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**Research Article**

Abdon Atangana*, Dumitru Baleanu, and Ahmed Alsaedi

# New properties of conformable derivative

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**Abstract:** Recently, the conformable derivative and its properties have been introduced. In this work we have investigated in more detail some new properties of this derivative and we have proved some useful related theorems. Also, some new definitions have been introduced.

**Keywords:** Conformable derivative, Conformable vectors, Conformable partial derivative

**MSC:** 47J25, 65Mxx

## 1 Introduction

Over the last decade, numerous explanations of fractional derivative have been recommended because of the rewards provided by this concept for modelling real world problems. We recall that the fractional partial derivatives are difficult to handle analytically, especially those describing real world processes, and the researchers sometimes have to rely on the numerical methods to solve these equations. One of the well-known fractional derivatives is the Riemann-Liouville fractional order derivative, which is not always appropriate for modelling real world problems. The second one is the so-called Caputo derivative; this one is opposite with relation to displaying physical field complications and has been intensively used for this purpose. However, new derivatives should be proposed in order to deal better with the dynamics of the complex systems [2–9]. We have noticed that recently a new derivative has been suggested in [1, 17, 21, 22] and it seems to satisfy all the requirements of the standard derivative. However, there is no discretization of this version in the literature. This paper will, therefore, be devoted to the discussion supporting the detailed investigation of the properties and useful theorems related to this derivative and its applications in real world problem [11–17]. The new derivative under investigation here is given in Definition.

## 2 Theorems, properties and definitions

**Definition 2.1.** *We denote a function $f : [0, \infty) \to R$. The conformable derivative of the function $f$ with order $\alpha$ is defined by [1]*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{1}$$

for all $t > 0$, $\alpha \in (0, 1)$ [1, 16].

We shall present some properties of this new derivative.

***Corresponding Author: Abdon Atangana:** Institute of Groundwater Studies, University of the Free State, Nelson Mandela Drive, 9300 Bloemfontein, South Africa, E-mail: abdonatangana@yahoo.fr

**Dumitru Baleanu:** Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankara University, 06530 Ankara, Turkey and Institute of Space Sciences, Magurele, Bucharest, Romania, E-mail: dumitru@cankaya.edu.tr

**Ahmed Alsaedi:** Department of Mathematics, Faculty of Science, King Abdulaziz University P. O. Box 80257, Jeddah 21589, Saudi Arabia, E-mail: aalsaedi@hotmail.com

**Properties** ([1]).

1. $T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$ *for all real constant a, b,*
2. $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$,
3. $T_\alpha(t^p)=pt^{p-\alpha}$ *for all p,*
4. $T_\alpha(\frac{f}{g})=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$,
5. $T_\alpha(c)=0$ *with c is a constant.*

**Remark 2.2.** *The new derivative is the generalized q-derivative or a fractal derivative*

*Proof.* By definition we have that

$$T_\alpha(f)(t)=\lim_{\varepsilon\to0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}.$$

□

From the above definition, let us put

$$q=1+\varepsilon t^{-\alpha},$$

such that when $\varepsilon\to0, q\to1$, then equation (1) can be converted to

$$T_\alpha(f)(t)=\lim_{q\to1}\frac{f(qt)-f(t)}{qt^\alpha-t^\alpha}.$$

**Theorem 2.3.** *Assuming that the function f is infinitly deferentiable at the point a, then the Taylor series of f with respect to the conformable derivative is given as [18–21]*

$$FT(f(x))=\sum_{n=0}^{\infty}a^{n-n\alpha}(x-a)^n\frac{f^{(n)}(a)}{n!},\tag{2}$$

*where $n!$ denotes the factorial of n and $f^n(a)$ is the nth derivative of f evaluated at the point a.*

*Proof.* Assuming that f is infinitely differentiable at the point a, then by definition we have:

$$T_\alpha(f(a))=\lim_{\varepsilon\to0}\frac{f(a+\varepsilon a^{1-\alpha})-f(a)}{\varepsilon}\tag{3}$$

