Computers and Mathematics with Applications 64 (2012) 3059–3064

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence of a periodic mild solution for a nonlinear fractional differential equation

Mohamed A.E. Herzallah [a,b], Dumitru Baleanu [c,d,*]

[a] College of Science in Zulfi, Majmaah University, Saudi Arabia
[b] Faculty of Science, Zagazig University, Zagazig, Egypt
[c] Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey
[d] Institute of Space Sciences, P.O.BOX, MG-23, R 76900,Magurele-Bucharest, Romania

ARTICLE INFO

Keywords:
Fractional derivative
Fractional nonlinear differential equations
Boundary value problem
Schaefer fixed point theorem

ABSTRACT

The aim of this manuscript is to analyze the existence of a periodic mild solution to the problem of the following nonlinear fractional differential equation

$${}_0^R D_t^\alpha u(t) - \lambda u(t) = f(t, u(t)), \qquad u(0) = u(1) = 0, \quad 1 < \alpha < 2,\ \lambda \in \mathbb{R},$$

where ${}_0^R D_t^\alpha$ denotes the Riemann–Liouville fractional derivative. We obtained the expressions of the general solution for the linear fractional differential equation by making use of the Laplace and inverse Laplace transforms. By making use of the Banach contraction mapping principle and the Schaefer fixed point theorem, the existence results of one or at least one mild solution for a nonlinear fractional differential equation were given.

1. Introduction

Fractional calculus is an emerging field in the area of the applied mathematics that deals with derivatives and integrals of arbitrary orders as well as with their applications. During the history of fractional calculus it was reported that the pure mathematical formulations of the investigated problems started to be dressed with more applications in various fields. As a result during the last decade fractional calculus has been applied successfully to almost every field of science and engineering. However, despite of the fact that several fields of application of fractional differentiation and integration are already well established, some others have just started.

Many applications of fractional calculus dynamics can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, etc. (see for more details Refs. [1–8] and the references therein).

One of the main applications of fractional calculus is in the field of modeling of the intermediate physical process, namely because fractional-order models are more accurate in many cases than the integer-order models. The description of memory and hereditary properties of various materials and processes are described more accurately with the fractional dynamics [2].

As a result, various phenomena are modeled with fractional differential equations. However, finding solutions to nonlinear fractional differential equations may not be possible in most of the cases. On the other hand, classes of boundary value problems have been analyzed. In [9,10] it was established the existence and uniqueness of the solution for a class of linear and superlinear fractional differential equations. Recently, there are some works dealing with the existence and

* Corresponding author at: Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey.
E-mail addresses: m_herzallah75@hotmail.com (M.A.E. Herzallah), dumitru@cankaya.edu.tr (D. Baleanu).

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.12.060

multiplicity of solutions of nonlinear fractional differential equations by the use of techniques of nonlinear analysis. For example, Belmekki et al. proved the existence of a mild solution of the problem [11]

$$ {}_0^R D_t^{\delta} u(t) - \lambda u(t) = f(t, u(t)), \qquad u(1) = \lim_{t \to 0^+} t^{1-\delta} u(t), \quad 0 < \delta < 1, \tag{1} $$

under some conditions on $f(t, u(t))$, where ${}_0^R D_t^{\delta}$ is the Riemann–Liouville derivative which is defined below, and presented the conditions that give the unique solution.

Having all above mentioned points in mind, in this manuscript, we studied the existence of a mild solution to the problem

$$ {}_0^R D_t^{\alpha} u(t) - \lambda u(t) = f(t, u(t)), \qquad u(0) = u(1) = 0, \quad 1 < \alpha < 2, \ \lambda \in \mathbb{R}, \tag{2} $$

and discussed when this solution is unique.

The organization of the manuscript is given below.

In Section 2, the definitions and theorems used in this manuscript are presented. In Section 3, we investigated the linear problem which is used in studying the nonlinear problem in Section 4. Finally, we presented our conclusion in Section 5.

2. Preliminaries

In this section, the basic definitions and theorems used in this paper are briefly presented. Let $f \in L(J, \mathbb{R})$, the set of all integrable functions from $J = [0, T]$ to the real numbers, and let α be a positive real number.

