

Research Article

Existence of Solutions for Fractional Differential Inclusions with Separated Boundary Conditions in Banach Space

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We discuss the existence of solutions for a class of some separated boundary differential inclusions of fractional orders $2 < \alpha < 3$ involving the Caputo derivative. In order to obtain necessary conditions for the existence result, we apply the fixed point technique, fractional calculus, and multivalued analysis.

1. Introduction

In recent years a great interest was devoted to the study of (singular and Neumann) boundary-value problems of fractional order [1–8], see also [9–12]. In the literature of fractional calculus, there are several definitions of fractional derivative that can be used. However, the most popular senses are the Riemann-Liouville and Caputo fractional derivatives; see, for instance, [13–19].

In this paper, we use the Caputo's fractional derivative since mathematical modeling of many physical problems requires initial and boundary conditions. These demands are satisfied using the Caputo fractional derivative. For more details we refer the reader to [20, 21] and references therein.

The importance of fractional boundary-value problems stems from the fact that they model various applications in fluid mechanics, viscoelasticity, physics, biology, and economics which cannot be modeled by differential equations with integer derivatives [21–23].

Delbosco and Rodino [24] considered the existence of a solution for the nonlinear fractional differential equation $d^\alpha x(t)/dt^\alpha = f(t, x)$, where $0 < \alpha < 1$ and $f : [0, a] \times \mathbb{R} \rightarrow$

\mathbb{R} , $0 < a \leq +\infty$, is a given function, continuous in $(0, a) \times \mathbb{R}$. They obtained results for solutions by using the Schauder fixed point theorem and the Banach contraction principle. Qiu and Bai [25] considered the existence of positive solution for equation:

$$\frac{d^\alpha x(t)}{dt^\alpha} + f(t, x(t)) = 0, \quad 0 < t < 1, \quad (1)$$

$$x(0) = x'(1) = x''(0) = 0,$$

where $2 < \alpha \leq 3$ and $f : (0, 1] \times [0, \infty) \rightarrow [0, \infty)$ with f being singular at $t = 0$ (i.e., $\lim_{t \rightarrow 0^+} f(t, \cdot) = +\infty$), by using Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type in a cone. Recently, Aitaliobrahim [26] considered the existence of solutions to the boundary-value problem:

$$x''(t) \in F(t, x(t), x'(t)), \quad \text{a.e. on } [0, 1], \quad (2)$$
$$x'(0) = r, \quad x'(1) = s,$$

where F is a closed multifunction, measurable in the first argument, and Lipschitz continuous in the second argument, by using fixed point theory for multivalued maps.

Motivated by the previous results, in this work we establish the existence result of a new version of fractional separated boundary-value problem:

$$\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} &\in F(t, x(t), x'(t), x''(t)), \quad \text{a.e. on } [0, 1], \\ x(0) &= 0, \quad x'(1) = s, \quad x''(0) = r, \end{aligned} \quad (3)$$

where $2 < \alpha \leq 3$, F is nonconvex, closed multifunction, measurable in the first argument, and Lipschitz continuous in the second argument, and r, s are in a Banach space X . The work is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel while in Section 3, we give the main result. Finally in Section 4 we give example to illustrate the application of our results.

2. Preliminaries

In this section, we present basic definitions of fractional calculus and some essential facts from multivalued analysis that will be used in this work to obtain our main results.

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2. The left Riemann-Liouville fractional integral of order $\alpha > 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0. \quad (4)$$

Definition 3. For $\alpha > 0$, $m-1 < \alpha < m$, $m \in \mathbb{N}$, $t > 0$, and $f \in C_{-1}^m$, the left Caputo fractional derivative is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-1-\alpha} f^{(m)}(s) ds, \quad (5)$$

where Γ is the well-known Gamma function.

The Caputo derivative defined in (5) is related to the Riemann-Liouville fractional integral, I^α , of order $\alpha \in \mathbb{R}^+$, by

$$D^\alpha f(t) = I^{m-\alpha} f^{(m)}(t). \quad (6)$$

It is known (see [20]) that

$$I^\alpha (D^\alpha f(t)) = f(t) - \sum_{k=0}^{m-1} c_k t^k, \quad (7)$$

$$D^\alpha I^\alpha f(t) = f(t), \quad (8)$$

where in (7), $c_k = f^{(k)}(0+)/k!$, $0 \leq k \leq m-1$.

