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Research Article

Analysis of Fractal Wave Equations by Local Fractional Fourier Series Method

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The fractal wave equations with local fractional derivatives are investigated in this paper. The analytical solutions are obtained by using local fractional Fourier series method. The present method is very efficient and accurate to process a class of local fractional differential equations.

1. Introduction

Fractional calculus deals with derivative and integrals of arbitrary orders [1]. During the last four decades, fractional calculus has been applied to almost every field of science and engineering [2–6]. In recent years, there has been a great deal of interest in fractional differential equations [7]. As a result, various kinds of analytical methods were developed [8–18]. For example, there are the exp-function method [8], the variational iteration method [9, 10], the homotopy perturbation method [11], the homotopy analysis method [12], the heat-balance integral method [13], the fractional variational iteration method [14, 15], the fractional difference method [16], the finite element method [17], the fractional Fourier and Laplace transforms [18], and so on.

Recently, local fractional calculus was applied to deal with problems for nondifferentiable functions; see [19–26] and the references therein. There are also analytical methods for solving the local fractional differential equations, which are referred to in [27–34]. The local fractional series method [32–34] was applied to process the local fractional wave equation in fractal vibrating [32] and local fractional heat-conduction equation [33].

More recently, the wave equation on the Cantor sets was considered as [21, 28]

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}}. \tag{1}$$

Local damped wave equation was written in the form [30]

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = f(x,t), \tag{2}$$

and local fractional dissipative wave equation in fractal strings was [31]

$$\begin{aligned} &\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} \\ &\quad - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} - \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = f(x,t). \end{aligned} \tag{3}$$

In this paper, we investigate the application of local fractional series method for solving the following local fractional wave equation:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0, \tag{4}$$

where initial and boundary conditions are presented as

$$u(0,t)=u(l,t)=\frac{\partial^{\alpha}u(l,0)}{\partial x^{\alpha}}=0,$$
$$u(x,0)=f(x), \tag{5}$$
$$\frac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}}=g(x).$$

The organization of the paper is as follows. In Section 2, the basic concepts of local fractional calculus and local fractional Fourier series are introduced. In Section 3, we present a local fractional Fourier series solution of wave equation with local fractional derivative. Two examples are shown in Section 4. Finally, Section 5 is devoted to our conclusions.

2. Mathematical Tools

In this section, we present some concepts of local fractional continuity, local fractional derivative, and local fractional Fourier series.

Definition 1 (see [21, 28, 30–32]). Suppose that there is

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}, \tag{6}$$

with $|x-x_0|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in R$. Then $f(x)$ is called local fractional continuous at $x=x_0$, where $\rho^{\alpha}|x-x_0|^{\alpha}\le|f(x_1)-f(x_2)|\le\tau^{\alpha}|x-x_0|^{\alpha}$ with $\rho,\tau>0$.

Suppose that the function $f(x)$ satisfies the above properties of the local fractional continuity. Then the condition (6) for $x\in(a,b)$ is denoted as

$$f(x)\in C_{\alpha}(a,b), \tag{7}$$

where $\dim_H f(x)=\alpha$.

Definition 2 (see [19–21]). Let $f(x)\in C_{\alpha}(a,b)$. Local fractional derivative of $f(x)$ of order α at $x=x_0$ is given by

$$D_x{}^{(\alpha)}f(x_0)=f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}}, \tag{8}$$

where $\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(1+\alpha)\Delta(f(x)-f(x_0))$.

Definition 3 (see [19, 20, 32–34]). Let $f(x)\in C_{\alpha}(-\infty,+\infty)$, and let $f(x)$ be $2l$-periodic. For $k\in Z$, local fraction Fourier series of $f(x)$ is defined as

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_n\cos_{\alpha}\frac{\pi^{\alpha}(kx)^{\alpha}}{l^{\alpha}}+b_n\sin_{\alpha}\frac{\pi^{\alpha}(kx)^{\alpha}}{l^{\alpha}}\right), \tag{9}$$

where the local fraction Fourier coefficients are

$$a_n=\frac{1}{l^{\alpha}}\int_{-l}^{l}f(x)\cos_{\alpha}\frac{\pi^{\alpha}(kx)^{\alpha}}{l^{\alpha}}(dx)^{\alpha},$$
$$b_n=\frac{1}{l^{\alpha}}\int_{-l}^{l}f(x)\sin_{\alpha}\frac{\pi^{\alpha}(kx)^{\alpha}}{l^{\alpha}}(dx)^{\alpha}, \tag{10}$$

with local fractional integral given by [21, 29–34]

$${}_aI_b{}^{(\alpha)}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}f(t_j)(\Delta t_j)^{\alpha}, \tag{11}$$

where $\Delta t_j=t_{j+1}-t_j, \Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_j,\ldots\}$ and $[t_j,t_{j+1}]$, $j=0,\ldots,N-1, t_0=a, t_N=b$, is a partition of the interval $[a,b]$.

