_3\}\|u\|_X + \bar{\theta}\max\{\kappa_2,\kappa_4\}\|v\|_Y].
\end{aligned}
\tag{3.15}
$$

Similarly, with the aid of notation (3.4), (3.5) and (3.9), one can obtain

$$
\begin{aligned}
\|v\|_Y \le\ & \bar{\varepsilon}_1[\varpi_0 + \lambda\max\{\varpi_1,\varpi_3\}\|u\|_X + \theta\max\{\varpi_2,\varpi_4\}\|v\|_Y] + \bar{\varepsilon}_2[\kappa_0\\
&+ \bar{\lambda}\max\{\kappa_1,\kappa_3\}\|u\|_X + \bar{\theta}\max\{\kappa_2,\kappa_4\}\|v\|_Y].
\end{aligned}
\tag{3.16}
$$

From (3.15) and (3.16), we find that

$$
\begin{aligned}
&\|u\|_X + \|v\|_Y\\
&\le \varpi_0(\varepsilon_1 + \bar{\varepsilon}_1) + \kappa_0(\varepsilon_2 + \bar{\varepsilon}_2)\\
&\quad + \Big[\lambda\max\{\varpi_1,\varpi_3\}(\varepsilon_1 + \bar{\varepsilon}_1) + \bar{\lambda}\max\{\kappa_1,\kappa_3\}(\varepsilon_2 + \bar{\varepsilon}_2)\Big]\|u\|_X\\
&\quad + \Big[\theta\max\{\varpi_2,\varpi_4\}(\varepsilon_1 + \bar{\varepsilon}_1) + \bar{\theta}\max\{\kappa_2,\kappa_4\}(\varepsilon_2 + \bar{\varepsilon}_2)\Big]\|v\|_Y,\\
&\le \Omega_1 + \max\{\Omega_2,\Omega_3\}\|(u,v)\|_{X\times Y},
\end{aligned}
\tag{3.17}
$$

which, in view of $\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y$, yields

$$
\|(u,v)\|_{X\times Y} \le \frac{\Omega_1}{1-\max\{\Omega_2,\Omega_3\}},
$$

where $\Omega_1, \Omega_2, \Omega_3$ are respectively given by (3.10), (3.11) and (3.12). This shows that $V(F)$ is bounded. Thus, by Lemma 3.1, the operator F has at least one fixed point. Consequently, the problem (1.1)-(1.2) has at least one solution on $[0,T]$. This completes the proof. □

Our next result deals with the uniqueness of solutions for problem (1.1)-(1.2) and relies on Banach's contraction mapping principle. For computational convenience, we introduce the notation:

$$\begin{aligned}\Lambda = \frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big\{\xi_1(1+\bar{\gamma_0}) + \xi_2(1+\bar{\delta_0}) + \frac{|\mu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\Big[\zeta_1(1+\gamma_0)\\ + \zeta_2(1+\delta_0)\Big]\Big\} + \bar{\zeta_1}(1+\gamma_0) + \bar{\zeta_2}(1+\delta_0),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\Lambda_1 = \frac{|\mu|\Gamma(\alpha)T^{\beta-1}}{|\Delta|}\Big\{\zeta_1(1+\gamma_0) + \zeta_2(1+\delta_0) + \frac{|\nu|\sigma^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\Big[\xi_1(1+\bar{\gamma_0})\\ + \xi_2(1+\bar{\delta_0})\Big]\Big\} + \bar{\xi_1}(1+\bar{\gamma_0}) + \bar{\xi_2}(1+\bar{\delta_0}),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\zeta_1 &= \max\Big\{|I^{\alpha+\beta-1}M_1(\sigma)|, |I^{\alpha+\beta-1}M_3(\sigma)|\Big\},\\ \zeta_2 &= \max\Big\{|I^{\alpha+\beta-1}M_2(\sigma)|, |I^{\alpha+\beta-1}M_4(\sigma)|\Big\},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\bar{\zeta_1} &= \sup_{t\in[0,T]}\Big\{|I^{\alpha}M_1(t)|, |I^{\alpha}M_3(t)|\Big\},\\ \bar{\zeta_2} &= \sup_{t\in[0,T]}\Big\{|I^{\alpha}M_2(t)|, |I^{\alpha}M_4(t)|\Big\},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\xi_1 &= \max\Big\{|I^{\alpha+\beta-1}N_1(\eta)|, |I^{\alpha+\beta-1}N_3(\eta)|\Big\},\\ \xi_2 &= \max\Big\{|I^{\alpha+\beta-1}N_2(\eta)|, |I^{\alpha+\beta-1}N_4(\eta)|\Big\},\end{aligned} \tag{3.22}$$