Definition 2.1. The fractional integral of order α of the function $f(t)$ is defined by [4,6,7,12]

$$ I^{\alpha} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds. $$

Definition 2.2. The Riemann–Liouville fractional derivative of the function $f(t)$ of order $\alpha \in (n-1, n)$ is defined by [4,6,7,12]

$$ {}_0^R D_t^{\alpha} f(t) = D^n I_a^{n-\alpha} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s) ds. $$

One of the useful ways in solving the fractional differential equation is using the Laplace transform [4,7] where the Laplace transform of the Riemann–Liouville derivative is given as

$$ L({}_0^R D_t^{\alpha} f(t)) = s^{\alpha} \hat{f}(s) - \sum_{k=0}^{n-1} s^{k} {}_0^R D_t^{\alpha - r - 1} f(0), $$

where $\hat{f}(s)$ is the Laplace transform of $f(t)$.

The Mittag-Leffler function with one and two parameters plays a similar role in fractional calculus as the role of the exponential function in the theory of integer-order differential equation.

The Mittag-Leffler function in two parameters is defined as [4,7],

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. \tag{3} $$

By inspection we obtained that $E_{1,1}(z) = e^z$, $E_{1,2}(z) = \frac{e^z - 1}{z}$, $E_{2,1}(z) = \cosh(\sqrt{z})$ and $E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}$, *respectively*.

The Laplace transform of the Mittag-Leffler function in two parameters has the form

$$ L(t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\pm a t^{\alpha})) = \frac{k! \lambda^{\alpha - \beta}}{(\lambda^{\alpha} \mp a)^{k+1}}, \qquad \mathrm{Re}(\lambda) > |a|^{\frac{1}{\alpha}}, \tag{4} $$

where $E_{\alpha,\beta}^{(k)}(y) = \frac{d^k}{dy^k} E_{\alpha,\beta}(y)$.

In proving the existence of the solution of our problem, the following fixed point theorem plays an important role [13].

Theorem 2.1 (*Schaefer Fixed Point Theorem*)**.** *Let X be a normed linear space, and let the operator $T : X \to X$ be compact. Then either*

(i) *the operator T has a fixed point in X, or*
(ii) *the set $A = \{u \in X : u = \mu T(u), \mu \in (0, 1)\}$ is unbounded.*

3. The linear problem

The starting point is to consider the linear fractional differential equation as given below

$$ {}_0^R D_t^{\alpha} u(t) - \lambda u(t) = y(t), \quad 1 < \alpha < 2, \qquad u(0) = u(1) = 0, \tag{5} $$

where $\lambda \in \mathbb{R}$ and y is a known function.

Taking the Laplace transform of (5) we obtain

$$s^{\alpha}\hat{u}(s) - {}_0^RD_t^{\alpha-1}u(0) - s{}_0^RD_t^{\alpha-2}u(0) - \lambda\hat{u}(s) = \hat{y}(s),$$

which gives

$$\hat{u}(s) = \frac{c_1}{s^{\alpha}-\lambda} + \frac{c_2 s}{s^{\alpha}-\lambda} + \frac{1}{s^{\alpha}-\lambda}\hat{y}(s),$$

where $c_1 = {}_0^RD_t^{\alpha-1}u(0)$ and $c_2 = {}_0^RD_t^{\alpha-2}u(0)u(0)$. By taking the inverse Laplace transform and by using the boundary conditions the solution is obtained as

$$u(t) = -\frac{t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{E_{\alpha,\alpha}(\lambda)}\int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})y(s)ds + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})y(s)ds, \tag{6}$$

which can be written in the form

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{7}$$

where

$$G_{\lambda,\alpha}(t,s) = \begin{cases} -\dfrac{t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{E_{\alpha,\alpha}(\lambda)}(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha}) + (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) & 0 \le s \le t \\ -\dfrac{t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{E_{\alpha,\alpha}(\lambda)}(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha}) & t \le s \le 1. \end{cases} \tag{8}$$

It is easy to see that $G(t,s)$ is bounded and continuous on $[0,1]\times[0,1]$. Thus, we proved our statement.

Theorem 3.1. *If y is a continuous function and λ is a real parameter, then the linear problem (5) has a continuous mild solution given by (7).*

Two special cases of (8) are presented below.

