Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 579740, 20 pages
http://dx.doi.org/10.1155/2013/579740

Research Article

Positive Solutions to Fractional Boundary Value Problems with Nonlinear Boundary Conditions

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Received 20 March 2013; Accepted 5 April 2013

Academic Editor: Juan J. Trujillo

We consider a system of boundary value problems for fractional differential equation given by $D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}u)(t) = \lambda_1 a_1(t) f_1(u(t), v(t))$, $t \in (0,1)$, $D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}v)(t) = \lambda_2 a_2(t) f_2(u(t), v(t))$, $t \in (0,1)$, where $1 < \alpha, \beta \le 2$, $2 < \alpha+\beta \le 4$, λ_1, λ_2 are eigenvalues, subject either to the boundary conditions $D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}u(1) = 0$, $u(0) = 0$, $D_{0^+}^{\beta_1}u(1) - \sum_{i=1}^{m-2} a_{1i} D_{0^+}^{\beta_1}u(\xi_{1i}) = 0$, $D_{0^+}^{\alpha}v(0) = D_{0^+}^{\alpha}v(1) = 0$, $v(0) = 0$, $D_{0^+}^{\beta_1}v(1) - \sum_{i=1}^{m-2} a_{2i} D_{0^+}^{\beta_1}v(\xi_{2i}) = 0$ or $D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}u(1) = 0$, $u(0) = 0$, $D_{0^+}^{\beta_1}u(1) - \sum_{i=1}^{m-2} a_{1i} D_{0^+}^{\beta_1}u(\xi_{1i}) = \psi_1(u)$, $D_{0^+}^{\alpha}v(0) = D_{0^+}^{\alpha}v(1) = 0$, $v(0) = 0$, $D_{0^+}^{\beta_1}v(1) - \sum_{i=1}^{m-2} a_{2i} D_{0^+}^{\beta_1}v(\xi_{2i}) = \psi_2(v)$, where $0 < \beta_1 < 1$, $\alpha - \beta_1 - 1 \ge 0$ and $\psi_1, \psi_2 : C([0,1]) \to [0,\infty)$ are continuous functions. The Krasnoselskiis fixed point theorem is applied to prove the existence of at least one positive solution for both fractional boundary value problems. As an application, an example is given to demonstrate some of main results.

1. Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order; the fractional calculus may be considered an old and yet novel topic.

Recently, fractional differential equations have found numerous applications in various fields of physics and engineering [1, 2]. It should be noted that most of the books and papers on fractional calculus are devoted to the solvability of initial value problems for differential equations of fractional order. In contrast, the theory of boundary value problems for nonlinear fractional differential equations has received attention quite recently and many aspects of this theory need to be explored. For more details and examples, see [3–9] and the references therein; moreover, fractional derivative arises from many physical processes, such as a charge transport in amorphous semiconductors [10]; electrochemistry and material science are also described by differential equations of fractional order [11–15]. In [16], Bai and Lü considered the boundary value problem of fractional order differential equation

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \le 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

In [17], Salem considered the following nonlinear m-point boundary value problem of fractional type:

$$\begin{gathered} D_{0^+}^{\alpha} x(t) + q(t) f(t, x(t)) = 0, \quad \text{a.e. on } [0,1], \\ \alpha \in (n-1, n], \ n \ge 2, \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = \sum_{i=1}^{m-2} \xi_i x(\eta_i), \end{gathered} \tag{2}$$

where $0 < \eta_1 < \cdots < \eta_{m-2} < 1$, $\xi_i > 0$ with $\sum_{i=1}^{m-2}\xi_i\eta_i^{\alpha-1} < 1$, q is a real valued continuous function, and f is a nonlinear Pettis integrable function.

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problems, Leibenson [18] introduced the p-Laplacian equation as follows:

$$\left(\phi_p\left(x'(t)\right)\right)' = f\left(t, x(t), x'(t)\right), \tag{3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $(1/q) + (1/p) = 1$.

Ahmad et al. [19] also considered the existence of solutions for the following three-point boundary value problem of Langevin equation with two different fractional orders:

$$\begin{gathered} {}^cD^\beta\left({}^cD^\beta + \lambda\right)x(t) = f(t, x(t)), \\ 0 < t < 1,\ 1 < \beta \le 2,\ 0 < \alpha \le 1, \\ x(0) = 0, \qquad x(\eta) = 0, \qquad x(1) = 0, \quad 0 < \eta < 1, \end{gathered} \tag{4}$$

where ${}^cD$ is the Caputo fractional derivative, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a given continuous function, and λ is a real number.

Dai [20] considered the following problem of ordinary differential equations:

$$\begin{gathered} \left(\phi_p\left(u'\right)\right)' = \lambda a(t) f(u), \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{gathered} \tag{5}$$

By means of global bifurcation techniques and the approximation of connected components, existence and multiplicity results for positive solutions were obtained.

Motivated by the works above, our purpose in this paper is to show the existence of at least one positive solution for the following fractional p-Laplacian system:

$$\begin{gathered} D_{0^+}^\beta\phi_p\left(D_{0^+}^\alpha u\right)(t) = \lambda_1 a_1(t) f_1(u(t), v(t)), \quad t \in (0,1), \\ D_{0^+}^\beta\phi_p\left(D_{0^+}^\alpha v\right)(t) = \lambda_2 a_2(t) f_2(u(t), v(t)), \quad t \in (0,1), \end{gathered} \tag{6}$$

where $1 < \alpha, \beta \le 2$, $2 < \alpha + \beta \le 4$, $D_{0^+}^\alpha$ is the Riemann-Liouville fractional derivative of order α, $\lambda_1, \lambda_2 > 0$, and $m > 2$ is integer.

We first consider the problem (6) with following boundary condition:

$$\begin{gathered} D_{0^+}^\alpha u(0) = D_{0^+}^\alpha u(1) = 0, \quad u(0) = 0, \\ D_{0^+}^{\beta_1} u(1) - \sum_{i=1}^{m-2} a_{1i} D_{0^+}^{\beta_1} u\left(\xi_{1i}\right) = 0, \\ D_{0^+}^\alpha v(0) = D_{0^+}^\alpha v(1) = 0, \quad v(0) = 0, \\ D_{0^+}^{\beta_1} v(1) - \sum_{i=1}^{m-2} a_{2i} D_{0^+}^{\beta_1} v\left(\xi_{2i}\right) = 0. \end{gathered} \tag{7}$$

We then consider the case in which the boundary conditions are changed to

$$\begin{gathered} D_{0^+}^\alpha u(0) = D_{0^+}^\alpha u(1) = 0, \quad u(0) = 0, \\ D_{0^+}^{\beta_1} u(1) - \sum_{i=1}^{m-2} a_{1i} D_{0^+}^{\beta_1} u\left(\xi_{1i}\right) = \psi_1(u), \\ D_{0^+}^\alpha v(0) = D_{0^+}^\alpha v(1) = 0, \quad v(0) = 0, \\ D_{0^+}^{\beta_1} v(1) - \sum_{i=1}^{m-2} a_{2i} D_{0^+}^{\beta_1} v\left(\xi_{2i}\right) = \psi_2(v), \end{gathered} \tag{8}$$

where $\psi_1, \psi_2 : C([0,1]) \to [0,\infty)$ are continuous functions, where $C([0,1])$ means the set of continuous, real valued functions on the unit interval $[0,1]$.

In the cases, we assume that $0 < \beta_1 < 1$, $\alpha - \beta_1 - 1 \ge 0$.

In the past few decades, many important results relative to (6) with certain boundary value conditions have been obtained; we refer the reader to [21–25] and the references therein.

The following conditions will be used in this paper:

(H1) $\phi_p(s) = |s|^{p-2}s$, $p > 1$ is a p-laplacian operator. Obviously, ϕ_p is invertible and $\phi_p^{-1} = \phi_q$, where $q > 1$ is a constant such that $(1/q) + (1/p) = 1$;

(H2) $0 < \xi_{j1} < \xi_{j2} < \cdots < \xi_{jm-2} < 1$, $a_{ji} > 0$ for $i = 1, 2, \ldots, m-2$ and $\sum_{i=1}^{m-2} a_{ji}\xi_{ji}^{\alpha-\beta_1-1} < 1$, $j = 1, 2$;

(H3) $f_j : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a given continuous function and a_j is a positive real valued continuous function, $j = 1, 2$.

The rest of the paper is organized as follows: in Section 2, we will recall certain results from the theory of the continuous fractional calculus; in Section 3, we will provide some conditions under which the problem (6) and (7) has at least one positive solution; in Section 4, by suitable conditions, we will prove that the problem (6) and (8) has at least one positive solution; finally, in Section 4, we will provide some numerical examples, which will explicate the applicability of our results.

2. Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in the proofs of the main results.

Definition 1. Let X be a real Banach space. A nonempty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:

(1) $x \in P$, $\mu \ge 0$, implies $\mu x \in P$,

(2) $x \in P$, $-x \in P$, implies $x = 0$.

Definition 2 (see [26, 27]). The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds, \tag{9}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 3 (see [26, 27]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f:(0,\infty) \to \mathbb{R}$ is defined as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad (10)$$

where $n = [\alpha] + 1$.

Lemma 4 (see [28]). *The equality $D_{0^+}^{\gamma} I_{0^+}^{\gamma} f(t) = f(t)$, $\gamma > 0$, holds for $f \in L^1(0,1)$.*

Lemma 5 (see [28]). *Let $\alpha > 0$. Then the differential equation*

$$D_{0^+}^{\alpha} u = 0 \quad (11)$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where $n - 1 < \alpha \le n$.

Lemma 6 (see [28]). *Let $\alpha > 0$. Then the following equality holds for $u \in L^1(0,1)$, $D_{0^+}^{\alpha} u \in L^1(0,1)$,*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad (12)$$

$c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where $n - 1 < \alpha \le n$.