In view of (10), the weights of the fractional trigonometric functions are expressed as follows:

$$a_n=\frac{1/(\Gamma(1+\alpha))\int_{-l}^{l}f(x)\cos_{\alpha}n^{\alpha}(\pi x/l)^{\alpha}(dx)^{\alpha}}{1/(\Gamma(1+\alpha))\int_{-l}^{l}\cos_{\alpha}^{2}n^{\alpha}(\pi x/l)^{\alpha}(dx)^{\alpha}},$$
$$b_n=\frac{1/(\Gamma(1+\alpha))\int_{-l}^{l}f(x)\sin_{\alpha}n^{\alpha}(\pi x/l)^{\alpha}(dx)^{\alpha}}{1/(\Gamma(1+\alpha))\int_{-l}^{l}\sin_{\alpha}^{2}n^{\alpha}(\pi x/l)^{\alpha}(dx)^{\alpha}}. \tag{12}$$

Lemma 4 (see [21]). *If m and h are constant coefficients, then local fractional differential equation with constant coefficients*

$$\frac{d^{2\alpha}y}{dx^{2\alpha}}+m\frac{d^{\alpha}y}{dx^{\alpha}}+hy=0\quad(m^2-4h<0) \tag{13}$$

has a family of solution

$$y(x)=AE_{\alpha}\left(\frac{-m-i^{\alpha}\sqrt{4h-m^2}}{2}x^{\alpha}\right)+BE_{\alpha}\left(\frac{-m+i^{\alpha}\sqrt{4h-m^2}}{2}x^{\alpha}\right) \tag{14}$$

with two constants A and B.

Proof. See [21]. □

3. Solution to Wave Equation with Local Fractional Derivative

If we have the particular solution of (4) in the form

$$u(x,t)=\phi(x)T(t), \tag{15}$$

then we get the equations

$$\phi^{(2\alpha)}+\lambda^{2\alpha}\phi=0; \tag{16}$$

$$T^{(2\alpha)}+T^{(\alpha)}+\lambda^{2\alpha}T=0. \tag{17}$$

with the boundary conditions

$$\phi(0)=\phi^{(\alpha)}(l)=0. \tag{18}$$

Equation (4) has the solution

$$\phi(x)=C_1\cos_\alpha\lambda^\alpha x^\alpha+C_2\sin_\alpha\lambda^\alpha x^\alpha, \tag{19}$$

where C_1 and C_2 are all constant numbers.

According to (19), for $x=0$ and $x=l$ we get

$$\begin{aligned}\phi(0)&=C_1=0,\\ \phi(l)&=\phi(x)\big|_{x=l}=C_2\sin_\alpha\lambda^\alpha l^\alpha=0.\end{aligned} \tag{20}$$

Obviously $C_2\neq 0$, since otherwise $\phi(x)\equiv 0$.

Hence, we arrive at

$$\lambda_n^\alpha l^\alpha=n^\alpha\pi^\alpha, \tag{21}$$

where n is an integer.

We notice

$$\begin{aligned}\lambda_n^\alpha&=\left(\frac{n\pi}{l}\right)^\alpha \quad (n=0,1,2,\dots),\\ \phi_n(x)&=\sin_\alpha\lambda_n^\alpha x^\alpha\\ &=\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha \quad (n=0,1,2,\dots).\end{aligned} \tag{22}$$