$$\begin{aligned}\bar{\xi_1} &= \sup_{t\in[0,T]}\Big\{|I^{\beta}N_1(t)|, |I^{\beta}N_3(t)|\Big\},\\ \bar{\xi_2} &= \sup_{t\in[0,T]}\Big\{|I^{\beta}N_2(t)|, |I^{\beta}N_4(t)|\Big\},\end{aligned} \tag{3.23}$$

$$\gamma_0 = \sup_{t\in[0,T]}\Big|\int_0^t \gamma_1(t,s)ds\Big|, \quad \bar{\gamma_0} = \sup_{t\in[0,T]}\Big|\int_0^t \gamma_2(t,s)ds\Big|, \tag{3.24}$$

$$\delta_0 = \sup_{t\in[0,T]}\Big|\int_0^t \delta_1(t,s)ds\Big|, \quad \bar{\delta_0} = \sup_{t\in[0,T]}\Big|\int_0^t \delta_2(t,s)ds\Big|, \tag{3.25}$$

$$\begin{aligned}\varepsilon = \varepsilon_1\varrho_1 + \varepsilon_2\varrho_2, \quad \bar{\varepsilon} = \bar{\varepsilon}_1\varrho_1 + \bar{\varepsilon}_2\varrho_2, \quad \varrho_1 = \sup_{t\in[0,T]}|f(t,0,0,0,0)|,\\ \varrho_2 = \sup_{t\in[0,T]}|g(t,0,0,0,0)|,\end{aligned} \tag{3.26}$$

where $\varepsilon_i, \bar{\varepsilon}_i (i = 1, 2)$ are respectively given by (3.6)–(3.9).

Theorem 3.3. *Let $f, g : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous functions and there exist positive functions $M_i(t)$, $N_i(t) \geq 0$ $(i = 1, \dots, 4)$ such that*

$$|f(t,u_1,u_2,u_3,u_4) - f(t,v_1,v_2,v_3,v_4)| \leq \sum_{i=1}^4 M_i(t)|u_i - v_i|,$$

$$g(t,u_1,u_2,u_3,u_4) - g(t,v_1,v_2,v_3,v_4)| \leq \sum_{i=1}^4 N_i(t)|u_i - v_i|,$$

for all $t \in [0,T]$, $u_i, v_i \in \mathbb{R}$. In addition, assume that $\Lambda < \frac{1}{2}$ and $\Lambda_1 < \frac{1}{2}$, where Λ and Λ_1 are given by (3.18)and (3.19) respectively. Then boundary value problem (1.1)-(1.2) has a unique solution on $[0,T]$.

Proof. Let us fix $r \geq \max\{2\varepsilon/(1-2\Lambda), 2\bar{\varepsilon}/(1-2\Lambda_1)\}$, where Λ, Λ_1, and $\varepsilon, \bar{\varepsilon}$ are respectively given by (3.18), (3.19) and (3.26). Firstly, we show that $FB_r \subset B_r$, where $B_r = \{(u,v) \in X \times Y : \|(u,v)\|_{X\times Y} \leq r\}$ and F is given by (3.1). For $(u,v) \in B_r$, note that