However, if we put $k=\varepsilon a^{1-\alpha}$, then we shall have the following

$$T_\alpha(f(a))=a^{1-\alpha}\lim_{k\to0}\frac{f(a+k)-f(a)}{k}=a^{1-\alpha}f'(a).\tag{4}$$

Henceforth, assume by induction, that for all $m\geq1$, we have the following

$$T_{m\alpha}(f(a))=a^{m-m\alpha}\frac{f^{(m)}(a)}{m!}.\tag{5}$$

We shall prove that this relation holds for the next step, which is at m+1. However by definition, we have the following formula

$$T_{(m+1)\alpha}(f(a))=T_\alpha\left(T_{m\alpha}(f(a))\right)$$

which by induction formula equation (5) can be converted to the following

$$T_{(m+1)\alpha}(f(a))=T_\alpha\left(T_{m\alpha}(f(a))=T_\alpha\left(a^{m-m\alpha}\frac{f^{(m)}(a)}{m!}\right)=a^{(1-\alpha)(m+1)}\frac{f^{(m+1)}(a)}{(m+1)!}\right.\tag{6}$$

The property is also verified at $m+1$, and then, the relation is true for all $n$. □

**Theorem 2.4.** *Let f and g be two function differentiable, such that g is differentiable at any $t$, and f is differentiable at any $g(t)$, and then the conformable derivative obeys the Chain rule, meaning*

$$T_\alpha((fog)(x))=x^{1-\alpha}g(x)^{1-\alpha}g'(x)T_\alpha(f(t))\mid_{t=g(x)}.\tag{7}$$

*Proof.* Assuming that $g$ is differentiable function at any given $t$, and $f$ is differentiable at any $g(t)$ then by definition we have the following

$$T_\alpha((fog)(x)) = \lim_{\varepsilon\to 0} \frac{(fog)\,(x+\varepsilon x^{1-\alpha}) - (fog)\,(x)}{\varepsilon} \tag{8}$$

Now if we put $k = \varepsilon x^{1-\alpha}$, we obtain

$$T_\alpha((fog)(x)) = x^{1-\alpha} \lim_{k\to 0} \frac{(fog)\,(x+k) - (fog)\,(x)}{k} \tag{9}$$

with

$$\lim_{k\to 0} \frac{(fog)\,(x+k) - (fog)\,(x)}{k}$$

as the derivative of composite function, so if we apply the chain rule for the ordinary derivative, we obtain

$$\lim_{k\to 0} \frac{(fog)\,(x+k) - (fog)\,(x)}{k} = \lim_{k\to 0} \frac{f(g(x)+k) - f(g(x))}{k}. \tag{10}$$

Now we put

$$k = \varepsilon g(x)^{1-\alpha}.$$

Then the above equation (10) can be transform into

$$\lim_{k\to 0} \frac{(fog)\,(x+k) - (fog)\,(x)}{k} = g(x)^{\alpha-1} \lim_{\varepsilon\to 0} \frac{f(g(x)+\varepsilon g(x)^{1-\alpha}) - f(g(x))}{\varepsilon} \tag{11}$$

Now replacing (11) in (9) we obtain the following

$$T_\alpha((fog)(x)) = x^{1-\alpha} g(x)^{\alpha-1} \lim_{\varepsilon\to 0} \frac{f(g(x)+\varepsilon g(x)^{1-\alpha}) - f(g(x))}{\varepsilon} \tag{12}$$

with of course

$$T_\alpha(f(t))\,|_{t=g(x)} = \lim_{\varepsilon\to 0} \frac{f(g(x)+\varepsilon g(x)^{1-\alpha}) - f(g(x))}{\varepsilon}.$$

This completes the proof. □

**Theorem 2.5.** *Let $\alpha$, $\beta$ be positive constants such, that $0 < \alpha, \beta < 1$, and $f$ be a function (non-constant) 2 times differentiable on a open real interval $(a, b)$, then the conformable derivative obeys to the following*

$$T_{\alpha+\beta}(f(x)) \neq T_\alpha\left(T_\beta(f(x))\right). \tag{13}$$

*Proof.* Under the condition that the function f is differentiable, we have the following

$$T_\alpha\left(T_\beta f(x)\right) = T_\alpha\left(\lim_{\varepsilon\to 0} \frac{f(x+\varepsilon x^{1-\beta}) - f(x)}{\varepsilon}\right).$$