Now, let X be a real separable Banach space with the norm $\|\cdot\|$. We denote by $\mathcal{C}([0, 1], X)$ the Banach space of continuous functions from $[0, 1]$ to E equipped with the norm $\|x(\cdot)\|_\infty := \sup\{\|x(t)\|; t \in [0, 1]\}$. For $x \in X$ and for nonempty sets A, B of X we denote $d(x, A) = \inf\{d(x, y); y \in A\}$, $e(A, B) := \sup\{d(x, B); x \in A\}$, and $H(A, B) := \max\{e(A, B), e(B, A)\}$. A multifunction is said to be measurable if its graph is measurable.

Also, we recall the following results that will be used in this paper.

Definition 4 (see [26]). Let $T : X \rightarrow 2^X$ be a multifunction with closed values:

- (i) T is k -Lipschitz if $H(T(x), T(y)) \leq k\|x - y\|$ for each $x, y \in X$,
- (ii) T is a contraction if it is k -Lipschitz with $k < 1$,
- (iii) T has a fixed point if there exists $x \in X$ such that $x \in T(x)$.

Lemma 5 (see [26]). *If $T : X \rightarrow 2^X$ is a contraction with nonempty closed values, then it has a fixed point.*

Lemma 6 (see [26]). *Assume that $F : [a, b] \times X \rightarrow 2^X$ is a multifunction with nonempty closed values satisfying the following:*

- (i) for every $x \in X$, $F(\cdot, x)$ is measurable on $[a, b]$;
- (ii) for every $t \in [a, b]$, $F(t, \cdot)$ is (Hausdorff) continuous on E .

Then for any measurable function $x(\cdot) : [a, b] \rightarrow X$, the multifunction $F(\cdot, x(\cdot))$ is measurable on $[a, b]$.

Definition 7 (see [26]). A measurable multivalued function $F : [0, 1] \rightarrow 2^X$ is said to be integrably bounded if there exists a function $h \in L^1([0, 1], X)$ such that for all $v \in F(t)$, $\|v\| \leq h(t)$ for almost every $t \in [0, 1]$.

Definition 8 (see [26]). A function $x(\cdot) : [0, 1] \rightarrow X$ is said to be a solution of (3) if $x(\cdot)$ is absolutely continuous on $[0, 1]$ and satisfies (3).

Lemma 9 (see [25]). *Given $\varphi \in C[0, 1]$ and $2 < \alpha \leq 3$, the unique solution of*

$$\frac{d^\alpha y(t)}{dt^\alpha} + \varphi(t) = 0, \quad 0 < t < 1, \quad (9)$$

$$y(0) = y'(1) = y''(0) = 0$$

is

$$y(t) = \int_0^1 G(t, s) \varphi(s) ds, \quad (10)$$

where

$$G(t, s) = \begin{cases} \frac{(\alpha - 1)t(1 - s)^{\alpha - 2} - (t - s)^{\alpha - 1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t(1 - s)^{\alpha - 2}}{\Gamma(\alpha - 1)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (11)$$

Obviously, $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$ and $0 < G(t, s) \leq \lambda$, for each $t, s \in [0, 1]$ and some λ .

3. Main Results

Now we are in a position to state and prove the main results of this paper.

Theorem 10. *Let $F : [0, 1] \times X \times X \times X \rightarrow 2^X$ be a set-valued map with nonempty closed values satisfying the following:*

- (i) *for each $(x, y, z) \in X \times X \times X$, $t \mapsto F(t, x, y, z)$ is measurable and integrably bounded;*
- (ii) *there exists a function $m(\cdot) \in L^1([0, 1], \mathbb{R}^+)$ such that for all $t \in [0, 1]$ and for all $x_1, x_2, y, z \in X$*

$$H(F(t, x_1, y, z), F(t, x_2, y, z)) \leq m(t) \|x_1 - x_2\|. \quad (12)$$

Then, if $\int_0^1 (1 + m(s)) ds < 1/\lambda$, for all $r, s \in X$, the problem (3) has at least one solution on $[0, 1]$.