Case 1: For $\alpha = 1$ one gets that

$$G_{\lambda,1}(t,s) = \begin{cases} -\dfrac{E_{1,1}(\lambda t)}{E_{1,1}(\lambda)}E_{1,1}(\lambda(1-s)) + E_{1,1}(\lambda(t-s)) & 0 \le s \le t \\ -\dfrac{E_{1,1}(\lambda t)}{E_{1,1}(\lambda)}E_{1,1}(\lambda(1-s)) & t \le s \le 1. \end{cases}$$

By using the properties of the Mittag-Leffler function we obtain that

$$G_{\lambda,1}(t,s) = \begin{cases} 0 & 0 \le s \le t \\ -e^{\lambda(t-s)} & t \le s \le 1, \end{cases}$$

which is precisely the Green function for the ordinary linear differential equation of first order.

Case 2:

For $\alpha = 2$ one gets that

$$G_{\lambda,2}(t,s) = \begin{cases} -\dfrac{tE_{2,2}(\lambda t^2)}{E_{2,2}(\lambda)}(1-s)E_{2,2}(\lambda(1-s)^2) + (t-s)E_{2,2}(\lambda(t-s)^2) & 0 \le s \le t \\ -\dfrac{tE_{2,2}(\lambda t^2)}{E_{2,2}(\lambda)}(1-s)E_{2,2}(\lambda(1-s)^2) & t \le s \le 1. \end{cases}$$

By using the properties of the Mittag-Leffler function we obtain that

$$G_{\lambda,2}(t,s) = \frac{1}{\sqrt{\lambda}\sinh(\sqrt{\lambda}t)}\begin{cases} -\sinh(\sqrt{\lambda}s)\sinh(\sqrt{\lambda}(1-t)) & 0 \le s \le t \\ -\sinh(\sqrt{\lambda}t)\sinh(\sqrt{\lambda}(1-s)) & t \le s \le 1, \end{cases}$$

which is precisely the Green function for the periodic boundary value problem considered in [14].

4. The nonlinear problem

Now we investigate the nonlinear problem (2). If u is a mild solution of (2), then it is given by

$$u(t) = \int_0^1 G_{\lambda,\alpha}(t,s)f(s,u(s))ds, \tag{9}$$

where $G_{\lambda,\alpha}(t,s)$ is the Green function given by (8).

Define the operator $T : C[0,1] \to C[0,1]$ by

$$Tu(t) = \int_0^1 G_{\lambda,\alpha}(t,s) f(s,u(s))ds. \tag{10}$$

Theorem 4.1. *Let the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfy the following*

(i) *$f(t,u)$ is bounded, that is there exists $M > 0$ such that*

$$|f(t,u)| \le M, \quad \forall t \in [0,1],\ u \in \mathbb{R}, \tag{11}$$

(ii) *$f(t,u)$ satisfies Lipschitz condition, that is there exists a constant $K > 0$ such that*

$$|f(t,u) - f(t,v)| \le K|u - v|, \quad t \in [0,1],\ u, v \in \mathbb{R}. \tag{12}$$

Then the operator T given by (10) is compact.

Proof. Let $t_1 < t_2 \in (0,1)$ and $u \in C[0,1]$. Then we have

$$\begin{aligned}
|Tu(t_1) - Tu(t_2)| &= \Bigg| -\frac{t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^\alpha)}{E_{\alpha,\alpha}(\lambda)} \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha) f(s,u(s))ds \\
&\quad + \int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^\alpha) f(s,u(s))ds \\
&\quad + \frac{t_2^{\alpha-1}E_{\alpha,\alpha}(\lambda t_2^\alpha)}{E_{\alpha,\alpha}(\lambda)} \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha) f(s,u(s))ds \\
&\quad - \int_0^{t_2} (t_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha) f(s,u(s))ds \Bigg| \\
&= \Bigg| \left(\frac{t_2^{\alpha-1}E_{\alpha,\alpha}(\lambda t_2^\alpha)}{E_{\alpha,\alpha}(\lambda)} - \frac{t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^\alpha)}{E_{\alpha,\alpha}(\lambda)} \right) \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha) f(s,u(s))ds \\
&\quad + \int_0^{t_1} ((t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^\alpha) - (t_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha)) f(s,u(s))ds \\
&\quad - \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha) f(s,u(s))ds \Bigg| \\
&\le M \Bigg[\left| \frac{t_2^{\alpha-1}E_{\alpha,\alpha}(\lambda t_2^\alpha)}{E_{\alpha,\alpha}(\lambda)} - \frac{t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^\alpha)}{E_{\alpha,\alpha}(\lambda)} \right| \int_0^1 |(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha)|ds \\
&\quad + \int_0^{t_1} |((t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^\alpha) - (t_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha))|ds \\
&\quad + \int_{t_1}^{t_2} |(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha)|ds \Bigg],
\end{aligned}$$