$$
\begin{aligned}
&|f(t,u(t),v(t),(\phi_1 u)(t),(\psi_1 v)(t))| \\
&\leq |f(t,u(t),v(t),(\phi_1 u)(t),(\psi_1 v)(t)) - f(t,0,0,0,0)| + |f(t,0,0,0,0)| \\
&\leq M_1(t)|u(t)| + M_2(t)|v(t)| + M_3(t)|(\phi_1 u)(t)| + M_4(t)|(\psi_1 v)(t)| + \varrho_1 \\
&\leq M_1(t)|u(t)| + M_2(t)|v(t)| + \gamma_0 M_3(t)|u(t)| + \delta_0 M_4(t)|v(t)| + \varrho_1 \\
&\leq \Big[M_1(t) + \gamma_0 M_3(t)\Big]|u(t)| + \Big[M_2(t) + \delta_0 M_4(t)\Big]|v(t)| + \varrho_1 \\
&\leq \Big(M_1(t) + M_2(t) + \gamma_0 M_3(t) + \delta_0 M_4(t)\Big)\|(u,v)\|_{X\times Y} + \varrho_1 \\
&\leq \Big(M_1(t) + M_2(t) + \gamma_0 M_3(t) + \delta_0 M_4(t)\Big) r + \varrho_1.
\end{aligned}
$$

Similarly, one can obtain

$$
|g(t,u(t),v(t),(\phi_2 u)(t),(\psi_2 v)(t))| \leq \Big(N_1(t) + N_2(t) + \bar{\gamma_0} N_3(t) + \bar{\delta_0} N_4(t)\Big) r + \varrho_2.
$$

Then, using the notation (3.20), (3.21), (3.22) and (3.26), we have

$$
\begin{aligned}
&|F_1(u,v)(t)| \\
&\leq \frac{\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big\{\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}\Big(\Big(N_1(\tau) + N_2(\tau) \\
&\quad + \bar{\gamma_0} N_3(\tau) + \bar{\delta_0} N_4(\tau)\Big) r + \varrho_2\Big) d\tau\Big) ds \\
&\quad + \frac{|\nu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\int_0^\sigma \frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)}\Big(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(\Big(M_1(\tau) + M_2(\tau) + \gamma_0 M_3(\tau) \\
&\quad + \delta_0 M_4(\tau)\Big) r + \varrho_1\Big) d\tau\Big) ds\Big\} \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\Big(M_1(s) + M_2(s) + \gamma_0 M_3(s) + \delta_0 M_4(s)\Big) r + \varrho_1\Big) ds \\
&\leq \frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big\{I^{\alpha+\beta-1}\Big(N_1(\eta) + \bar{\gamma_0} N_3(\eta)\Big) r + I^{\alpha+\beta-1}\Big(N_2(\eta) + \bar{\delta_0} N_4(\eta)\Big) r \\
&\quad + \frac{\eta^{\alpha+\beta-1}\varrho_2}{\Gamma(\alpha+\beta)} \\
&\quad + \frac{|\mu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\Big[I^{\alpha+\beta-1}\Big(M_1(\sigma) + \gamma_0 M_3(\sigma)\Big) r + I^{\alpha+\beta-1}\Big(M_2(\sigma) + \delta_0 M_4(\sigma)\Big) r\Big] \\
&\quad + \frac{\rho_1|\mu|(\eta\sigma)^{\alpha+\beta-1}}{\eta\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-1)}\Big\} + I^\alpha\Big(M_1(t) + \gamma_0 M_3(t)\Big) r + I^\alpha\Big(M_2(t) + \delta_0 M_4(t)\Big) r \\
&\quad + \frac{\varrho_1 T^\alpha}{\Gamma(\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
&\le r\Big\{\frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big(\xi_1(1+\bar{\gamma_0})+\xi_2(1+\bar{\delta_0})+\frac{|\mu|\zeta_1\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}(1+\gamma_0)\\
&\quad+\frac{|\mu|\zeta_2\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}(1+\delta_0)\Big)+\bar{\zeta_1}(1+\gamma_0)+\bar{\zeta_2}\Big(1+\delta_0\Big)\Big\}\\
&\quad+\varrho_1\Big[\frac{|\mu\nu|\Gamma(\beta)(\eta\sigma)^{\alpha+\beta-1}T^{\alpha-1}}{\eta|\Delta|\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-1)}+\frac{T^\alpha}{\Gamma(\alpha+1)}\Big]+\varrho_2\frac{|\nu|\Gamma(\beta)\eta^{\alpha+\beta-1}T^{\alpha-1}}{|\Delta|\Gamma(\alpha+\beta)},
\end{aligned}
$$