In the following, we present the Green function of fractional differential equation boundary value problem.

Let

$$y(t) = -\phi_p\left(D_{0^+}^{\alpha} u\right)(t), \quad (13)$$

then, the problem

$$\begin{gathered} D_{0^+}^{\beta} \phi_p\left(D_{0^+}^{\alpha} u\right)(t) = h(t), \quad 1 < \beta \le 2,\ t \in (0,1), \\ D_{0^+}^{\alpha} u(0) = D_{0^+}^{\alpha} u(1) = 0, \end{gathered} \quad (14)$$

where $h \in C[0,1]$, is turned into problem

$$\begin{gathered} D_{0^+}^{\beta} y(t) + h(t) = 0, \quad 1 < \beta \le 2,\ t \in (0,1), \\ y(0) = y(1) = 0. \end{gathered} \quad (15)$$

Lemma 7. *Suppose that $h \in C[0,1]$, then the boundary value problem* (15) *has a unique solution*

$$y(t) = \int_0^1 H(t,s) h(s)\, ds, \quad (16)$$

where

$$H(t,s) = \begin{cases} \dfrac{t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le t \le s \le 1. \end{cases} \quad (17)$$

Proof. The proof is similar to that of Lemma 2.3 in [16], so we omit it here. □

Lemma 8 (see [16]). *For $\lambda > -1$ and $\alpha > 0$,*

$$D_{0^+}^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}. \quad (18)$$

Lemma 9 (see [29]). *Suppose that $g \in L^1(0,1)$ and α, β are two constants such that $0 \le \beta \le 1 < \alpha$; then,*

$$D_{0^+}^{\beta} \int_0^t (t-s)^{\alpha-1} g(s)\, ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s)\, ds. \quad (19)$$

Lemma 10. *Suppose that (H1) and (H2) hold. Then, for $y \in C[0,1]$, the boundary value problem*

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + \phi_q\left(y(t)\right) = 0, \quad t \in (0,1),\ 1 < \alpha \le 2, \\ u(0) = 0, \qquad D_{0^+}^{\beta_1} u(1) - \Sigma_{i=1}^{m-2} a_{ji} D_{0^+}^{\beta_1} u\left(\xi_{ji}\right) = 0, \\ 0 < \beta_1 < 1, \end{gathered} \quad (20)$$

has a unique solution

$$\begin{aligned} u(t) = {} & \int_0^1 G(t,s) \phi_q\left(y(s)\right) ds \\ & + \frac{t^{\alpha-1}}{\left(1-\Delta_j\right)} \Sigma_{i=1}^{m-2} a_{ji} \int_0^1 G_1\left(\xi_{ji}, s\right) \phi_q\left(y(s)\right) ds, \end{aligned} \quad (21)$$

where $\Delta_j = \Sigma_{i=1}^{m-2} a_{ji} \xi_{ji}^{\alpha-\beta_1-1} \ne 1$, for $j = 1, 2$,

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta_1-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-\beta_1-1}(1-s)^{\alpha-\beta_1-1} - (t-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-\beta_1-1}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \quad (22)$$

Proof. By applying Lemma 6, (20) is equivalent to the following integral equation:

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\left(y(s)\right) ds - c_1 t^{\alpha-1} - c_2 t^{\alpha-2}, \quad (23)$$

for some arbitrary constants $c_1, c_2 \in \mathbb{R}$.

By the boundary condition $u(0) = 0$, we conclude that $c_2 = 0$; then we have

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\left(y(s)\right) ds - c_1 t^{\alpha-1}. \quad (24)$$

It follows from Lemmas 8 and 9 that

$$D_{0^+}^{\beta_1}u(t) = -\frac{1}{\Gamma(\alpha-\beta_1)} \times \int_0^t (t-s)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds - c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta_1)}t^{\alpha-\beta_1-1}. \tag{25}$$

So, by the boundary condition $D_{0^+}^{\beta_1}u(1)-\Sigma_{i=1}^{m-2}a_{ji}D_{0^+}^{\beta_1}u(\xi_{ji}) = 0$, we obtain that

$$\begin{aligned} c_1 = &-\frac{1}{\Gamma(\alpha)\left(1-\Delta_j\right)}\int_0^1(1-s)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds \\ &+\frac{1}{\Gamma(\alpha)\left(1-\Delta_j\right)}\Sigma_{i=1}^{m-2}a_{ji} \\ &\times\int_0^{\xi_{ji}}\left(\xi_{ji}-s\right)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds. \end{aligned} \tag{26}$$

Then, the unique solution of (20) is given by the formula

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(y(s))\,ds \\ &+\frac{1}{\left(1-\Delta_j\right)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds \\ &-\frac{t^{\alpha-1}}{\left(1-\Delta_j\right)\Gamma(\alpha)} \\ &\times\sum_{i=1}^{m-2}a_{ji}\int_0^{\xi_{ji}}\left(\xi_{ji}-s\right)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds \\ = &-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(y(s))\,ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds \\ &+\frac{\Delta_j}{\left(1-\Delta_j\right)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds \\ &-\frac{t^{\alpha-1}}{\left(1-\Delta_j\right)\Gamma(\alpha)} \\ &\times\sum_{i=1}^{m-2}a_{ji}\int_0^{\xi_{ji}}\left(\xi_{ji}-s\right)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds \\ = &\frac{1}{\Gamma(\alpha)}\left[\int_0^t\left[t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}-(t-s)^{\alpha-1}\right]\right. \\ &\times\phi_q(y(s))\,ds \\ &\left.+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds\right] \\ &+\frac{t^{\alpha-1}}{\left(1-\Delta_j\right)\Gamma(\alpha)} \\ &\times\sum_{i=1}^{m-2}a_{ji}\int_0^1\xi_{ji}^{\alpha-\beta_1-1}(1-s)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds \\ &-\frac{t^{\alpha-1}}{\left(1-\Delta_j\right)\Gamma(\alpha)} \\ &\times\sum_{i=1}^{m-2}a_{ji}\int_0^{\xi_{ji}}\left(\xi_{ji}-s\right)^{\alpha-\beta_1-1}\phi_q(y(s))\,ds \\ = &\frac{1}{\Gamma(\alpha)}\left[\int_0^t\left[t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}-(t-s)^{\alpha-1}\right]\right. \\ &\times\phi_q(y(s))\,ds \\ &\left.+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-\beta-1}\phi_q(y(s))\,ds\right] \\ &+\frac{t^{\alpha-1}}{\left(1-\Delta_j\right)\Gamma(\alpha)} \\ &\times\sum_{i=1}^{m-2}a_{ji}\left[\int_0^{\xi_{ji}}\left[(1-s)^{\alpha-\beta-1}\xi_{ji}{}^{\alpha-\beta_1-1}\right.\right. \\ &\left.-\left(\xi_{ji}-s\right)^{\alpha-\beta_1-1}\right]\phi_q(y(s))\,ds \\ &+\int_{\xi_{ji}}^1(1-s)^{\alpha-\beta_1-1} \\ &\left.\times\,\xi_{ji}{}^{\alpha-\beta_1-1}\phi_q(y(s))\,ds\right] \\ = &\int_0^1 G(t,s)\,\phi_q(y(s))\,ds \\ &+\frac{t^{\alpha-1}}{\left(1-\Delta_j\right)}\sum_{i=1}^{m-2}a_{ji}\int_0^1 G_1\left(\xi_{ji},s\right)\phi_q(y(s))\,ds. \end{aligned} \tag{27}$$

Then, the proof is completed. □

Lemma 11. *Assume $\alpha-\beta_1-1\ge 0$, then; for all $(t,s)\in[0,1]\times[0,1]$, we have*

(i) $0 \le G_1(t,s) \le (1/\Gamma(\alpha))t^{\alpha-\beta_1-1}(1-s)^{\alpha-\beta_1-1}$, $0 \le G(t,s) \le G(s,s)$, *for any $t,s\in[0,1]$;*

(ii) *there exists a positive function $g\in C[0,1]$ such that* $\min_{\gamma\le t\le\delta}G(t,s)\ge g(s)G(s,s)$, $s\in(0,1)$, *where*

$$g(s)=\begin{cases}\dfrac{\delta^{\alpha-1}(1-s)^{\alpha-\beta_1-1}-(\delta-s)^{\alpha-1}}{s^{\alpha-1}(1-s)^{\alpha-\beta_1-1}}, & s\in(0,r],\\ \left(\dfrac{\gamma}{s}\right)^{\alpha-1}, & s\in[r,1),\end{cases}\tag{28}$$

with $\gamma<r<\delta$.

Proof. (i) If $0\le s\le t\le 1$, we have

$$\begin{aligned}G_1(t,s)&=\frac{1}{\Gamma(\alpha)}t^{\alpha-\beta_1-1}(1-s)^{\alpha-\beta_1-1}-(t-s)^{\alpha-\beta_1-1}\\&\ge\frac{1}{\Gamma(\alpha)}t^{\alpha-\beta_1-1}(1-s)^{\alpha-\beta_1-1}-(t-ts)^{\alpha-\beta_1-1}\\&=\frac{1}{\Gamma(\alpha)}t^{\alpha-\beta_1-1}\left((1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-\beta_1-1}\right)=0.\end{aligned}\tag{29}$$

If $0\le t\le s\le 1$, we get

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}t^{\alpha-\beta_1-1}(1-s)^{\alpha-\beta_1-1}\ge 0.\tag{30}$$

Thus, $G_1(t,s)\ge 0$, for any $(t,s)\in[0,1]\times[0,1]$. It is obvious that $G_1(t,s)\le(1/\Gamma(\alpha))t^{\alpha-\beta_1-1}(1-s)^{\alpha-\beta_1-1}$.