using the definition and properties of the Mittag-Leffler function to find the values of the three integrations in the last inequality one gets the following.

For the first integration:

$$\begin{aligned}
\int_0^1 |(1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha)|ds &\le \int_0^1 (1-s)^{\alpha-1} \sum_{k=0}^\infty \frac{|\lambda|^k(1-s)^{\alpha k}}{\Gamma(\alpha k+\alpha)} ds \\
&= \sum_{k=0}^\infty \frac{|\lambda|^k}{\Gamma(\alpha k+\alpha)} \int_0^1 (1-s)^{\alpha k+\alpha-1} ds = \sum_{k=0}^\infty \frac{|\lambda|^k}{\Gamma(\alpha k+\alpha+1)} \\
&= E_{\alpha,\alpha+1}(|\lambda|).
\end{aligned}$$

Similar for the third integration one can prove

$$\int_{t_1}^{t_2} |(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_2-s)^\alpha)|ds \le (t_2-t_1)^\alpha E_{\alpha,\alpha+1}(|\lambda|(t_2-t_1)^\alpha).$$

Now for the second integration one gets

$$\begin{aligned}
&\int_0^{t_1} |((t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha})-(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_2-s)^{\alpha}))|ds \\
&\quad = \int_0^{t_1}\left|\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+\alpha)}\left[(t_1-s)^{\alpha k+\alpha-1}-(t_2-s)^{\alpha k+\alpha-1}\right]\right|ds \\
&\quad \le \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(\alpha k+\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha k+\alpha-1}-(t_1-s)^{\alpha k+\alpha-1}\right]ds \\
&\quad \le \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma(\alpha k+\alpha)}[-(t_2-t_1)^{\alpha k+\alpha}+t_2^{\alpha k+\alpha}-t_1^{\alpha k+\alpha}] \\
&\quad \le -(t_2-t_1)^{\alpha}E_{\alpha,\alpha+1}(|\lambda|(t_2-t_1)^{\alpha})+t_2^{\alpha}E_{\alpha,\alpha+1}(|\lambda|t_2^{\alpha})-t_1^{\alpha}E_{\alpha,\alpha+1}(|\lambda|t_1^{\alpha}).
\end{aligned}$$

From these results one gets that

$$|Tu(t_1)-Tu(t_2)| \le M\left[\frac{E_{\alpha,\alpha+1}(|\lambda|)}{E_{\alpha,\alpha}(|\lambda|)}(t_2^{\alpha-1}E_{\alpha,\alpha}(|\lambda|t_2^{\alpha})-t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|t_1^{\alpha}))+t_2^{\alpha}E_{\alpha,\alpha+1}(|\lambda|t_2^{\alpha})-t_1^{\alpha}E_{\alpha,\alpha+1}(|\lambda|t_1^{\alpha})\right],$$

which gives that

$$|Tu(t_1)-Tu(t_2)| \to 0 \quad \forall |t_2-t_1| \to 0. \tag{13}$$

Now we prove that T is continuous. For $u, v \in C[0, 1]$ one gets

$$\begin{aligned}
|Tu(t)-Tv(t)| &\le \int_0^1 |G_{\lambda,\alpha}(t,s)||f(s,u(s))-f(s,v(s))|ds \\
&\le K\int_0^1 |G_{\lambda,\alpha}(t,s)||u(s)-v(s)|ds \\
\Rightarrow \|Tu-Tv\| \le K\|u-v\| &\int_0^1 |G_{\lambda,\alpha}(t,s)|ds.
\end{aligned}$$