which, in view of (3.18) and (3.26), implies

$$
\|F_1(u,v)\|_X\le\Lambda r+\varepsilon\le\frac{r}{2}. \tag{3.27}
$$

Analogously, using (3.19) and (3.26), we can obtain

$$
\|F_2(u,v)\|_Y=\Lambda_1 r+\bar{\varepsilon}\le\frac{r}{2}. \tag{3.28}
$$

From the estimates (3.27) and (3.28), it clearly follows that

$$
\|F(u,v)\|_{X\times Y}=\|F_1(u,v)\|_X+\|F_2(u,v)\|_Y\le r,
$$

and hence $FB_r\subset B_r$.

Now we show that the operator F is a contraction. For that, let $u_i,v_i\in\mathbb{R}$, $i=1,2$. Then, for each $t\in[0,T]$, it follows by (3.20)-(3.22) that

$$
\begin{aligned}
&|F_1(u_1,v_1)(t)-F_1(u_2,v_2)(t)|\\
&\le\frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big\{\int_0^\eta\frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|g(\tau,u_1(\tau),v_1(\tau),(\phi_2u_1)(\tau),\\
&\quad(\psi_2v_1)(\tau))-g(\tau,u_2(\tau),v_2(\tau),(\phi_2u_2)(\tau),(\psi_2v_2)(\tau))|d\tau\Big)ds\\
&\quad+\frac{|\mu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\int_0^\sigma\frac{(\sigma-s)^{\beta-2}}{\Gamma(\beta-1)}\Big(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f(\tau,u_1(\tau),v_1(\tau),(\phi_1u_1)(\tau),\\
&\quad(\psi_1v_1)(\tau))-f(\tau,u_2(\tau),v_2(\tau),(\phi_1u_2)(\tau),(\psi_1v_2)(\tau))|d\tau\Big)ds\Big\}\\
&\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u_1(s),v_1(s),(\phi_1u_1)(s),(\psi_1v_1)(s))\\
&\quad-f(s,u_2(s),v_2(s),(\phi_1u_2)(s),(\psi_1v_2)(s))|ds\\
&\le\frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big\{I^{\alpha+\beta-1}\Big(N_1(\eta)+\bar{\gamma_0}N_3(\eta)\Big)\|u_1-u_2\|_X\\
&\quad+I^{\alpha+\beta-1}\Big(N_2(\eta)+\bar{\delta_0}N_4(\eta)\Big)\|v_1-v_2\|_Y\\
&\quad+\frac{|\mu|\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\Big[I^{\alpha+\beta-1}\Big(M_1(\sigma)+\gamma_0M_3(\sigma)\Big)\|u_1-u_2\|_X\\
&\quad+I^{\alpha+\beta-1}\Big(M_2(\sigma)+\delta_0M_4(\sigma)\Big)\|v_1-v_2\|_Y\Big]\Big\}\\
&\quad+I^\alpha\Big(M_1(t)+\gamma_0M_3(t)\Big)\|u_1-u_2\|_X+I^\alpha\Big(M_2(t)+\delta_0M_4(t)\Big)\|v_1-v_2\|_Y\\
&\le\Big\{\frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big[\xi_1(1+\bar{\gamma_0})+\frac{|\mu|\eta^{\alpha+\beta-2}\zeta_1}{\Gamma(\alpha+\beta-1)}(1+\gamma_0)\Big]+\bar{\zeta_1}(1+\gamma_0)\Big\}\|u_1-u_2\|_X\\
&\quad+\Big\{\frac{|\nu|\Gamma(\beta)T^{\alpha-1}}{|\Delta|}\Big[\xi_2(1+\bar{\delta_0})+\frac{|\mu|\eta^{\alpha+\beta-2}\zeta_2}{\Gamma(\alpha+\beta-1)}(1+\delta_0)\Big]+\bar{\zeta_2}(1+\delta_0)\Big\}\|v_1-v_2\|_Y,
\end{aligned}
$$