Now, we show that $0\le G(t,s)\le G(s,s)$ for any $t,s\in[0,1]$. We define

$$\begin{aligned}g_1(t,s)&=\frac{t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)},\\&\qquad 0\le s\le t\le 1,\\ g_2(t,s)&=\frac{t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}}{\Gamma(\alpha)},\\&\qquad 0\le t\le s\le 1.\end{aligned}\tag{31}$$

One can get

$$\begin{aligned}g_1(t,s)&\ge\frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}(1-s)^{\alpha-\beta_1-1}-(t-ts)^{\alpha-1}\right]\\&=\frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}\left((1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-1}\right)\right]\ge 0,\end{aligned}\tag{32}$$

on the other hand, it is obvious that $g_2(t,s)\ge 0, 0\le t\le s\le 1$.

Thus

$$G(t,s)\ge 0,\quad\forall(t,s)\in[0,1]\times[0,1].\tag{33}$$

For any $0\le s\le t\le 1$,

$$\begin{aligned}\frac{\partial g_1(t,s)}{\partial t}&=\frac{1}{\Gamma(\alpha)}\left[(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\beta_1-1}\right.\\&\qquad\left.-(\alpha-1)(t-s)^{\alpha-2}\right]\\&=\frac{1}{\Gamma(\alpha)}(\alpha-1)t^{\alpha-2}\\&\quad\times\left[(1-s)^{\alpha-\beta_1-1}-\left(1-\frac{s}{t}\right)^{\alpha-2}\right]\\&\le\frac{1}{\Gamma(\alpha)}(\alpha-1)t^{\alpha-2}\\&\quad\times\left[(1-s)^{\alpha-\beta_1-1}-(1-s)^{\alpha-2}\right]\\&\le 0,\end{aligned}\tag{34}$$

then, $g_1(t,s)$ is nonincreasing with respect to t on $[s,1]$; hence, we obtain that

$$g_1(t,s)\le g_1(s,s),\quad\forall 0\le s\le t\le 1.\tag{35}$$

Also, we have

$$\frac{\partial g_2(t,s)}{\partial t}=\frac{1}{\Gamma(\alpha)}\left[(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\beta_1-1}\right]\ge 0,\tag{36}$$

then, $g_2(t,s)$ is increasing with respect to t on $[0,s]$. Then, by the fact that $g_1(s,s)=g_2(s,s)$, we have

$$G(t,s)\le G(s,s),\quad\forall t,s\in[0,1].\tag{37}$$

(ii) Since $g_1(\cdot,s)$ is nonincreasing and $g_2(\cdot,s)$ is nondecreasing, for all $s,t\in[0,1]$, we have

$$\begin{aligned}\min_{\gamma\le t\le\delta}G(t,s)&=\begin{cases}\min\limits_{\gamma\le t\le\delta}g_1(t,s), & s\in[0,\gamma],\\ \min\limits_{\gamma\le t\le\delta}\{g_1(t,s),g_2(t,s)\}, & s\in[\gamma,\delta],\\ \min\limits_{\gamma\le t\le\delta}g_2(t,s), & s\in[\delta,1],\end{cases}\\&=\begin{cases}\min\limits_{\gamma\le t\le\delta}g_1(t,s), & s\in[0,r],\\ \min\limits_{\gamma\le t\le\delta}g_2(t,s), & s\in[r,1],\end{cases}\\&\ge\begin{cases}g_1(\delta,s), & s\in[0,r],\\ g_2(\gamma,s), & s\in[r,1],\end{cases}\\&=\begin{cases}\delta^{\alpha-1}(1-s)^{\alpha-\beta_1-1}-(\delta-s)^{\alpha-1}, & s\in[0,r],\\ \gamma^{\alpha-1}(1-s)^{\alpha-\beta_1-1}, & s\in[r,1],\end{cases}\end{aligned}\tag{38}$$

where $\gamma<r<\delta$ is the solution of

$$\delta^{\alpha-1}(1-r)^{\alpha-\beta_1-1}-(\delta-r)^{\alpha-1}=\gamma^{\alpha-1}(1-r)^{\alpha-\beta_1-1},\tag{39}$$

so, we get

$$\min_{\gamma\le t\le\delta}G(t,s)\ge g(s)G(s,s),\quad\forall s\in[0,1],\tag{40}$$

where $g(s)$ is given in (28). This completes the proof. □

Remark 12. If $\gamma \in (0, 1/4)$ and $\delta = 1 - \gamma$, then Lemma 11 satisfies.

In this paper, we assume that $\gamma \in (0, 1/4)$ and $\delta = 1 - \gamma$.

Now, we consider system (6). Assume that (H1), (H2), and (H3) hold; then, by applying Lemmas 7 and 10, $(u, v) \in C(0, 1) \times C(0, 1)$ is a solution of system (6) if and only if $(u, v) \in C[0, 1] \times C[0, 1]$ is a solution of the following nonlinear integral system:

$$\begin{aligned} u(t) = {} & \int_0^1 G(t,s)\,\phi_q \\ & \times\left(\lambda_1 \int_0^1 H(s,r)\,a_1(r) f_1(u(r), v(r))\,dr\right) ds \\ & + \frac{t^{\alpha-1}}{(1-\Delta_1)} \Sigma_{i=1}^{m-2} a_{1i} \\ & \times \int_0^1 G_1(\xi_{1i}, s)\,\phi_q \\ & \times\left(\lambda_1 \int_0^1 H(s,r)\,a_1(r)\right. \\ & \left. \times f_1(u(r), v(r))\,dr\right) ds, \\ v(t) = {} & \int_0^1 G(t,s)\,\phi_q \\ & \times\left(\lambda_2 \int_0^1 H(s,r)\,a_2(r)\, f_2(u(r), v(r))\,dr\right) ds \\ & + \frac{t^{\alpha-1}}{(1-\Delta_2)} \Sigma_{i=1}^{m-2} a_{2i} \\ & \times \int_0^1 G_1(\xi_{2i}, s)\,\phi_q \\ & \times\left(\lambda_2 \int_0^1 H(s,r)\,a_2(r)\right. \\ & \left. \times f_2(u(r), v(r))\,dr\right) ds. \end{aligned} \tag{41}$$

We next recall the Krasnoselskii's fixed point theorem (see [30]). This lemma will be of use in Sections 3 and 4 of this paper.

Theorem 13. *Let E be a Banach space and let $K \subseteq E$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : K\cap(\overline{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator. If eigher*

(1) $\|Ty\| \le \|y\|$ *for* $y \in K \cap \partial\Omega_1$ *and* $\|Ty\| \ge \|y\|$ *for* $y \in K \cap \partial\Omega_2$ *or*

(2) $\|Ty\| \ge \|y\|$ *for* $y \in K \cap \partial\Omega_1$ *and* $\|Ty\| \le \|y\|$ *for* $y \in K \cap \partial\Omega_2$,

then, T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence of a Positive Solution: Case I

Let $\gamma \in (0, 1/4)$ and $\delta = 1 - \gamma$, and define

$$\begin{aligned} \eta &= \min_{\gamma \le t \le \delta} \{g(t)\}, \\ \sigma &= \min\left\{\eta, \gamma^{\alpha-1}\right\}. \end{aligned} \tag{42}$$

In this section, we consider the following assumption:

(L1) There exist numbers f_1^* and f_2^*, with $f_1^*, f_2^* \in (0, +\infty)$, such that

$$\begin{aligned} \lim_{(u,v)\to(0^+,0^+)} \frac{f_1(u,v)}{\phi_p(u+v)} &= f_1^*, \\ \lim_{(u,v)\to(0^+,0^+)} \frac{f_2(u,v)}{\phi_p(u+v)} &= f_2^*. \end{aligned} \tag{43}$$

(L2) There exist numbers f_1^{**} and f_2^{**}, with $f_1^{**}, f_2^{**} \in (0, +\infty)$, such that

$$\begin{aligned} \lim_{(u,v)\to(\infty,\infty)} \frac{f_1(u,v)}{\phi_p(u+v)} &= f_1^{**}, \\ \lim_{(u,v)\to(\infty,\infty)} \frac{f_2(u,v)}{\phi_p(u+v)} &= f_2^{**}. \end{aligned} \tag{44}$$

(L3) There are numbers M_j and m_j, where

$$\begin{aligned} M_j = {} & \frac{1}{2}\left[\int_0^1 G(s,s)\,\phi_q\left(\int_0^1 H(s,r)\,a_j(r)\left(f_j^*\right)dr\right)ds\right. \\ & + \frac{1}{\left(1-\Delta_j\right)} \Sigma_{i=1}^{m-2} a_{ji} \\ & \times \int_0^1 G_1\left(\xi_{ji}, s\right)\phi_q \\ & \left.\times\left(\int_0^1 H(s,r)\,a_j(r)\left(f_j^*\right)dr\right)ds\right]^{-1}, \\ m_j = {} & \frac{3}{2\sigma}\left[\int_0^1 G(s,s)\,\phi_q\left(\int_\gamma^\delta H(s,r)\,a_j(r)\left(f_j^{**}\right)dr\right)ds\right. \\ & + \frac{1}{\left(1-\Delta_j\right)} \Sigma_{i=1}^{m-2} a_{ji} \\ & \times \int_0^1 G_1\left(\xi_{ji}, s\right)\phi_q \\ & \left.\times\left(\int_\gamma^\delta H(s,r)\,a_j(r)\left(f_j^{**}\right)dr\right)ds\right]^{-1}, \end{aligned} \tag{45}$$

such that $\phi_p(m_j) < \lambda_j < \phi_p(M_j)$, $j = 1, 2$.

The basic space used in this paper is a real Banach space $E = C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ with the norm $\|(u,v)\| := \|u\| + \|v\|$, where $\|u\| = \max_{t\in[0,1]}|u(t)|$.