Proof. For the proof of this theorem, we use the similar steps as those of [26, Theorem 2.6] together with the theory of fractional calculus. Let r, s be in X . We introduce first the function $\rho : [0, 1] \rightarrow X$ defined by

$$\rho(t) = (r + s)t - \frac{1}{2}rt^2, \quad \forall t \in [0, 1], \quad (13)$$

and the multifunction $H : [0, 1] \times \mathcal{C}([0, 1], X) \rightarrow 2^X$ defined by

$$H(t, y(\cdot)) = \varphi(t) - F(t, y(t) + \rho(t), y'(t) + \rho'(t), y''(t) + \rho''(t)), \quad (14)$$

for all $(t, y(\cdot)) \in [0, 1] \times \mathcal{C}([0, 1], X)$. Consider the following problem:

$$-\frac{d^\alpha y(t)}{dt^\alpha} + \varphi(t) \in H(t, y(\cdot)), \quad \text{a.e. on } [0, 1], \quad (15)$$

$$y(0) = y'(1) = y''(0) = 0.$$

We should note that the function $y(\cdot)$ is a solution of (15), if and only if the function $x(t) = y(t) + \rho(t)$ is a solution of (3), for all $t \in [0, 1]$.

Next, by Lemma 6, for $y(\cdot) \in \mathcal{C}([0, 1], X)$, $F(\cdot, y(\cdot), y'(\cdot), y''(\cdot))$ is closed and measurable; then it has a measurable

selection which, by hypothesis (i), belongs to $L^1([0, 1], X)$. Thus the set

$$S_{F, y(\cdot)} := \left\{ f \in L^1([0, 1], X) : f(t) \in F(t, y(t), y'(t), y''(t)) \text{ for a.e. } t \in [0, 1] \right\} \quad (16)$$

is nonempty. Let us transform problem (15) into a fixed point problem. Consider the multivalued map

$$T : \mathcal{C}([0, 1], X) \rightarrow 2^{\mathcal{C}([0, 1], X)} \quad (17)$$

defined as follows, for $y(\cdot) \in \mathcal{C}([0, 1], X)$:

$$T(y(\cdot)) = \left\{ z(\cdot) \in \mathcal{C}([0, 1], X) : z(t) = \int_0^1 G(t, s) h(s) ds, \forall t \in [0, 1], h \in S_{H, y(\cdot)} \right\}, \quad (18)$$

where

$$S_{H, y(\cdot)} := \left\{ h \in L^1([0, 1], X) : h(t) \in H(t, y(\cdot)) \text{ for a.e. } t \in [0, 1] \right\}. \quad (19)$$

We will show that T satisfies the assumptions of Lemma 5. The proof will be given in two steps.

Step 1 (T has nonempty closed values). Indeed, let $(y_p(\cdot))_{p \geq 0} \in T(y(\cdot))$ such that $(y_p(\cdot))_{p \geq 0}$ converges to $\bar{y}(\cdot)$ in $\mathcal{C}([0, 1], X)$. Then $\bar{y}(\cdot) \in \mathcal{C}([0, 1], X)$ and for each $t \in [0, 1]$,

$$y_p(t) \in \int_0^1 G(t, s) H(s, y(\cdot)) ds, \quad (20)$$

where $\int_0^1 G(t, s) H(s, y(\cdot)) ds$ is the Aumann integral of $G(t, \cdot) H(\cdot, y)$, which is defined as

$$\int_0^1 G(t, s) H(s, y(\cdot)) ds = \left\{ \int_0^1 G(t, s) h(s) ds, h \in S_{H, y(\cdot)} \right\}. \quad (21)$$

Using the fact that the set-valued map F is closed and by (14), we conclude that the set

$$\int_0^1 G(t, s) H(s, y(\cdot)) ds \quad (22)$$

is closed for all $t \in [0, 1]$. Then

$$\bar{y}(t) \in \int_0^1 G(t, s) H(s, y(\cdot)) ds. \quad (23)$$

So, there exists $h \in S_{H, y(\cdot)}$ such that

$$\bar{y}(t) = \int_0^1 G(t, s) h(s) ds. \quad (24)$$

Hence $\bar{y}(\cdot) \in T(y(\cdot))$. So $T(y(\cdot))$ is closed for each $y(\cdot) \in \mathcal{C}([0, 1], X)$.

Step 2 (T is a contraction). Indeed, let $y_1(\cdot), y_2(\cdot) \in \mathcal{C}([0, 1], X)$ and consider $z_1(\cdot) \in T(y_1(\cdot))$. Then there exists $h_1 \in S_{H, y_1(\cdot)}$ such that

$$z_1(t) = \int_0^1 G(t, s) h_1(s) ds, \quad \forall t \in [0, 1]. \quad (25)$$

Using (14), there exists $f_1 \in S_{F, y_1(\cdot)}$ such that

$$h_1(t) = y_1(t) - f_1(t), \quad \forall t \in [0, 1]. \quad (26)$$

On the other hand, let $\varepsilon > 0$ and consider the valued map $U_\varepsilon : [0, 1] \rightarrow 2^X$, given by

$$U_\varepsilon(t) = \{x \in X : \|f_1(t) - x\| \leq m(t) \|y_1(t) - y_2(t)\| + \varepsilon\}. \quad (27)$$