For $\lambda^\alpha=\lambda_n^\alpha$ and $0<\rho$, following (17) implies that

$$\begin{aligned}\sum_{n=1}^{\infty}T_n(t)=\sum_{n=1}^{\infty}E_\alpha\left(-\frac{t^\alpha}{2}\right)\\ \times\left(A_n\cos_\alpha\rho t^\alpha+B_n\sin_\alpha\rho t^\alpha\right),\end{aligned} \tag{23}$$

where

$$\rho=\frac{\sqrt{4(n\pi/l)^{2\alpha}-1}}{2}. \tag{24}$$

Therefore,

$$\begin{aligned}u_n(x,t)&=\phi_n(x)T_n(t)\\ &=A_n\cos_\alpha\rho t^\alpha\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha E_\alpha\left(-\frac{1}{2}t^\alpha\right)\\ &\quad+B_n\sin_\alpha\rho t^\alpha\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha E_\alpha\left(-\frac{1}{2}t^\alpha\right).\end{aligned} \tag{25}$$

We now suppose a local fractional Fourier series solution of (4):

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}u_n(x,t)\\ &=\sum_{n=1}^{\infty}E_\alpha\left(-\frac{t^\alpha}{2}\right)\\ &\quad\times\left(A_n\cos_\alpha\rho t^\alpha+B_n\sin_\alpha\rho t^\alpha\right)\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha.\end{aligned} \tag{26}$$

Therefore,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\sum_{n=1}^{\infty}\frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha}, \tag{27}$$

where

$$\begin{aligned}&\frac{\partial u_n(x,t)}{\partial t^\alpha}\\ &=-\frac{1}{2}E_\alpha\left(-\frac{t^\alpha}{2}\right)\left(A_n\cos_\alpha\rho t^\alpha+B_n\sin_\alpha\rho t^\alpha\right)\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha\\ &\quad+\rho E_\alpha\left(-\frac{t^\alpha}{2}\right)\left(-A_n\sin_\alpha\rho t^\alpha+B_n\cos_\alpha\rho t^\alpha\right)\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha,\end{aligned} \tag{28}$$

with $\rho=\sqrt{(4(n\pi/l)^{2\alpha}-1)/2}$.

Submitting (26) to (5), we have

$$\begin{aligned}u(x,0)&=\sum_{n=1}^{\infty}u_n(x,0)\\ &=\sum_{n=1}^{\infty}A_n\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha=f(x),\end{aligned} \tag{29}$$

$$\begin{aligned}&\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\\ &=\sum_{n=1}^{\infty}\left(-\frac{1}{2}A_n+\rho B_n\right)\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha=g(x).\end{aligned} \tag{30}$$

So,

$$\begin{aligned}&\sum_{n=1}^{\infty}\rho B_n\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha\\ &=g(x)+\sum_{n=1}^{\infty}\frac{1}{2}A_n\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha=g(x)+\frac{1}{2}f(x).\end{aligned} \tag{31}$$

Let

$$G(x)=g(x)+\frac{1}{2}f(x). \tag{32}$$

In view of (30) and (31), we rewrite

$$\begin{aligned}\sum_{n=1}^{\infty}A_n\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha&=f(x),\\ \sum_{n=1}^{\infty}\rho B_n\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha&=G(x).\end{aligned} \tag{33}$$

We now find the local fractional Fourier coefficients of $f(x)$ and $G(x)$, respectively,

$$A_n = \frac{1/(\Gamma(1+\alpha))\int_0^l f(x)\sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{1/(\Gamma(1+\alpha))\int_0^l \sin_\alpha^2 n^\alpha (\pi x/l)^\alpha (dx)^\alpha} \quad (n=0,1,2,\ldots),$$
$$\rho B_n = \frac{1/(\Gamma(1+\alpha))\int_0^l G(x)\sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{1/(\Gamma(1+\alpha))\int_0^l \sin_\alpha^2 n^\alpha (\pi x/l)^\alpha (dx)^\alpha} \quad (n=0,1,2,\ldots). \tag{34}$$

Following (34), we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^l \sin_\alpha^2 n^\alpha \left(\frac{\pi x}{l}\right)^\alpha (dx)^\alpha = \frac{l^\alpha}{2\Gamma(1+\alpha)}, \tag{35}$$

such that

$$A_n = \frac{2\int_0^l f(x)\sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{l^\alpha},$$
$$B_n = \frac{2\int_0^l G(x)\sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{\rho l^\alpha}. \tag{36}$$

Thus, we get the solution of (4):

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t), \tag{37}$$

where

$$u_n(x,t) = E_\alpha\left(-\frac{t^\alpha}{2}\right) \times \left(A_n \cos_\alpha \rho t^\alpha + B_n \sin_\alpha \rho t^\alpha\right)\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha, \tag{38}$$

with

$$A_n = \frac{2\int_0^l f(x)\sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{l^\alpha} \quad (n=0,1,2,\ldots),$$
$$B_n = \frac{2\int_0^l G(x)\sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{\rho l^\alpha} \quad (n=0,1,2,\ldots), \tag{39}$$

with

$$G(x) = g(x) + \frac{1}{2}f(x). \tag{40}$$

4. Illustrative Examples

In order to illustrate the above result in this section, we give two examples.