Then, choose a cone $K \subset E$, by

$$K = \Big\{ (u,v) \in E \mid u(t) \geq 0, v(t) \geq 0, \min_{\gamma \leq t \leq \delta} (u(t) + v(t)) \geq \frac{\sigma}{3} \|(u,v)\| \Big\}, \tag{46}$$

and define an operator $T : E \to E$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \quad \forall t \in (0,1), \tag{47}$$

where

$$\begin{aligned} T_1(u,v)(t) = &\int_0^1 G(t,s)\phi_q \times \Big(\lambda_1 \int_0^1 H(s,r) a_1(r) \times f_1(u(r), v(r))\, dr\Big) ds \\ &+ \frac{t^{\alpha-1}}{(1-\Delta_1)} \Sigma_{i=1}^{m-2} a_{1i} \times \int_0^1 G_1(\xi_{1i}, s)\phi_q \times \Big(\lambda_1 \int_0^1 H(s,r) a_1(r) \times f_1(u(r), v(r))\, dr\Big) ds, \\ T_2(u,v)(t) = &\int_0^1 G(t,s)\phi_q \times \Big(\lambda_2 \int_0^1 H(s,r) a_2(r) \times f_2(u(r), v(r))\, dr\Big) ds \\ &+ \frac{t^{\alpha-1}}{(1-\Delta_2)} \Sigma_{i=1}^{m-2} a_{2i} \times \int_0^1 G_1(\xi_{2i}, s)\phi_q \times \Big(\lambda_2 \int_0^1 H(s,r) a_2(r) \times f_2(u(r), v(r))\, dr\Big) ds. \end{aligned} \tag{48}$$

Lemma 14. *Suppose that (H1), (H2), and (H3) hold. Then, the operator $T : K \to K$ is well defined, that is, $T(K) \subseteq K$.*

Proof. For any $(u,v) \in K$, by (H1), (H2), (H3) and Lemma 11, $T_1(u,v)(t) \geq 0$, $T_2(u,v)(t) \geq 0$, $t \in [0,1]$, and it follows from (47) that

$$\begin{aligned} \|T_1(u,v)\| = &\int_0^1 G(s,s)\phi_q \times \Big(\lambda_1 \int_0^1 H(s,r) a_1(r) \times f_1(u(r), v(r))\, dr\Big) ds \\ &+ \frac{1}{(1-\Delta_1)} \Sigma_{i=1}^{m-2} a_{1i} \times \int_0^1 G_1(\xi_{1i}, s)\phi_q \times \Big(\lambda_1 \int_0^1 H(s,r) a_1(r) \times f_1(u(r), v(r))\, dr\Big) ds \\ = &\Big(\int_0^\gamma + \int_\gamma^\delta + \int_\delta^1\Big) \times \Big(G(s,s)\phi_q \times \Big(\int_0^1 H(s,r) f_1(u(r), v(r))\, dr\Big)\Big) ds \\ &+ \frac{1}{(1-\Delta_1)} \Sigma_{i=1}^{m-2} a_{1i} \times \Big(\int_0^\gamma + \int_\gamma^\delta + \int_\delta^1\Big) \times \Big(G_1(\xi_{1i}, s)\phi_q \times \Big(\lambda_1 \int_0^1 H(s,r) a_1(r) \times f_1\ (u(r), v(r))\, dr\Big)\Big) ds \\ \leq 3 \Big[&\int_\gamma^\delta G(s,s)\phi_q \Big(\lambda_1 \int_0^1 H(s,r)\ a_1(r) \times f_1(u(r), v(r))\, dr\Big) ds \\ &+ \frac{1}{(1-\Delta_1)} \Sigma_{i=1}^{m-2} a_{1i} \times \int_\gamma^\delta G_1(\xi_{1i}, s)\phi_q \end{aligned}$$

$$\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)\right.$$
$$\left.\left.\times f_1(u(r),v(r))\,dr\right)ds\right]. \tag{49}$$

Thus, for any $(u,v)\in K$, it follows from Lemma 11 and (49) that

$$\begin{aligned}
&\min_{\gamma\le t\le\delta}T_1(u,v)(t)\\
&=\min_{\gamma\le t\le\delta}\left\{\int_0^1 G(t,s)\,\phi_q\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)\times f_1(u(r),v(r))\,dr\right)ds\right.\\
&\quad+\frac{t^{\alpha-1}}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\left.\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)\times f_1(u(r),v(r))\,dr\right)ds\right\}\\
&\ge\int_0^1 g(s)\,G(s,s)\,\phi_q\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{\gamma^{\alpha-1}}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\ge\eta\int_\gamma^\delta G(s,s)\,\phi_q\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{\gamma^{\alpha-1}}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_\gamma^\delta G_1(\xi_{1i},s)\,\phi_q\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\ge\sigma\left[\int_\gamma^\delta G(s,s)\,\phi_q\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_\gamma^\delta G_1(\xi_{1i},s)\,\phi_q\left.\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)\times f_1(u(r),v(r))\,dr\right)ds\right]\\
&\ge\frac{\sigma}{3}\left\|T_1(u,v)\right\|.
\end{aligned}\tag{50}$$

In the same way, for any $(u,v)\in K$, we have

$$\min_{\gamma\le t\le\delta}T_2(u,v)(t)\ge\frac{\sigma}{3}\left\|T_2(u,v)\right\|. \tag{51}$$

Therefore,

$$\begin{aligned}
&\min_{\gamma\le t\le\delta}\left(T_1(u,v)(t)+T_2(u,v)(t)\right)\\
&\quad\ge\frac{\sigma}{3}\left\|T_1(u,v)\right\|+\frac{\sigma}{3}\left\|T_2(u,v)\right\|\\
&\quad=\frac{\sigma}{3}\left\|\left(T_1(u,v),T_2(u,v)\right)\right\|.
\end{aligned}\tag{52}$$

From the above, we conclude that $T(u,v)(t)=(T_1(u,v)(t),T_2(u,v)(t))\in K$, that is, $T(K)\subset K$. This completes the proof. □

It is clear that the existence of a positive solution for system (6) is equivalent to the existence of a nontrivial fixed point of T in K.

Theorem 15. *Assume that (L1), (L2), and (L3) are satisfied. Then, system* (6) *and* (7) *has at least one positive solution.*

Proof. It follows from Lemma 14 that $T:K\to K$. Furthermore, by the application of the Ascoli-Arzela theorem, which we omit, T is a completely continuous operator.

Condition (L3) implies that there is $\epsilon>0$ sufficiently small such that

$$\begin{aligned}
\phi_q(\lambda_j)\le\frac{1}{2}&\left[\int_0^1 G(s,s)\,\phi_q\left(\int_0^1 H(s,r)\,a_j(r)\left(f_j^*+\epsilon\right)dr\right)ds\right.\\
&+\frac{1}{(1-\Delta_j)}\Sigma_{i=1}^{m-2}a_{ji}
\end{aligned}$$

$$\times \int_0^1 G_1(\xi_{ji}, s)\,\phi_q \left(\int_0^1 H(s,r)\,a_j(r) \times \left(f_j^* + \epsilon\right) dr\right) ds\Bigg]^{-1}, \quad j = 1, 2,$$

$$\phi_q(\lambda_j) \geq \frac{3}{2\sigma}\Bigg[\int_0^1 G(s,s)\,\phi_q \times \left(\int_\gamma^\delta H(s,r)\,a_j(r)\left(f_j^{**} - \epsilon\right) dr\right) ds + \frac{1}{(1-\Delta_j)}\Sigma_{i=1}^{m-2} a_{ji} \times \int_0^1 G_1(\xi_{ji}, s)\,\phi_q \times \left(\int_\gamma^\delta H(s,r)\,a_j(r) \times \left(f_j^{**} - \epsilon\right) dr\right) ds\Bigg]^{-1}, \quad j = 1, 2. \tag{53}$$

Now given this ϵ, it follows from condition (L1) that there exists some number $r_1^* > 0$ such that

$$f_1(u,v) \leq \left(f_1^* + \epsilon\right)\phi_p(u+v), \tag{54}$$

whenever $\|(u,v)\| < r_1^*$. Similarly, by condition (L1), for the same ϵ, there exists some number $r_2^* > 0$ such that

$$f_2(u,v) \leq \left(f_2^* + \epsilon\right)\phi_p(u+v), \tag{55}$$

whenever $\|(u,v)\| < r_2^*$. In particular, by putting $r^* = \min\{r_1^*, r_2^*\}$, we conclude that both (54) and (55) hold whenever $\|(u,v)\| < r^*$. So, define Ω_1 by