which yields

$$\|F_1(u_1,v_1)-F_1(u_2,v_2)\|_X \le \Lambda[\|u_1-u_2\|_X+\|v_1-v_2\|_Y], \tag{3.29}$$

where we have used (3.18). Similarly, we can find that

$$\|F_2(u_1,v_1)-F_2(u_2,v_2)\|_Y \le \Lambda_1[\|u_1-u_2\|_X+\|v_1-v_2\|_Y], \tag{3.30}$$

where we have used (3.19). Thus, from (3.29) and (3.30), we have

$$\begin{aligned}
&\|F(u_1,v_1)-F(u_2,v_2)\|_{X\times Y}\\
&=\|F_1(u_1,v_1)-F_1(u_2,v_2)\|_X+\|F_2(u_1,v_1)-F_2(u_2,v_2)\|_Y\\
&\le(\Lambda+\Lambda_1)[\|u_1-u_2\|_X+\|v_1-v_2\|_Y],
\end{aligned}$$

which implies that F is a contraction in view of the given condition $\Lambda+\Lambda_1<1$. Hence, by Banach's fixed point theorem, the operator F has a unique fixed point which corresponds to the unique solution of the problem (1.1)-(1.2) on $[0,T]$. This completes the proof. □

Example. Consider the boundary-value problem

$$\begin{gathered}
D^{3/2}u(t)=\frac{\sqrt{t^2+1}}{10}+\frac{t^2}{15}|u(t)|+\frac{t^2}{10}\tan^{-1}v(t)\\
+\frac{t}{25}\int_0^t\frac{(t-s)^{1/2}}{\Gamma(\frac32)}u(s)ds+\frac{1}{25}\int_0^t\frac{(t-s)^{1/3}}{\Gamma(\frac43)}v(s)ds,\\
D^{5/4}v(t)=\frac{e^t}{80}(|\sin u(t)|+1)+\frac{1}{20}|v(t)|\\
+\frac{1}{80}\int_0^t\frac{e^{-(s-t)}}{50}u(s)ds+\frac{t}{20}\int_0^t\frac{e^{-(s-t)/2}}{140}v(s)ds,\quad 0<t<1\\
D^{-1/2}u(0^+)=0,\quad D^{1/2}u(0^+)=-I^{1/2}(1/2),\\
D^{-3/4}v(0^+)=0,\quad D^{1/4}v(0^+)=-2I^{1/4}(1/4).
\end{gathered}\tag{3.31}$$

Here, $\alpha=3/2$, $\beta=5/4$, $v=-1$, $\mu=-2$, $\eta=1/2$, $\sigma=1/4$, $T=1$, $\gamma_1=(t-s)^{1/2}/\Gamma(3/2)$, $\delta_1=(t-s)^{1/3}/\Gamma(4/3)$, $\gamma_2=e^{-(s-t)}/50$, $\delta_2=e^{-(s-t)/2}/140$, $M_1(t)=t^2/15$, $M_2(t)=t^2/10$, $M_3(t)=t/25$, $M_4(t)=1/25$, $N_1(t)=e^t/80$, $N_2(t)=1/20$, $N_3(t)=1/80$, $N_4(t)=t/20$. Using the given data, we find that $\gamma_0\simeq0.75225$, $\delta_0\simeq0.83988$, $\bar\gamma_0\simeq0.03437$, $\bar\delta_0\simeq0.00927$, $\Delta\simeq1.20312$, $\zeta_1\simeq0.00489$, $\zeta_2\simeq0.00733$, $\bar\zeta_1\simeq0.05015$, $\bar\zeta_2\simeq0.07523$, $\xi_1\simeq0.01006$, $\xi_2\simeq0.01480$, $\bar\xi_1\simeq0.11841$, $\bar\xi_2\simeq0.04413$. Further, $\Lambda\simeq0.26689<1/2$, and $\Lambda_1\simeq0.21388<1/2$. Thus, by Theorem 3.3, problem (3.31) has a unique solution on $[0,1]$.

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