However, by changing the variable $\varepsilon x^{1-\alpha} = h$, we obtain the following relation

$$T_\alpha\left(T_\beta f(x)\right) = T_\alpha\left(x^{1-\beta} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) = T_\alpha\left(x^{1-\beta} f' f(x)\right). \tag{14}$$

Now, making use of the second property we arrive at

$$T_\alpha\left(T_\beta(x)\right) = T_\alpha\left(x^{1-\beta}\right) f'(x) + x^{1-\beta} T_\alpha\left(f'(x)\right) = x^{1-\beta-\alpha}\left[(1-\beta) f'(x) + x f''(x)\right]. \tag{15}$$

On the other hand,

$$T_{\alpha+\beta}(f(x)) = \lim_{\varepsilon\to 0} \frac{f'(x+\varepsilon x^{2-\alpha-\beta}) - f'(x)}{\varepsilon} \tag{16}$$

Again making the change of variable $l = \varepsilon x^{2-\alpha-\beta}$ and using the fact that the function is 2-times differentiable, we have

$$T_{\alpha+\beta}(f(x)) = x^{2-\alpha-\beta} \lim_{l\to 0} \frac{f'(x+l) - f'(x)}{l} = x^{2-\alpha-\beta} f''(x),$$

and we have that $T_{\alpha+\beta}(f(x)) \neq T_\alpha\left(T_\beta(f(x))\right)$, and then the proof is completed. □

**Corollary 2.6.** *Let $\alpha$, $\beta$ be positive constants, such that $0 < \alpha < 1$, $\beta = 1$, and $f$ be a function 2 times differentiable on a open real interval $(a, b)$, then the conformable derivative obeys to the following*

$$T_{\alpha+\beta}(f(x)) = T_\alpha\left(T_\beta(f(x))\right).$$

*The proof follows directly from the one of Theorem 2.5, by letting $\beta = 1$.*

**Theorem 2.7.** *Let $\alpha$, $\beta$ be positive constants, such that $0 < \alpha, \beta < 1$, and $f$ be a function (non-constant) 2 times differentiable on a open real interval $(a, b)$, then the conformable derivative obeys to the following*

$$T_\alpha\left(T_\beta f(x))\right) \neq T_\beta\left(T_\alpha(f(x))\right). \tag{17}$$

*Proof.* By using the definition, we have the following relation

$$\begin{aligned} T_\alpha\left(T_\beta f(x))\right) &= T_\alpha(x^{1-\beta})f'(x) + x^{1-\beta}T_\alpha(f'(x)) = x^{1-\beta-\alpha}[(1-\beta)f'(x) + xf''(x)], \\ T_\beta\left(T_\alpha f(x))\right) &= T_\beta(x^{1-\alpha})f'(x) + x^{1-\alpha}T_\beta(f'(x)) = x^{1-\beta-\alpha}[(1-\alpha)f'(x) + xf''(x)]. \end{aligned} \tag{18}$$

Therefore, the proof is completed. □

**Theorem 2.8.** *Allow $f$ to be a function $m$-time ($m \geq 2$) differentiable on an open real interval $(a, b)$ and $0 < \alpha \leq 1$, then the conformable derivative obeys the following relation*

$$T_\alpha(T_\alpha(\ldots\ldots\ldots(T_\alpha(f(x))))) \neq T_{m\alpha}(f(x)). \tag{19}$$

*Proof.* We can verify the above by choosing $m = 2$. The proof can be achieved using the routine used in Theorem 2.7. It is important to mention that many physical problems are described with partial differential equation, in order to fit the new derivative in this folder; we shall propose the following definition for the case of a function with many variables. □

**Definition 2.9.** *Let $f$ be a function with $m$ variables $x_1, \ldots\ldots\ldots x_m$, and the conformable partial derivative of $f$ of order $0 < \alpha \leq 1$ in $x_i$ is defined as follows*

$$\frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1, \ldots, x_m) = \lim_{\varepsilon \to 0} \frac{f(x_1, \ldots x_{i-1}, x_i + \varepsilon x_i^{1-\alpha}, \ldots, x_m) - f(x_1, \ldots, x_m)}{\varepsilon}. \tag{20}$$