$$\Omega_1 = \left\{(u,v) \in K : \|(u,v)\| < r^*\right\}. \tag{56}$$

Then, for $(u,v) \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|T_1(u,v)\| &= \max_{0\leq t\leq 1} T_1(u,v)(t) \\
&= \max_{0\leq t\leq 1}\Bigg\{\int_0^1 G(t,s)\,\phi_q \times \left(\lambda_1 \int_0^1 H(s,r)\,a_1(r) \times f_1(u(r), v(r))\,dr\right) ds \\
&\quad + \frac{t^{\alpha-1}}{(1-\Delta_1)}\Sigma_{i=1}^{m-2} a_{1i} \times \int_0^1 G_1(\xi_{1i}, s)\,\phi_q \times \left(\lambda_1 \int_0^1 H(s,r)\,a_1(r) \times f_1(u(r), v(r))\,dr\right) ds\Bigg\} \\
&= \phi_q(\lambda_1)\Bigg\{\int_0^1 G(s,s)\,\phi_q \times \left(\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r), v(r))\,dr\right) ds \\
&\quad + \frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2} a_{1i} \times \int_0^1 G_1(\xi_{1i}, s)\,\phi_q \times \left(\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r), v(r))\,dr\right) ds\Bigg\} \\
&\leq \phi_q(\lambda_1)\Bigg\{\int_0^1 G(s,s)\,\phi_q \times \left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^* + \epsilon\right) \times \phi_p(u(r) + v(r))\,dr\right) ds \\
&\quad + \frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2} a_{1i} \times \int_0^1 G_1(\xi_{1i}, s)\,\phi_q \times \left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^* + \epsilon\right) \times \phi_p(u(r) + v(r))\,dr\right) ds\Bigg\} \\
&\leq \phi_q(\lambda_1)\Bigg\{\int_0^1 G(s,s)\,\phi_q \times \left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^* + \epsilon\right) \times \phi_p(\|u\| + \|v\|)\,dr\right) ds
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\
&\quad\times\left(\int_0^1 H(s,r)\,a_1(r)\,(f_1^*+\epsilon)\,\phi_p\right.\\
&\qquad\left.\times(\|u\|+\|v\|)\,dr\right)ds\Big\}\\
=\ &\phi_q(\lambda_1)\Big\{\int_0^1 G(s,s)\,\phi_q\\
&\times\left(\int_0^1 H(s,r)\,a_1(r)\,(f_1^*+\epsilon)\,dr\right)\\
&\times(\|u\|+\|v\|)\,ds\\
&+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\
&\quad\times\left(\int_0^1 H(s,r)\,a_1(r)(f_1^*+\epsilon)\,dr\right)\\
&\quad\times(\|u\|+\|v\|)\,ds\Big\}\\
=\ &\phi_q(\lambda_1)\Big\{\int_0^1 G(s,s)\,\phi_q\\
&\quad\times\left(\int_0^1 H(s,r)\,a_1(r)(f_1^*+\epsilon)\,dr\right)ds\\
&+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\
&\quad\times\left(\int_0^1 H(s,r)\,a_1(r)\right.\\
&\qquad\left.\times(f_1^*+\epsilon)\,dr\right)ds\Big\}\\
&\cdot\|(u,v)\|\\
\le\ &\frac{1}{2}\|(u,v)\|.
\end{aligned}\tag{57}$$

Also, by similarly argument, we get

$$\|T_2(u,v)\|\le\frac{1}{2}\|(u,v)\|,\tag{58}$$

for $(u,v)\in K\cap\partial\Omega_1$. Thus, for $(u,v)\in K\cap\partial\Omega_1$, we have

$$\begin{aligned}
\|T(u,v)\| &= \|(T_1(u,v),T_2(u,v))\|\\
&= \|T_1(u,v)\|+\|T_2(u,v)\|\\
&\le\frac{1}{2}\|(u,v)\|+\frac{1}{2}\|(u,v)\|=\|(u,v)\|.
\end{aligned}\tag{59}$$

On the other hand, letting $\epsilon>0$ be the same number selected at the beginning this proof, it follows from condition (L2) that there exists number $r^{**}>0$ such that

$$\begin{aligned}
f_1(u,v)&\ge(f_1^{**}-\epsilon)\,\phi_p(u+v),\\
f_2(u,v)&\ge(f_2^{**}-\epsilon)\,\phi_p(u+v),
\end{aligned}\tag{60}$$

whenever $u+v\ge r^{**}$. Let

$$r=\max\left\{2r^*,\frac{3r^{**}}{\sigma}\right\}.\tag{61}$$

Moreover, let

$$\Omega_2=\{(u,v)\in K:\|(u,v)\|<r\}.\tag{62}$$

Then, $\overline{\Omega}_1\subseteq\Omega_2$.

If $(u,v)\in K\cap\partial\Omega_2$, then it follows that for any $t\in[\gamma,\delta]$,

$$u(t)+v(t)\ge\min_{t\in[\gamma,\delta]}[u(t)+v(t)]\ge\frac{\sigma}{3}\|(u,v)\|\ge r^{**}.\tag{63}$$

Thus, (63) shows that for $(u,v)\in K\cap\partial\Omega_2$, (60) holds, whenever $t\in[\gamma,\delta]$.

So, for each $(u,v)\in K\cap\partial\Omega_2$, we have

$$\begin{aligned}
\|T_1(u,v)\| &= \max_{0\le t\le1}T_1(u,v)(t)\\
&=\phi_q(\lambda_1)\Big\{\int_0^1 G(s,s)\,\phi_q\\
&\quad\times\left(\int_0^1 H(s,r)\,a_1(r)\right.\\
&\qquad\left.\times f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\quad\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\
&\qquad\times\left(\int_0^1 H(s,r)\,a_1(r)\right.\\
&\qquad\quad\left.\times f_1(u(r),v(r))\,dr\right)ds\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\geq \phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\,\phi_q\left(\int_\gamma^\delta H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i} \times \int_0^1 G_1(\xi_{1i},s)\,\phi_q \left.\times\left(\int_\gamma^\delta H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)ds\right\}\\
&\geq \phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\,\phi_q \times\left(\int_\gamma^\delta H(s,r)\,a_1(r)\,(f_1^{**}-\epsilon)\,\phi_p \times (u(r)+v(r))\,dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i} \times \int_0^1 G_1(\xi_{1i},s)\,\phi_q \left.\times\left(\int_\gamma^\delta H(s,r)\,a_1(r)\,(f_1^{**}-\epsilon)\,\phi_p \times (u(r)+v(r))\,dr\right)ds\right\}\\
&\geq \phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\,\phi_q \times\left(\int_\gamma^\delta H(s,r)\,a_1(r)\,(f_1^{**}-\epsilon) \times\phi_p\left(\frac{\sigma}{3}\|(u,v)\|\right)dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i} \times \int_0^1 G_1(\xi_{1i},s)\,\phi_q \left.\times\left(\int_\gamma^\delta H(s,r)\,a_1(r)\,(f_1^{**}-\epsilon) \times\phi_p\left(\frac{\sigma}{3}\|(u,v)\|\right)dr\right)ds\right\}\\
&= \phi_q(\lambda_1)\cdot\frac{\sigma}{3}\left\{\int_0^1 G(s,s)\,\phi_q \times\left(\int_\gamma^\delta H(s,r)\,a_1(r) \times (f_1^{**}-\epsilon)\,dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i} \times \int_0^1 G_1(\xi_{1i},s)\,\phi_q \left.\times\left(\int_\gamma^\delta H(s,r)\,a_1(r) \times (f_1^{**}-\epsilon)\,dr\right)ds\right\}\\
&\quad\cdot\|(u,v)\|\\
&\geq \frac{1}{2}\|(u,v)\|.
\end{aligned}
\tag{64}
$$

Similarly, we obtain that

$$
\left\|T_2(u,v)\right\| \geq \frac{1}{2}\|(u,v)\|, \tag{65}
$$

for $(u,v)\in K\cap\partial\Omega_2$. Thus, for $(u,v)\in K\cap\partial\Omega_2$, we have

$$
\begin{aligned}
\|T(u,v)\| &= \left\|(T_1(u,v),T_2(u,v))\right\|\\
&= \left\|T_1(u,v)\right\| + \left\|T_2(u,v)\right\|\\
&\geq \frac{1}{2}\|(u,v)\| + \frac{1}{2}\|(u,v)\| = \|(u,v)\|.
\end{aligned}
\tag{66}
$$

Thus, all conditions of Theorem 13 are satisfied. Consequently, we conclude that T has a fixed point on K. This is a positive solution of systems (6) and (7). The proof is completed. □

4. Existence of a Positive Solution: Case II

In this section, we assume that $\gamma\in(0,1/4)$ and $\delta=1-\gamma$. We now provide a set of conditions under which the problem (6) and (8) will have at least one positive solution. we need conditions (L1) and (L2) in this section. furthermore, we use notations η,σ, and K which were defined in Section 3. we will introduce new conditions.

(L4) The functionals $\psi_1(u)$ and $\psi_2(v)$ are continuous in u and v and nonnegative for $u, v \geq 0$ and satisfy

$$\lim_{\|u\| \to 0^+} \frac{\psi_1(u)}{\|u\|} = 0, \tag{67}$$

$$\lim_{\|v\| \to 0^+} \frac{\psi_2(v)}{\|v\|} = 0. \tag{68}$$

(L5) There are numbers N_j and n_j, where

$$\begin{aligned} N_j = \frac{1}{4}\Bigg[& \int_0^1 G(s,s)\,\phi_q \\ & \times \left(\int_0^1 H(s,r)\,a_j(r)\left(f_j^*\right) dr \right) ds \\ & + \frac{1}{\left(1-\Delta_j\right)} \Sigma_{i=1}^{m-2} a_{ji} \\ & \times \int_0^1 G_1\left(\xi_{ji}, s\right) \phi_q \\ & \times \left(\int_0^1 H(s,r)\,a_j(r)\left(f_j^*\right) dr \right) ds \Bigg]^{-1}, \\ n_j = \frac{3}{2\sigma}\Bigg[& \int_0^1 G(s,s)\,\phi_q \\ & \times \left(\int_\gamma^\delta H(s,r)\,a_j(r)\left(f_j^{**}\right) dr \right) ds \\ & + \frac{1}{\left(1-\Delta_j\right)} \Sigma_{i=1}^{m-2} a_{ji} \\ & \times \int_0^1 G_1\left(\xi_{ji}, s\right) \phi_q \\ & \times \left(\int_\gamma^\delta H(s,r)\,a_j(r)\left(f_j^{**}\right) dr \right) ds \Bigg]^{-1}, \end{aligned} \tag{69}$$

such that $\phi_p(n_j) < \lambda_j < \phi_p(N_j)$, $j = 1, 2$.

Remark 16. Condition (67) in (L4) is true only if for each $\mu > 0$ there is $r > 0$ such that whenever $0 < \|u\| \leq r$, it follows that $0 < \psi_1(u)/\|u\| < \mu$. The same is true for condition (68) involving ψ_2.