Let us consider (4) subject to initial and boundary conditions

$$u(0,t) = u(l,t) = \frac{\partial^\alpha u(l,0)}{\partial x^\alpha} = 0,$$
$$u(x,0) = f(x) = \frac{x^\alpha}{\Gamma(1+\alpha)},$$
$$\frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = g(x) = \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{41}$$

In view of (40), we have

$$G(x) = g(x) + \frac{1}{2}f(x) = \frac{3}{2}\frac{x^\alpha}{\Gamma(1+\alpha)}, \tag{42}$$

such that

$$\begin{aligned} A_n &= \frac{2\int_0^l \left(x^\alpha/\Gamma(1+\alpha)\right)\sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{l^\alpha} \\ &= \frac{2\Gamma(1+\alpha)}{l^\alpha}\, {}_0I_l^{(\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha \\ &= \frac{2\Gamma(1+\alpha)}{(n\pi)^\alpha}\left\{\frac{l^\alpha}{\Gamma(1+\alpha)}\cos_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha - \left(\frac{l}{n\pi}\right)^\alpha\left[\cos_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha - 1\right]\right\}, \end{aligned} \tag{43}$$

$$\begin{aligned} B_n &= \frac{3\int_0^l \left(x^\alpha/\Gamma(1+\alpha)\right)\sin_\alpha n^\alpha (\pi x/l)^\alpha (dx)^\alpha}{\rho l^\alpha} \\ &= \frac{3\Gamma(1+\alpha)}{\rho l^\alpha}\, {}_0I_l^{(\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha \\ &= -\frac{3\Gamma(1+\alpha)}{\rho(n\pi)^\alpha}\left[\frac{l^\alpha}{\Gamma(1+\alpha)}\cos_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha - \left(\frac{l}{n\pi}\right)^\alpha \sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha\right]. \end{aligned} \tag{44}$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} E_\alpha\left(-\frac{t^\alpha}{2}\right) \times \left(A_n \cos_\alpha \rho t^\alpha + B_n \sin_\alpha \rho t^\alpha\right)\sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha, \tag{45}$$

where

$$A_n = -\frac{2\Gamma(1+\alpha)}{\rho(n\pi)^\alpha}\left[\frac{l^\alpha}{\Gamma(1+\alpha)}\cos_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha - \left(\frac{l}{n\pi}\right)^\alpha \sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha\right],$$
$$B_n = -\frac{3\Gamma(1+\alpha)}{\rho(n\pi)^\alpha}\left[\frac{l^\alpha}{\Gamma(1+\alpha)}\cos_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha - \left(\frac{l}{n\pi}\right)^\alpha \sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha\right]. \tag{46}$$

In view of (4), our second example is initial and boundary conditions as follows:

$$u(0,t)=u(l,t)=\frac{\partial^{\alpha}u(l,0)}{\partial x^{\alpha}}=0,$$
$$u(x,0)=f(x)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}, \tag{47}$$
$$\frac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}}=g(x)=0.$$