*Before we continue with our investigation we shall first verify the Clairaut's theorem for partial derivatives of conformable fractional orders.*

**Theorem 2.10.** *Assume that $f(x, y)$ is function for which $\partial_x^\alpha\left[\partial_y^\beta(f(x, y))\right]$ and $\partial_y^\beta\left[\partial_x^\alpha(f(x, y))\right]$ exist and are continous over the domain $D \subset R_2$ then*

$$\partial_x^\alpha\left[\partial_y^\beta(f(x, y))\right] = \partial_y^\beta\left[\partial_x^\alpha(f(x, y))\right]. \tag{21}$$

*Proof.* By definition as in equation (2.21) we have that

$$\partial_x^\alpha\left[\partial_y^\beta(f(x, y))\right] = \partial_x^\alpha\left[\lim_{\varepsilon \to 0} \frac{f(x, y + \varepsilon y^{1-\beta}) - f(x, y)}{\varepsilon}\right] \tag{22}$$

Making the change of variable $\varepsilon y^{1-\beta} = k$ we obtain

$$\partial_x^\alpha\left[\partial_y^\beta(f(x, y))\right] = y^{1-\beta}\partial_x^\alpha\left[\lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}\right].$$

Since $f$ is differentiable in $y$ direction we have that,

$$\partial_x^\alpha\left[\partial_y^\beta(f(x, y))\right] = y^{1-\beta}\partial_x^\alpha[\frac{\partial f}{\partial y}(x, y)] \tag{23}$$

Again by definition we have the following

$$\partial_x^{\alpha}\left[\partial_y^{\beta}(f(x,y))\right] = y^{1-\beta}\lim_{\varepsilon\to 0}\frac{\frac{\partial f}{\partial y}(x+\varepsilon x^{1-\alpha},y)-\frac{\partial f}{\partial y}(x,y)}{\varepsilon}. \tag{24}$$

After making a change of variable, we obtain

$$\partial_x^{\alpha}\left[\partial_y^{\beta}(f(x,y))\right] = y^{1-\beta}x^{1-\alpha}\lim_{l\to 0}\frac{\frac{\partial f}{\partial y}(x+l,y)-\frac{\partial f}{\partial y}(x,y)}{l}. \tag{25}$$

Since the function $f$ is differentiable in $x-$direction, we have that

$$\partial_x^{\alpha}\left[\partial_y^{\beta}(f(x,y))\right] = y^{1-\beta}x^{1-\alpha}\frac{\partial^2 f(x,y)}{\partial x\partial y}. \tag{26}$$

Since $f$ is continuous, by using the Clairaut's theorem for partial derivatives, we have that,

$$\frac{\partial^2 f(x,y)}{\partial x\partial y} = \frac{\partial^2 f(x,y)}{\partial y\partial x}, \tag{27}$$

then the equation (26) becomes

$$\partial_x^{\alpha}\left[\partial_y^{\beta}(f(x,y))\right] = y^{1-\beta}x^{1-\alpha}\frac{\partial^2 f(x,y)}{\partial x\partial y} = x^{1-\alpha}y^{1-\beta}\lim_{l\to 0}\frac{\frac{\partial f}{\partial x}(x,y+l)-\frac{\partial f}{\partial x}(x,y)}{l} \tag{28}$$

By making a change of variable meaning $\varepsilon = ly^{1-\beta}$ and later $\varphi = hx^{1-\alpha}$ we arrive at

$$\partial_x^{\alpha}\left[\partial_y^{\beta}(f(x,y))\right] = \partial_y^{\beta}\left[\lim_{l\to 0}\frac{\frac{\partial f}{\partial x}(x,y+l)-\frac{\partial f}{\partial x}(x,y)}{l}\right] = \partial_y^{\beta}\left[\partial_x^{\alpha}(f(x,y))\right],$$

and this completes the proof. □

In this case $f$ has a partial conformable derivative with respect to each variable, $y$. At the point $a$, these partial conformable derivatives define the conformable vector

$$\nabla_{\alpha} f(a) = \left(\partial_x^{\alpha}(f(a)), \partial_y^{\beta}(f(a))\right), \tag{29}$$