By repeating the way that we used in Section 3, with a minor modification, we can get that $(u, v) \in C(0,1) \times C(0,1)$ is a solution of systems (6) and (8) if and only if $(u, v) \in C[0,1] \times C[0,1]$ is a solution of the following nonlinear integral system:

$$\begin{aligned} u(t) = & \int_0^1 G(t,s)\,\phi_q \\ & \times \left(\lambda_1 \int_0^1 H(s,r)\,a_1(r) f_1(u(r), v(r))\,dr \right) ds \\ & + \frac{t^{\alpha-1}}{(1-\Delta_1)} \Sigma_{i=1}^{m-2} a_{1i} \\ & \times \int_0^1 G_1(\xi_{1i}, s)\,\phi_q \\ & \times \left(\lambda_1 \int_0^1 H(s,r)\,a_1(r) \right. \\ & \left. \times f_1(u(r), v(r))\,dr \right) ds \\ & + \frac{\Gamma(\alpha-\beta_1)\,t^{\alpha-1}}{(1-\Delta_1)\,\Gamma(\alpha)} \psi_1(u), \\ v(t) = & \int_0^1 G(t,s)\,\phi_q \left(\lambda_2 \int_0^1 H(s,r) \right. \\ & \left. \times a_2(r)\, f_2(u(r), v(r))\,dr \right) ds \\ & + \frac{t^{\alpha-1}}{(1-\Delta_2)} \Sigma_{i=1}^{m-2} a_{2i} \\ & \times \int_0^1 G_1(\xi_{2i}, s)\,\phi_q \\ & \times \left(\lambda_2 \int_0^1 H(s,r)\,a_2(r) \right. \\ & \left. \times f_2(u(r), v(r))\,dr \right) ds \\ & + \frac{\Gamma(\alpha-\beta_1)\,t^{\alpha-1}}{(1-\Delta_2)\,\Gamma(\alpha)} \psi_2(v). \end{aligned} \tag{70}$$

Thus, we define $S : E \to E$ defined by $S(u, v) = (S_1(u, v), S_2(u, v))$, where

$$\begin{aligned} S_1(u,v) = & \int_0^1 G(t,s)\,\phi_q \\ & \times \left(\lambda_1 \int_0^1 H(s,r)\,a_1(r) f_1(u(r), v(r))\,dr \right) ds \\ & + \frac{t^{\alpha-1}}{(1-\Delta_1)} \Sigma_{i=1}^{m-2} a_{1i} \\ & \times \int_0^1 G_1(\xi_{1i}, s)\,\phi_q \\ & \times \left(\lambda_1 \int_0^1 H(s,r)\,a_1(r) \right. \\ & \left. \times f_1(u(r), v(r))\,dr \right) ds \\ & + \frac{\Gamma(\alpha-\beta_1)\,t^{\alpha-1}}{(1-\Delta_1)\,\Gamma(\alpha)} \psi_1(u), \end{aligned} \tag{71}$$

$$S_2(u,v) = \int_0^1 G(t,s)\,\phi_q \times\left(\lambda_2\int_0^1 H(s,r)\,a_2(r)\,f_2(u(r),v(r))\,dr\right)ds + \frac{t^{\alpha-1}}{(1-\Delta_2)}\Sigma_{i=1}^{m-2}a_{2i} \times\int_0^1 G_1(\xi_{2i},s)\,\phi_q \times\left(\lambda_2\int_0^1 H(s,r)\,a_2(r) \times f_2(u(r),v(r))\,dr\right)ds + \frac{\Gamma(\alpha-\beta_1)\,t^{\alpha-1}}{(1-\Delta_2)\,\Gamma(\alpha)}\psi_2(v). \tag{72}$$

Lemma 17. *Suppose that (H1), (H2), and (H3) hold. Then, the operator $S: K\to K$ is well defined, that is, $S(K)\subseteq K$.*

Proof. For any $(u,v)\in K$, by (H1), (H2), (H3), and Lemma 11, $S_1(u,v)(t)\ge 0$, $S_2(u,v)(t)\ge 0$, $t\in[0,1]$, and it follows from (71) that

$$\begin{aligned}
\|S_1(u,v)\| &= \int_0^1 G(s,s)\,\phi_q \times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)ds \\
&\quad + \frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i} \times\int_0^1 G_1(\xi_{1i},s)\,\phi_q \times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)ds \\
&\quad + \frac{\Gamma(\alpha-\beta_1)}{(1-\Delta_1)\,\Gamma(\alpha)}\psi_1(u) \\
&= \left(\int_0^\gamma+\int_\gamma^\delta+\int_\delta^1\right) \times\left(G(s,s)\,\phi_q \times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)\right)ds \\
&\quad + \frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i} \times\left[\left(\int_0^\gamma+\int_\gamma^\delta+\int_\delta^1\right) \times\left(G_1(\xi_{1i},s)\,\phi_q \times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)\right)ds\right] \\
&\quad + \frac{\Gamma(\alpha-\beta_1)}{(1-\Delta_1)\,\Gamma(\alpha)}\psi_1(u) \\
&\le 3\left[\int_\gamma^\delta G(s,s)\,\phi_q \times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)ds\right. \\
&\quad + \frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i} \times\int_\gamma^\delta G_1(\xi_{1i},s)\,\phi_q \times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)ds \\
&\quad \left.+\frac{\Gamma(\alpha-\beta_1)}{3(1-\Delta_1)\,\Gamma(\alpha)}\psi_1(u)\right].
\end{aligned} \tag{73}$$

Thus, for any $(u,v)\in K$, it follows from Lemma 11 and (73) that

$$\begin{aligned}
&\min_{\gamma\le t\le\delta} S_1(u,v)(t) \\
&\quad = \min_{\gamma\le t\le\delta}\left\{\int_0^1 G(t,s)\,\phi_q \times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r) \times f_1(u(r),v(r))\,dr\right)ds\right. \\
&\qquad + \frac{t^{\alpha-1}}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i} \times\int_0^1 G_1(\xi_{1i},s)\,\phi_q
\end{aligned}$$

$$\begin{aligned}
&\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)\right.\\
&\qquad\left.\times f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{\Gamma(\alpha-\beta_1)\,t^{\alpha-1}}{(1-\Delta_1)\,\Gamma(\alpha)}\psi_1(u)\Bigg\}\\
&\geq\int_0^1 g(s)\,G(s,s)\,\phi_q\\
&\quad\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{\gamma^{\alpha-1}}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\quad\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\
&\quad\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{\Gamma(\alpha-\beta_1)\,\gamma^{\alpha-1}}{(1-\Delta_1)\,\Gamma(\alpha)}\psi_1(u)\\
&\geq\eta\int_\gamma^\delta G(s,s)\,\phi_q\\
&\quad\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{\gamma^{\alpha-1}}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\quad\times\int_\gamma^\delta G_1(\xi_{1i},s)\,\phi_q\\
&\quad\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{\Gamma(\alpha-\beta_1)\,\gamma^{\alpha-1}}{3(1-\Delta_1)\,\Gamma(\alpha)}\psi_1(u)\\
&\geq\sigma\Bigg[\int_\gamma^\delta G(s,s)\,\phi_q\\
&\quad\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\quad\times\int_\gamma^\delta G_1(\xi_{1i},s)\,\phi_q\\
&\quad\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)f_1(u(r),v(r))\,dr\right)ds\\
&\quad+\frac{\Gamma(\alpha-\beta_1)}{3(1-\Delta_1)\,\Gamma(\alpha)}\psi_1(u)\Bigg]\\
&\geq\frac{\sigma}{3}\left\|S_1(u,v)\right\|.
\end{aligned}\tag{74}$$

In the same way, for any $(u,v)\in K$, we have

$$\min_{\gamma\leq t\leq\delta}S_2(u,v)(t)\geq\frac{\sigma}{3}\left\|S_2(u,v)\right\|.\tag{75}$$

Therefore,

$$\begin{aligned}
&\min_{\gamma\leq t\leq\delta}\left(S_1(u,v)(t)+S_2(u,v)(t)\right)\\
&\qquad\geq\frac{\sigma}{3}\left\|S_1(u,v)\right\|+\frac{\sigma}{3}\left\|S_2(u,v)\right\|\\
&\qquad=\frac{\sigma}{3}\left\|\left(S_1(u,v),S_2(u,v)\right)\right\|.
\end{aligned}\tag{76}$$

From the above, we conclude that $S(u,v)(t)=(S_1(u,v)(t),S_2(u,v)(t))\in K$, that is, $S(K)\subset K$. This completes the proof. □

It is obvious that existence of a solution for the problem (6) and (8) is equivalent to existence of a fixed point for S on K.

Theorem 18. *Suppose that conditions (L1), (L2), (L4), and (L5) hold. Then, systems* (6) *and* (8) *has at least one positive solution.*

Proof. Lemma 17 shows that $S:K\to K$. Moreover, from continuity of ψ_1,ψ_2,G, and f, it is obvious that both S_1 and S_2 are completely continuous operators by the application of the Ascoli-Arzela theorem. Then, S is a completely continuous operator. Also, condition (L3) implies that there is $\epsilon>0$ sufficiently small such that

$$\begin{aligned}
\phi_q\left(\lambda_j\right)\leq\frac{1}{4}\Bigg[&\int_0^1 G(s,s)\,\phi_q\\
&\times\left(\int_0^1 H(s,r)\,a_j(r)\left(f_j^*+\epsilon\right)dr\right)ds\\
&+\frac{1}{\left(1-\Delta_j\right)}\Sigma_{i=1}^{m-2}a_{ji}\\
&\times\int_0^1 G_1\left(\xi_{ji},s\right)\phi_q\\
&\times\left(\int_0^1 H(s,r)\,a_j(r)\right.\\
&\qquad\left.\times\left(f_j^*+\epsilon\right)dr\right)ds\Bigg]^{-1},\\
&\qquad j=1,2,
\end{aligned}$$

$$\begin{aligned}\phi_q(\lambda_j) \geq \frac{3}{2\sigma}\Bigg[\int_0^1 G(s,s)\,\phi_q &\times\left(\int_\gamma^\delta H(s,r)\,a_j(r)\left(f_j^{**}-\epsilon\right)dr\right)ds\\ &+\frac{1}{(1-\Delta_j)}\Sigma_{i=1}^{m-2}a_{ji}\\ &\times\int_0^1 G_1(\xi_{ji},s)\,\phi_q\\ &\times\left(\int_\gamma^\delta H(s,r)\,a_j(r)\right.\\ &\left.\times\left(f_j^{**}-\epsilon\right)dr\right)ds\Bigg]^{-1},\\ &j=1,2.\end{aligned} \tag{77}$$