Following (40), we get

$$G(x)=\frac{1}{2}\frac{x^{\alpha}}{\Gamma(1+\alpha)}. \tag{48}$$

Hence, we obtain

$$\begin{aligned}A_n&=\frac{2\int_0^l\left(x^{\alpha}/\Gamma(1+\alpha)\right)\sin_{\alpha}n^{\alpha}(\pi x/l)^{\alpha}(dx)^{\alpha}}{l^{\alpha}}\\&=\frac{2\Gamma(1+\alpha)}{l^{\alpha}}\,{}_0I_l^{(\alpha)}\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\\&=-\frac{2\Gamma(1+\alpha)}{\rho(n\pi)^{\alpha}}\left[\frac{l^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\right.\\&\qquad\left.-\left(\frac{l}{n\pi}\right)^{\alpha}\sin_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\right],\\B_n&=\frac{\int_0^l\left(x^{\alpha}/\Gamma(1+\alpha)\right)\sin_{\alpha}n^{\alpha}(\pi x/l)^{\alpha}(dx)^{\alpha}}{\rho l^{\alpha}}\\&=\frac{\Gamma(1+\alpha)}{\rho l^{\alpha}}\,{}_0I_l^{(\alpha)}\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\\&=-\frac{\Gamma(1+\alpha)}{\rho(n\pi)^{\alpha}}\left[\frac{l^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\right.\\&\qquad\left.-\left(\frac{l}{n\pi}\right)^{\alpha}\sin_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\right].\end{aligned} \tag{49}$$

So,

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}E_{\alpha}\left(-\frac{t^{\alpha}}{2}\right)\\&\quad\times\left(A_n\cos_{\alpha}\rho t^{\alpha}+B_n\sin_{\alpha}\rho t^{\alpha}\right)\sin_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha},\end{aligned} \tag{50}$$

with

$$\begin{aligned}A_n&=\frac{2\Gamma(1+\alpha)}{(n\pi)^{\alpha}}\left\{\frac{l^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\right.\\&\qquad\left.-\left(\frac{l}{n\pi}\right)^{\alpha}\left[\cos_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}-1\right]\right\},\\B_n&=-\frac{\Gamma(1+\alpha)}{\rho(n\pi)^{\alpha}}\left[\frac{l^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\right.\\&\qquad\left.-\left(\frac{l}{n\pi}\right)^{\alpha}\sin_{\alpha}n^{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}\right].\end{aligned} \tag{51}$$

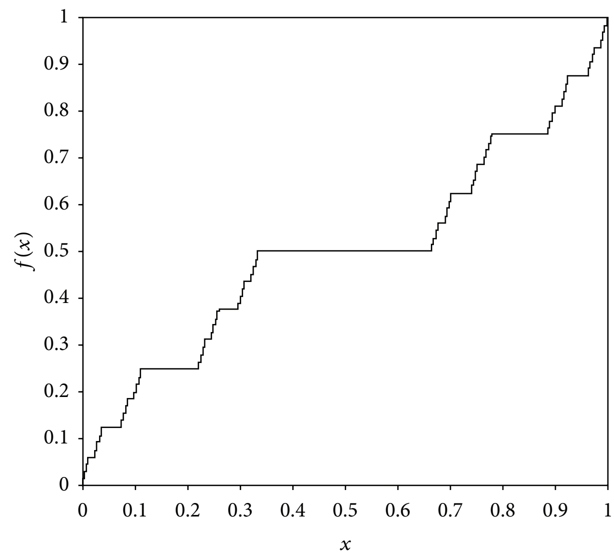

Figure 1: For $\alpha=\ln 2/\ln 3$, graph of a Lebesgue-Cantor staircase function shown at $x\in[0,1]$.

We notice that fraction boundary condition is expressed as a Lebesgue-Cantor staircase function [21, 32]; namely,

$$\begin{aligned}u(x,0)&=f(x)=H_{\alpha}(C\cap(0,x))\\&={}_0I_x^{(\alpha)}1=\frac{x^{\alpha}}{\Gamma(1+\alpha)},\end{aligned} \tag{52}$$

where C is any fractal set and the fractal dimension of $x^{\alpha}/\Gamma(1+\alpha)$ is α. For $x\in[0,1]$ the graph of the Lebesgue-Cantor staircase function (52) is shown in Figure 1 when fractal dimension is $\alpha=\ln 2/\ln 3$.

5. Conclusions

The present work expresses the local fractional Fourier series solution to wave equations with local fractional derivative. Two examples are given to illustrat approximate solutions for wave equations with local fractional derivative resulting from local fractional Fourier series method. The results obtained from the local fractional analysis seem to be general since the obtained solutions go back to the classical one when fractal dimension $\alpha=1$; namely, it is a process from fractal geometry to Euclidean geometry. Local fractional Fourier series method is one of very efficient and powerful techniques for finding the solutions of the local fractional differential equations. It is also worth noting that the advantage of the local fractional differential equations displays the nondifferential solutions, which show the fractal and local behaviors of moments. However, the classical Fourier series is used to handle the continuous functions.

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