We shall present the next definition as

**Definition 2.11.** *Consider the scalar field $f(\underline{x})$ and the vector field $\underline{F}(\underline{x})$ that are assumed to possess partial conformable derivative of order α with respect to all the Cartesian coordinates $x_i, i = 1,2,3$. We define the conformable gradient of order α as being the vector field*

$$\nabla_{\underline{x}}^{\alpha} f = \sum_{i=1}^{3}(\partial_{x_i}^{\alpha} f)\underline{e_i}, \tag{30}$$

*with, of course, $\underline{e_i}$ the unit vector in the "$i$" direction.*

The conformable of order $\alpha$ as the scalar field is defined as

$$\nabla_{\underline{x}}^{\alpha}.f = \sum_{i=1}^{3}(\partial_{x_i}^{\alpha} F_i), \tag{31}$$

and the conformable curl of order $\alpha$ as the vector field is hence defined as follows

$$\nabla_{\underline{x}}^{\alpha}\times f = \sum_{i=1}^{3}\left[\sum_{j=1}^{3}\sum_{k=1}^{3} e_{ijk}\partial_{x_i}^{\alpha} F_k\right] \tag{32}$$

where of course $e_{ijk}$ is the Levi-Civita symbol.

Having these definitions in hand we can present the following properties. Assuming that the different operations for the scalar and vector fields exist, we have

1. $\nabla^{\alpha}_{\underline{x}}(af+bg) = a\nabla^{\alpha}_{\underline{x}}(f) + b\nabla^{\alpha}_{\underline{x}}(g)$,
2. $\nabla^{\alpha}_{\underline{x}}.(a\underline{F}+b\underline{G}) = a\nabla^{\alpha}_{\underline{x}}.(\underline{F}) + b\nabla^{\alpha}_{\underline{x}}.(\underline{G})$
3. $\nabla^{\alpha}_{\underline{x}} \times (a\underline{F}+b\underline{G}) = a\nabla^{\alpha}_{\underline{x}} \times (\underline{F}) + b\nabla^{\alpha}_{\underline{x}} \times (\underline{G})$,
4. $\nabla^{\alpha}_{\underline{x}}.[\nabla^{\alpha}_{\underline{x}} \times \underline{F}] = 0$
5. $\nabla^{\alpha}_{\underline{x}} \times [\nabla^{\alpha}_{\underline{x}}.\underline{F}] = 0.$

We shall present the proof for the above properties, but the first three properties are a direct effect of the linear representative of the different operations and are fairly trivial to obtain. Nonetheless, the last two relations rely on Clairaut's result established in Theorem 2.10. We shall present the proof for the relation 4, which from the definition of the conformable divergence and conformable curl of order $U(3b1)$ is

$$\nabla^{\alpha}_{\underline{x}}.[\nabla^{\alpha}_{\underline{x}} \times \underline{F}] = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} e_{ijk}\partial^{\alpha}_{x_i}\nabla^{\alpha}_{x_j}F_k. \tag{33}$$

However, the Levi-Civita symbol equals to zero whenever two or more indexes take the same value, that is

$$e_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \text{ is } (1,2,3), (3,1,2) \text{ or } (2,3,1), \\ -1 \text{if } (i,j,k) \text{ is } (1,3,2), (3,1,2) \text{ or } (2,1,3), \\ 0 \text{ if } i=j \text{ or } j=k \text{ or } i=k, \end{cases} \tag{34}$$

and the triple sum in the relation is reduced to

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} e_{ijk}\partial^{\alpha}_{x_i}\nabla^{\alpha}_{x_j}F_k = \sum_{i=1}^{3}(e_{ijk}\partial^{\alpha}_{x_i}[\partial^{\alpha}_{x_j}F_k - \partial^{\alpha}_{x_k}F_j]), i \neq j \neq k, \tag{35}$$

or the above can be converted to

$$\nabla^{\alpha}_{\underline{x}}.[\nabla^{\alpha}_{\underline{x}} \times \underline{F}] = \partial^{\alpha}_{x_i}[\partial^{\alpha}_{x_j}F_k - \partial^{\alpha}_{x_k}F_j] + \partial^{\alpha}_{x_j}[\partial^{\alpha}_{x_k}F_i - \partial^{\alpha}_{x_i}F_k] + \partial^{\alpha}_{x_k}[\partial^{\alpha}_{x_i}F_j - \partial^{\alpha}_{x_j}F_i] = 0, \tag{36}$$

by direct application of the conformable Clairaut's theorem. The last property is established using similar arguments as for property 4.