Now given this ϵ, just as before, conditions (54) and (55) remain true whenever $\|(u,v)\|<r^*$, exactly as in the proof of Theorem 15. It follows from (L4) and Remark 16 that there is r_1^{***} such that $\psi_1(u)\leq\mu\|u\|$ whenever $0<\|u\|\leq r_1^{***}$, and there is r_2^{***} such that $\psi_2(v)\leq\mu\|v\|$ whenever $0<\|v\|\leq r_2^{***}$. In particular and without loss of generality, let us suppose that

$$0<\mu<\frac{(1-\Delta_j)\,\Gamma(\alpha)}{2\Gamma(\alpha-\beta_1)},\quad j=1,2. \tag{78}$$

Now, let $r_1 = \min\{r^*, r_1^{***}, r_2^{***}\}$. Observe that for any $(u,v)\in K$ we have that $\|u\|,\|v\|\leq\|(u,v)\|$. Then, we obtain that for all $(u,v)\in K$ satisfying $0<\|(u,v)\|<r_1$ we have

$$\begin{aligned} f_1(u,v)&\leq\left(f_1^*+\epsilon\right)\phi_p(u+v),\\ f_2(u,v)&\leq\left(f_2^*+\epsilon\right)\phi_p(u+v),\\ \psi_1(u)&\leq\frac{(1-\Delta_1)\,\Gamma(\alpha)}{2\Gamma(\alpha-\beta_1)}\|u\|,\\ \psi_2(v)&\leq\frac{(1-\Delta_2)\,\Gamma(\alpha)}{2\Gamma(\alpha-\beta_1)}\|v\|.\end{aligned} \tag{79}$$

So, define Ω_1 by $\Omega_1=\{(u,v)\in K: 0<\|(u,v)\|<r_1\}$. We obtain for $(u,v)\in K\cap\partial\Omega_1$ that

$$\begin{aligned}\|S_1(u,v)\| &= \max_{0\leq t\leq 1} S_1(u,v)(t)\\ &= \max_{0\leq t\leq 1}\Bigg\{\int_0^1 G(t,s)\,\phi_q\\ &\quad\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)\right.\\ &\quad\left.\times f_1(u(r),v(r))\,dr\right)ds\\ &\quad+\frac{t^{\alpha-1}}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\ &\quad\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\ &\quad\times\left(\lambda_1\int_0^1 H(s,r)\,a_1(r)\right.\\ &\quad\left.\times f_1(u(r),v(r))\,dr\right)ds\\ &\quad+\frac{t^{\alpha-1}\Gamma(\alpha-\beta_1)}{\Gamma(\alpha)(1-\Delta_1)}\psi_1(u)\Bigg\}\\ &\leq\phi_q(\lambda_1)\Bigg\{\int_0^1 G(s,s)\,\phi_q\\ &\quad\times\left(\int_0^1 H(s,r)\,a_1(r)\right.\\ &\quad\left.\times f_1(u(r),v(r))\,dr\right)ds\\ &\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\ &\quad\times\int_0^1 G_1(\xi_{1i},s)\\ &\quad\times\phi_q\left(\int_0^1 H(s,r)\,a_1(r)\right.\\ &\quad\left.\times f_1(u(r),v(r))\,dr\right)ds\Bigg\}\\ &\quad+\frac{\Gamma(\alpha-\beta_1)}{\Gamma(\alpha)(1-\Delta_1)}\psi_1(u)\\ &\leq\phi_q(\lambda_1)\Bigg\{\int_0^1 G(s,s)\,\phi_q\\ &\quad\times\left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^*+\epsilon\right)\right.\\ &\quad\left.\times\phi_p(u(r)+v(r))\,dr\right)ds\\ &\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\ &\quad\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\ &\quad\times\left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^*+\epsilon\right)\right.\\ &\quad\left.\times\phi_p(u(r)+v(r))\,dr\right)ds\Bigg\}\\ &\quad+\frac{1}{2}\|u\|\end{aligned}$$

$$\begin{aligned}
&\leq \phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\,\phi_q\right.\\
&\qquad\times\left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^*+\epsilon\right)\right.\\
&\qquad\left.\times\phi_p(\|u\|+\|v\|)\,dr\right)ds\\
&\qquad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\qquad\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\
&\qquad\times\left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^*+\epsilon\right)\phi_p\right.\\
&\qquad\left.\left.\times(\|u\|+\|v\|)\,dr\right)ds\right\}\\
&\quad+\frac{1}{2}\|u\|\\
&=\phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\,\phi_q\right.\\
&\qquad\times\left(\int_0^1 H(s,r)\,a_1(r)\right.\\
&\qquad\left.\times\left(f_1^*+\epsilon\right)dr\right)(\|u\|+\|v\|)\,ds\\
&\qquad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\qquad\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\
&\qquad\times\left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^*+\epsilon\right)dr\right)\\
&\qquad\left.\times(\|u\|+\|v\|)\,ds\right\}\\
&\quad+\frac{1}{2}\|u\|\\
&=\phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\,\phi_q\right.\\
&\qquad\times\left(\int_0^1 H(s,r)\,a_1(r)\left(f_1^*+\epsilon\right)dr\right)ds\\
&\qquad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\\
&\qquad\times\int_0^1 G_1(\xi_{1i},s)\,\phi_q\\
&\qquad\times\left(\int_0^1 H(s,r)\,a_1(r)\right.\\
&\qquad\left.\left.\times\left(f_1^*+\epsilon\right)dr\right)ds\right\}\\
&\qquad\cdot\|(u,v)\|+\frac{1}{2}\|u\|\\
&\leq\frac{1}{4}\|(u,v)\|+\frac{1}{2}\|u\|.
\end{aligned}\tag{80}$$

Also, by similarly argument, we obtain that

$$\left\|S_2(u,v)\right\|\leq\frac{1}{4}\|(u,v)\|+\frac{1}{2}\|v\|,\tag{81}$$

for $(u,v)\in K\cap\partial\Omega_1$. Thus, for $(u,v)\in K\cap\partial\Omega_1$, we have

$$\begin{aligned}
\|S(u,v)\| &= \left\|\left(S_1(u,v),S_2(u,v)\right)\right\|\\
&=\left\|S_1(u,v)\right\|+\left\|S_2(u,v)\right\|\\
&\leq\frac{1}{4}\|(u,v)\|+\frac{1}{4}\|(u,v)\|+\frac{1}{2}\|u\|+\frac{1}{2}\|v\|\\
&=\frac{1}{2}\|(u,v)\|+\frac{1}{2}(\|u\|+\|v\|)\\
&=\frac{1}{2}\|(u,v)\|+\frac{1}{2}\|(u,v)\|=\|(u,v)\|.
\end{aligned}\tag{82}$$

This implies that for $(u,v)\in K\cap\partial\Omega_1$, we have

$$\|S(u,v)\|\leq\|(u,v)\|.\tag{83}$$

On the other hand, letting $\epsilon>0$ be the same number selected at the beginning this proof, as before, condition (L2) implies that there exists number $r^{**}>0$ such that

$$\begin{aligned}
f_1(u,v)&\geq\left(f_1^{**}-\epsilon\right)\phi_p(u+v),\\
f_2(u,v)&\geq\left(f_2^{**}-\epsilon\right)\phi_p(u+v),
\end{aligned}\tag{84}$$

whenever $u+v\geq r^{**}$. Let

$$r_2=\max\left\{2r_1,\frac{3r^{**}}{\sigma}\right\}.\tag{85}$$

Moreover, if we let

$$\Omega_2=\left\{(u,v)\in K:\|(u,v)\|<r_2\right\},\tag{86}$$

then, $\overline{\Omega}_1\subseteq\Omega_2$.

If $(u,v)\in K\cap\partial\Omega_2$, then, it follows that for any $t\in[\gamma,\delta]$,

$$u(t)+v(t)\geq\min_{t\in[\gamma,\delta]}[u(t)+v(t)] \geq\frac{\sigma}{3}\|(u,v)\|\geq r^{**}. \tag{87}$$

Thus, (87) shows that for $(u,v)\in K\cap\partial\Omega_2$, (84) holds, whenever $t\in[\gamma,\delta]$.