We claim that $U_\varepsilon(t)$ is nonempty, for each $t \in [0, 1]$. Indeed, let $t \in [0, 1]$; then we have

$$\begin{aligned} & H(F(t, y_1(t), y_1'(t), y_1''(t)), F(t, y_2(t), y_2'(t), y_2''(t))) \\ & \leq m(t) \|y_1(t) - y_2(t)\|. \end{aligned} \quad (28)$$

Hence, there exists $x \in F(t, y_2(t), y_2'(t), y_2''(t))$, such that

$$\|f_1(t) - x\| \leq m(t) \|y_1(t) - y_2(t)\| + \varepsilon. \quad (29)$$

The multifunction

$$V : t \longrightarrow U_\varepsilon(t) \cap F(t, y_2(t), y_2'(t), y_2''(t)) \text{ is measurable.} \quad (30)$$

Then there exists a measurable selection for V denoted by f_2 such that, for all $t \in [0, 1]$,

$$f_2(t) \in F(t, y_2(t), y_2'(t), y_2''(t)) \quad (31)$$

and

$$\|f_1(t) - f_2(t)\| \leq m(t) \|y_1(t) - y_2(t)\| + \varepsilon. \quad (32)$$

Now, for all $t \in [0, 1]$, set $h_2(t) = y_2(t) - f_2(t)$ and

$$z_2(t) = \int_0^1 G(t, s) h_2(s) ds. \quad (33)$$

We have

$$\begin{aligned} & \|z_1(t) - z_2(t)\| \\ & \leq \int_0^1 \|G(t, s)\| \|h_1(s) - h_2(s)\| ds \\ & \leq \lambda \int_0^1 \|y_1(s) - y_2(s)\| ds + \lambda \int_0^1 \|f_1(s) - f_2(s)\| ds \\ & \leq \lambda \int_0^1 \|y_1(s) - y_2(s)\| ds \\ & \quad + \lambda \int_0^1 m(s) \|y_1(s) - y_2(s)\| ds + \lambda \varepsilon \\ & \leq \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s)) ds + \lambda \varepsilon. \end{aligned} \quad (34)$$

So, we conclude that

$$\|z_1(\cdot) - z_2(\cdot)\|_\infty \leq \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s)) ds + \lambda \varepsilon. \quad (35)$$

By an analogous relation, obtained by interchanging the roles of $y_1(\cdot)$ and $y_2(\cdot)$, it follows that

$$\begin{aligned} & H(T(y_1(\cdot)), T(y_2(\cdot))) \\ & \leq \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s)) ds + \lambda \varepsilon. \end{aligned} \quad (36)$$

By letting $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} & H(T(y_1(\cdot)), T(y_2(\cdot))) \\ & \leq \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s)) ds. \end{aligned} \quad (37)$$

Consequently, if $\int_0^1 (1 + m(s)) ds < 1/\lambda$, T is a contraction. By Lemma 5, T has a fixed point which is a solution of (15). \square

4. Example

In this section we present an example to illustrate the applications of our main results, and we consider the following fractional inclusion boundary-value problem:

$$\begin{aligned} & \frac{d^\alpha x(t)}{dt^\alpha} \in F(t, x(t), x'(t), x''(t)), \quad t \in [0, 1], \quad 2 < \alpha \leq 3, \\ & x(0) = 0, \quad x'(1) = s, \quad x''(0) = r, \end{aligned} \quad (38)$$

where r and s are arbitrary real numbers and $F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$F(t, x, y, z) = \left[0, \frac{\sin x}{(1+t)^2} + \cos(yz) \right]. \quad (39)$$

Then we have

$$\sup\{|u| : u \in F(t, x, y, z)\} \leq 1 + \frac{1}{(1+t)^2}, \quad (40)$$

$$H(F(t, x, y, z), F(t, \bar{x}, y, z)) \leq \frac{1}{(1+t)^2} |x - \bar{x}|.$$

Let $m(t) = 1/(1+t)^2$. Then $\int_0^1 (1+m(s))ds = 3/2 < 1/\lambda$ for some $\lambda > 2/3$.

By Theorem 10 the problem (38) has at least one solution on $[0, 1]$.

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