## 2.1 The conformable divergence theorem

For remembrance, let the vector field $\underline{V}$ have a continuous derivative on an open region space [8–10] $D$ containing the volume $V$ and surface $S$ of $V$ positively outward orientated

$$\int\int\int \nabla.\underline{F}dV = \int\int \underline{F}.\underline{n}dS. \tag{37}$$

Therefore within the framework of the conformable fractional derivatives, this theorem can be rephrased; we shall present the following definition.

**Definition 2.12.** *Let the vector field $\underline{F}$ have the partial conformable derivatives of order $\beta$ with respect to all variables $\underline{x}=(x_1,\ldots,x_m)^T$ on $D$. Then we denote by $H^{\beta}_{\underline{x}}\underline{F}$ the vector*

$$H^{\beta}_{\underline{x}}\underline{F} = \sum_{i=1}^{m}\left\{e^T_{x_i}\left[\nabla^{\beta}_{\underline{x}}(\underline{F})^T\right].\underline{e_{x_i}}\right\}\underline{e_{x_i}} = \sum_{i=1}^{m}\partial^{\beta}_{x_i}F_{x_i}\underline{e_{x_i}}. \tag{38}$$

*We shall then propose the following theorem.*

**Theorem 2.13.** *Let the vector field $\underline{F}$ have the continuous partial conformable derivatives of order $\alpha$ on an open region of the space $D$ containing the volume $V$ and $S$ is the boundary surface of $V$ positively outward oriented. Then we have the following relation*

$$\int\int\int_V nabla^{\alpha}_{\underline{x}}\underline{F}dV^{alpha} = \int\int_S H^{\alpha-1}_{\underline{x}}\underline{F}.\underline{n}dS^{\alpha}, \tag{39}$$

*with*

$$dV^{\alpha} = x^{\alpha-1}y^{\alpha-1}z^{\alpha-1}dxdydz \text{ and } dS^{\alpha} = x^{\alpha-1}y^{\alpha-1}dxdy,$$

*which is in accordance with the conformable derivative as anti-derivative of conformable derivative.*

*Proof.* To achieve this we must first remark that

$$\int\int\int_{V} \nabla_{\underline{x}}^{\alpha}.\underline{F}dV^{\alpha} = \int\int_{S} H_{\underline{x}}^{\alpha-1}\underline{F}.\underline{n}dS^{\alpha}$$

and this completes the proof. □

In an analogous way, the distinguished theorem of Green and Stokes of the vector calculus can perhaps be appropriately modified to acclimatise straightforwardly the perception of conformable derivative; we therefore have the following theorems.

**Theorem 2.14** (The conformable Green's theorem). *Let $C$ be a simple positively oriented, piecewise smooth and closed curve in $R_2$, say for instance the $x-y$ plane, furthermore assume $D$ is the interior of $C$. If $f(x,y)$ and $g(x,y)$ are two functions having continuous partial conformable fractional derivative on $D$ then*

$$\int\int_{D} (\partial_{x}^{\alpha}g - \partial_{y}^{\alpha}f)dS^{\alpha} = \int_{C} \partial_{y}^{\alpha-1}fdx^{\alpha} + \partial_{x}^{\alpha-1}gdy^{\alpha}. \tag{40}$$