In addition, recall that by condition (L4), ψ_1 and ψ_2 are assumed to be nonnegative for $(u,v)\in K$. So, for each $(u,v)\in K\cap\partial\Omega_2$, we have

$$\begin{aligned}
\|S_1(u,v)\| &= \max_{0\leq t\leq 1}S_1(u,v)(t)\\
&=\phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\phi_q\left(\int_0^1 H(s,r)a_1(r)\times f_1(u(r),v(r))\,dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_0^1 G_1(\xi_{1i},s)\phi_q\left.\left(\int_0^1 H(s,r)a_1(r)\times f_1(u(r),v(r))\,dr\right)ds\right\}\\
&\quad+\frac{\Gamma(\alpha-\beta_1)}{\Gamma(\alpha)(1-\Delta_1)}\psi_1(u)\\
&\geq\phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\phi_q\left(\int_\gamma^\delta H(s,r)a_1(r)\times f_1(u(r),v(r))\,dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_0^1 G_1(\xi_{1i},s)\phi_q\left.\left(\int_\gamma^\delta H(s,r)a_1(r)\times f_1(u(r),v(r))\,dr\right)ds\right\}\\
&\geq\phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\phi_q\left(\int_\gamma^\delta H(s,r)a_1(r)(f_1^{**}-\epsilon)\times\phi_p(u(r)+v(r))\,dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_0^1 G_1(\xi_{1i},s)\phi_q\left.\left(\int_\gamma^\delta H(s,r)a_1(r)(f_1^{**}-\epsilon)\phi_p\times(u(r)+v(r))\,dr\right)ds\right\}\\
&\geq\phi_q(\lambda_1)\left\{\int_0^1 G(s,s)\phi_q\left(\int_\gamma^\delta H(s,r)a_1(r)(f_1^{**}-\epsilon)\phi_p\times\left(\frac{\sigma}{3}\|u,v\|\right)dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_0^1 G_1(\xi_{1i},s)\phi_q\left.\left(\int_\gamma^\delta H(s,r)a_1(r)(f_1^{**}-\epsilon)\times\phi_p\left(\frac{\sigma}{3}\|u,v\|\right)dr\right)ds\right\}\\
&=\phi_q(\lambda_1)\cdot\frac{\sigma}{3}\left\{\int_0^1 G(s,s)\phi_q\left(\int_\gamma^\delta H(s,r)a_1(r)\times(f_1^{**}-\epsilon)\,dr\right)ds\right.\\
&\quad+\frac{1}{(1-\Delta_1)}\Sigma_{i=1}^{m-2}a_{1i}\times\int_0^1 G_1(\xi_{1i},s)\phi_q
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_\gamma^\delta H(s,r)\,a_1(r)\right.\\
&\qquad\left.\left.\times\left(f_1^{**}-\epsilon\right)dr\right)ds\right\}\\
&\cdot\|(u,v)\|\\
&\geq\frac{1}{2}\|(u,v)\|.
\end{aligned} \tag{88}
$$

Similarly, one can get

$$\left\|S_2(u,v)\right\|\geq\frac{1}{2}\|(u,v)\|, \tag{89}$$

for $(u,v)\in K\cap\partial\Omega_2$. Thus, for $(u,v)\in K\cap\partial\Omega_2$, we have

$$
\begin{aligned}
\|S(u,v)\|&=\left\|\left(S_1(u,v),S_2(u,v)\right)\right\|\\
&=\left\|S_1(u,v)\right\|+\left\|S_2(u,v)\right\|\\
&\geq\frac{1}{2}\|(u,v)\|+\frac{1}{2}\|(u,v)\|=\|(u,v)\|.
\end{aligned} \tag{90}
$$

Thus, all conditions of Theorem 13 are satisfied. Consequently, we conclude that S has a fixed point on K. This is a positive solution of systems (6) and (8). The proof is completed. □

5. Application

Example 19. Consider the following singular boundary value problem:

$$
\begin{aligned}
&D_{0^+}^{3/2}\phi_p\left(D_{0^+}^{3/2}u\right)(t)=\lambda_1 a_1(t)\,f_1(u(t),v(t)),\\
&\qquad t\in(0,1),\\
&D_{0^+}^{3/2}\phi_p\left(D_{0^+}^{3/2}v\right)(t)=\lambda_2 a_2(t)\,f_2(u(t),v(t)),\\
&\qquad t\in(0,1),
\end{aligned} \tag{91}
$$

subject to the boundary conditions

$$
\begin{aligned}
&D_{0^+}^{3/2}u(0)=D_{0^+}^{3/2}u(1)=0,\quad u(0)=0,\\
&D_{0^+}^{1/2}u(1)-\frac{1}{2}D_{0^+}^{1/2}u\left(\frac{1}{4}\right)-\frac{1}{3}D_{0^+}^{1/2}u\left(\frac{1}{2}\right)=0,\\
&D_{0^+}^{3/2}v(0)=D_{0^+}^{3/2}v(1)=0,\quad v(0)=0,\\
&D_{0^+}^{1/2}v(1)-\frac{3}{4}D_{0^+}^{1/2}v\left(\frac{1}{3}\right)-\frac{1}{5}D_{0^+}^{1/2}v\left(\frac{2}{3}\right)=0.
\end{aligned} \tag{92}
$$

Here, $\alpha=\beta=3/2$, $\beta_1=1/2$, $p=2$, $m=4$, $a_{11}=1/2$, $a_{12}=1/3$, $a_{21}=3/4$, $a_{22}=1/5$, $\xi_{11}=1/4$, $\xi_{12}=1/2$, $\xi_{21}=1/3$ and $\xi_{22}=2/3$. Now, we have $q=2$,

$$
\begin{aligned}
&f_1(u,v)=(u+v)\left(500-\frac{495}{u^2+v^2+1}\right),\\
&f_2(u,v)=(u+v)\left(1000-\frac{990}{u^2+v^2+1}\right),\\
&a_1(t)=t,\quad a_2(t)=2t.
\end{aligned} \tag{93}
$$

It is easy to check that $f_1,f_2:[0,\infty)\times[0,\infty)\to[0,\infty)$ are continuous. The functions $a_1(t)$ and $a_2(t)$ are obviously nonnegative for all $t\in[0,1]$. We now check that all conditions of Theorem 15 hold. By definition of functions f_1 and f_2, we get

$$
\begin{aligned}
&\lim_{(u,v)\to(0^+,0^+)}\frac{f_1(u,v)}{\phi_p(u+v)}\\
&\quad=\lim_{(u,v)\to(0^+,0^+)}\left(500-\frac{495}{u^2+v^2+1}\right)=5,\\
&\lim_{(u,v)\to(0^+,0^+)}\frac{f_2(u,v)}{\phi_p(u+v)}\\
&\quad=\lim_{(u,v)\to(0^+,0^+)}\left(1000-\frac{990}{u^2+v^2+1}\right)=10.
\end{aligned} \tag{94}
$$

Thus, let

$$f_1^*=5,\qquad f_2^*=10. \tag{95}$$

On the other hand, we have

$$
\begin{aligned}
&\lim_{(u,v)\to(\infty,\infty)}\frac{f_1(u,v)}{\phi_p(u+v)}\\
&\quad=\lim_{(u,v)\to(\infty,\infty)}\left(500-\frac{495}{u^2+v^2+1}\right)=500,\\
&\lim_{(u,v)\to(\infty,\infty)}\frac{f_2(u,v)}{\phi_p(u+v)}\\
&\quad=\lim_{(u,v)\to(\infty,\infty)}\left(1000-\frac{990}{u^2+v^2+1}\right)=1000.
\end{aligned} \tag{96}
$$

Thus, let

$$f_1^{**}=500,\qquad f_2^{**}=1000. \tag{97}$$

It follows from (94)–(97) that conditions (L1) and (L2) hold.

Choose $\gamma=1/4$, $\delta=3/4$. Then, by direct calculations, we can obtain that $\eta=0.3780$, $\sigma=0.3536$, and

$$
\begin{aligned}
&\Delta_1=0.8333,\qquad \Delta_2=0.9500,\\
&m_1=0.0168,\qquad m_2=0.0014,\\
&M_1=0.1976,\qquad M_2=0.0164.
\end{aligned} \tag{98}
$$

Then, for $0.0168<\lambda_1<0.1976$ and $0.0014<\lambda_2<0.0164$, condition (L3) holds. Thus, all conditions of Theorem 15 hold. Hence, system (91) with boundary conditions (92) has at least one positive solution.

Now, consider the problem (91) with following boundary conditions:

$$
\begin{aligned}
&D_{0^+}^{3/2}u(0)=D_{0^+}^{3/2}u(1)=0,\quad u(0)=0,\\
&D_{0^+}^{1/2}u(1)-\frac{1}{2}D_{0^+}^{1/2}u\left(\frac{1}{4}\right)-\frac{1}{3}D_{0^+}^{1/2}u\left(\frac{1}{2}\right)=\psi_1(u),\\
&D_{0^+}^{3/2}v(0)=D_{0^+}^{3/2}v(1)=0,\quad v(0)=0,\\
&D_{0^+}^{1/2}v(1)-\frac{3}{4}D_{0^+}^{1/2}v\left(\frac{1}{3}\right)-\frac{1}{5}D_{0^+}^{1/2}v\left(\frac{2}{3}\right)=\psi_2(v),
\end{aligned} \tag{99}
$$

where

$$\psi_1(u) = \left(u\left(\frac{1}{4}\right)\right)^2,$$
$$\psi_2(v) = \left(v\left(\frac{3}{4}\right)\right)^3. \tag{100}$$

In this case, we check that all conditions of Theorem 18 hold. It follows from (94)–(97) that conditions (L1) and (L2) hold. We now show that (L4) and (L5) hold:

$$\begin{aligned}\lim_{\|u\|\to 0^+}\frac{\psi_1(u)}{\|u\|} &= \lim_{\|u\|\to 0^+}\frac{(u(1/4))^2}{\|u\|}\\ &\le \lim_{\|u\|\to 0^+}\frac{\|u\|^2}{\|u\|} = \lim_{\|u\|\to 0^+}\|u\| = 0,\\ \lim_{\|v\|\to 0^+}\frac{\psi_2(v)}{\|v\|} &= \lim_{\|v\|\to 0^+}\frac{(v(3/4))^3}{\|v\|}\\ &\le \lim_{\|v\|\to 0^+}\frac{\|v\|^3}{\|v\|} = \lim_{\|v\|\to 0^+}\|v\|^2 = 0.\end{aligned} \tag{101}$$

So, condition (L4) is satisfied. Now, by direct calculation, one can get

$$n_1 = 0.0168, \qquad n_2 = 0.0014,$$
$$N_1 = 0.0998, \qquad N_2 = 0.0082. \tag{102}$$

Then, for $0.0168 < \lambda_1 < 0.0998$ and $0.0014 < \lambda_2 < 0.0082$, condition (L5) holds. Thus, all conditions of Theorem 18 hold. Hence, system (91) with the boundary conditions (99) has at least one positive solution.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable suggestions and comments.

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