*Proof.* This is clearly a direct application of the classical version of Green's theorem. Now, since

$$\int\int_{D} (\partial_{x}^{\alpha}g - \partial_{y}^{\alpha}f)dS^{\alpha} = \int\int_{D} \partial_{x}(\partial_{x}^{\alpha-1}g) - \partial_{y}(\partial_{y}^{\alpha-1}f)dS^{\alpha}, \tag{41}$$

then, by applying the Green function theorem, we have

$$\int\int_{D} (\partial_{x}^{\alpha}g - \partial_{y}^{\alpha}f)dS^{\alpha} = \int_{C} \partial_{y}^{\alpha-1}fdx^{\alpha} + \partial_{x}^{\alpha-1}gdy^{\alpha}. \tag{42}$$

It should be reminded that for supplementary determinations this relation into the plane can be articulated in terms of the conformable fractional of the vector field $\underline{F} = (f,g,h)$, $h$ being an arbitrary function in $D$. □

**Theorem 2.15** (Conformable Stockes's theorem). *Let $S$ be a regular surface of class $C^2$ described by the parametric equations*

$$\underline{P}(u,v) = (x(u,v), y(u,v), z(u,v)),\ u,v \in S, \tag{43}$$

where $x,y,z$ are the Cartesian coordinates. Consider a simple positively oriented, piecewise smooth, closed curve $C_k$ in the plane $u-v$, assume $K$ to be the interior of $C_k$. $S$ and $C$ are images of the domain $K$ and its boundary $C_k$ in the Cartesian space, and $\underline{F}$ is a vector field having continuous partial derivatives of conformable order $\alpha$ on $S$, then

$$\int\int_{S} [\nabla_{\underline{x}}^{\alpha} \times \underline{F} + (\partial_x[K_{\underline{x}}^{\alpha-1}].F_y\underline{e_y})\underline{e_x} + (\partial_y[K_{\underline{x}}^{\alpha-1}].F_z\underline{e_z})\underline{e_y} + (\partial_z[K_{\underline{x}}^{\alpha-1}].F_x\underline{e_x})\underline{e_z}].\underline{n}dS^{\alpha}$$
$$= \int_{C} \partial_{z}^{\alpha-1}F_xdx^{\alpha} + \partial_{x}^{\alpha-1}F_ydy^{\alpha} + \partial_{y}^{\alpha-1}F_zdz^{\alpha},$$

with

$$[K_{\underline{x}}^{\alpha-1}] = -\nabla_{\underline{x}}^{\alpha} = (\partial_{y}^{\alpha} - \partial_{z}^{\alpha})\underline{e_x} + (\partial_{z}^{\alpha} - \partial_{x}^{\alpha})\underline{e_y} + (\partial_{x}^{\alpha} - \partial_{y}^{\alpha})\underline{e_z},$$

where $\underline{n}$ is the unit positive outside normal to $S$.

*Proof.* To prove this theorem we shall firstly spit the line of the integral

$$\int_C \partial_z^{\alpha-1} F_x dx^\alpha + \partial_x^{\alpha-1} F_y dy^\alpha + \partial_y^{\alpha-1} F_z dz^\alpha,$$

into the three following components:

$$\int_C \partial_z^{\alpha-1} F_x dx^\alpha, \quad \int_C \partial_x^{\alpha-1} F_y dy^\alpha \text{ and } \int_C \partial_y^{\alpha-1} F_z dz^\alpha.$$

Therefore, without the loss of generality, we shall consider the first component. In the meantime, we shall presume that the boundary $C_k$ can be explained by the mean of one single parametric relation $C_k : (u(t), v(t)),\ t \in [a, b]$. Then the corresponding curve $C$ in the Cartesian space of coordinates is given by

$$\underline{P}(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))). \tag{44}$$

Consequently:

$$g_1 = \partial_z^{\alpha-1} F_x x^{\alpha-1}. \tag{45}$$

Thus, we shall have the following

$$\int_C g_1 dx = \int_a^b g_1\,(u(t), v(t))\,\frac{dx(u(t), v(t))}{dt} dt \tag{46}$$

$$= \int_a^b g_1\,(u(t), v(t))\,\partial_u x \frac{du}{dt} dt + g_1\,(u(t), v(t))\,\partial_v x \frac{du}{dt} dt = \int_{C_k} g_1\,(u, v)\,\partial_u x du + g_1\,(u, v)\,\partial_v x dv.$$

Henceforward, employing the Green's theorem in its conventional form to this last outcome contributes to

$$\begin{aligned}\int_C g_1 dx &= \int\int_S \partial_u (g_1\,(u, v)\,\partial_v x) + \partial_u (g_1\,(u, v)\,\partial_u x) du dv,\\ &= \int\int_S [\partial_x g_1 \partial_u x + \partial_y g_1 \partial_y y + \partial_z g_1 \partial_u z]\,\partial_v x - [\partial_x g_1 \partial_u x + \partial_y g_1 \partial_y y + \partial_z g_1 \partial_u z]\,\partial_u x du dv\\ &= \int\int_S \left[\partial_z g_1 \left|\frac{\partial(z, x)}{\partial(u, v)}\right| - \partial_z g_1 \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\right] du dv = \int\int_S [\partial_z g_1 n_y - \partial_z g_1 n_z]\, du dv.\end{aligned} \tag{47}$$

Note that, n_y and n_z are referred as the components of the normal to S, that is

$$\underline{n} = (n_x, n_y, n_z) = \frac{\partial_u \underline{x} \times \partial_v \underline{x}}{|\partial_u \underline{x} \times \partial_v \underline{x}|}. \tag{48}$$

Now, repeating the same for two other integral components and adding those three relations, the rearrangement of the terms with respect to the components of the normal vector yields

$$\begin{aligned}&\int_C \partial_z^{\alpha-1} F_x dx^\alpha + \partial_x^{\alpha-1} F_y dy^\alpha + \partial_y^{\alpha-1} F_z dz^\alpha\\ &= \int\int_S [\partial_z f_1 n_y - \partial_z f_1 n_y] + [\partial_y f_1 n_x - \partial_x f_1 n_y] + [\partial_z f_1 n_z - \partial_x f_1 n_y] du dv.\end{aligned} \tag{49}$$

This, in other terms, gives

$$\int_C \partial_z^{\alpha-1} F_x dx^\alpha + \partial_x^{\alpha-1} F_y dy^\alpha + \partial_y^{\alpha-1} F_z dz^\alpha \tag{50}$$

$$\int\int_S [\nabla_{\underline{x}}^\alpha \times \underline{F} + (\partial_x [K_{\underline{x}}^{\alpha-1}].F_y \underline{e_y})\underline{e_x} + (\partial_y [K_{\underline{x}}^{\alpha-1}].F_z \underline{e_z})\underline{e_y} + (\partial_z [K_{\underline{x}}^{\alpha-1}].F_x \underline{e_x})\underline{e_z}].\underline{n} dS^\alpha.$$

This completes the proof. □

We shall conclude this paper by proposing the following Jacobian matrices.

Let $f$ and $g$ be two functions of two variables, such that their respective partial derivatives exist and are continuous, then the conformable fractional Jacobian matrix associate is given as:

$$\begin{pmatrix} \frac{\partial^\alpha f}{\partial x^\alpha} & \frac{\partial^\alpha f}{\partial y^\alpha} \\ \frac{\partial^\alpha g}{\partial x^\alpha} & \frac{\partial^\alpha g}{\partial y^\alpha} \end{pmatrix} = \begin{pmatrix} x^{1-\alpha}\frac{\partial f}{\partial x} & y^{1-\alpha}\frac{\partial f}{\partial y} \\ x^{1-\alpha}\frac{\partial g}{\partial x} & y^{1-\alpha}\frac{\partial g}{\partial y} \end{pmatrix} \tag{51}$$

This can be generalized in higher dimension.

# 3 Conclusion

The concept of derivative with fractional parameter has recently captured the minds of scholars. For instance, Riemann-Liouville proposed the idea of fractional order derivative. Caputo and others later modified this fractional derivative. These fractional derivatives have been employed to model many real world problems with success due to their physical properties. A derivative based upon the classical derivative with fractional parameter has been proposed recently. The derivative has very interesting properties and, therefore, in this work we have undertaken an investigation regarding more details around this new derivative. We have first remarked that, there is a relationship between this derivative, the well-known q-derivative and fractal derivatives. Furthermore, we presented novel theorems, properties and remarks in connection with the classical derivatives. As a result, the new derivative obeys several properties followed by the classical derivative.

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