About the Green function as in proving (13) we have

$$\int_0^1 |G_{\lambda,\alpha}(t,s)|ds \le \left(\frac{E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,\alpha}(\lambda)|}+1\right)E_{\alpha,\alpha+1}(|\lambda|), \tag{14}$$

thus we get

$$\|Tu-Tv\| \le K\left(\frac{E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,\alpha}(\lambda)|}+1\right)E_{\alpha,\alpha+1}(|\lambda|)\|u-v\|, \tag{15}$$

which proves that T is a continuous operator. □

Now let D be a bounded set in $C[0, 1]$. By continuity of the operator T we get that $TD = \{Tu : u \in D\}$ is bounded, and by (13) we have that TD is an equicontinuous set in the space $C[0, 1]$. By the Arzela–Ascoli theorem [13], we get that TD is relatively compact set, which proves that T is a compact operator.

Theorem 4.2. *If the hypotheses in Theorem 4.1 are satisfied and*

$$K\left(\frac{E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,\alpha}(\lambda)|}+1\right)E_{\alpha,\alpha+1}(|\lambda|) < 1, \tag{16}$$

then there is a unique mild solution of problem (2).

Proof. From (15) if we have (16) then T is a contraction mapping. Then by the Banach fixed point theorem there is a unique fixed point of the operator T which is the unique mild solution of (2). □

Theorem 4.3. *Assume that the hypotheses of Theorem 4.1 are satisfied. Then there is at least one mild solution in $C[0, 1]$ to the problem (2).*

Proof. Consider the set $A = \{u \in C[0,1] : u = \mu T(u), \mu \in (0,1)\}$. Let $u \in A$ then $u = \mu Tu$ which gives

$$|u(t)| \le \mu \int_0^1 |G_{\lambda,\alpha}(t,s)||f(s,u(s))|ds$$
$$\le \mu M \left(\frac{E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,\alpha}(\lambda)|} + 1 \right) E_{\alpha,\alpha+1}(|\lambda|) < \infty.$$

Thus, $\|u\| < \infty$ which proves that the set A is bounded. By Theorem 2.1 we get that T has a fixed point in $C[0,1]$ which is the required mild solution. □

Finally, we finish this section by giving an example which illustrate our results.

Example 4.1. Consider the nonlinear fractional differential equation

$$ {}_0^R D_t^\alpha u(t) - \lambda u(t) = \frac{A}{2} t \sin(u(t)) + t^4, \qquad u(0) = u(1) = 0, \quad 1 < \alpha < 2,\ \lambda \in \mathbb{R},\ A = \text{constant}. \tag{17}$$

It is obvious that $f(t, u(t)) = \frac{A}{2} t \sin(u(t)) + t^4$ satisfies the conditions of Theorem 4.1 with $M = A/2 + 1$ and $K = A/2$. Using the results of Theorem 4.1 we get that (17) has at least one mild solution given by (9). Moreover if $AE_{\alpha,\alpha+1}(|\lambda|) < 1$ then this solution will be unique.

5. Conclusion

In this manuscript, we discussed the existence of at least one mild solution for a nonlinear fractional differential equation with the Riemann–Liouville fractional derivative with $\alpha \in (0, 2)$. We determined the conditions to get a unique solution by proving that the Green function of the ordinary linear differential equation of first order and the periodic boundary value problem can be given as special cases of the fractional Green function of our problem. One example is analyzed in order to illustrate our results.

References

[1] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Vienna, New York, 1997.
[2] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional Calculus Models and Numerical Methods, World Scientific Publishing, New York, 2012.
[3] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, New Jersey, 2000.
[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematical Studies, vol. 204, Elsevier, Amsterdam, 2006.
[5] R.L. Magin, Fractional calculus in bioengineering, parts 1–3, Crit. Rev. Biomed. Eng. 32 (1) (2004) 1–377.
[6] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, New York, 1993.
[7] I. Podlubny, Fractional Differential Equations, Acad. Press, San Diego, New York, London, 1999.
[8] J.R. Wang, Y. Zhou, Analysis of nonlinear fractional control systems in Banach spaces, Nonlinear Anal. TMA 74 (17) (2011) 5929–5942.
[9] D. Baleanu, O. Mustafa, R. Agarwal, Asymptotically linear solutions for some linear fractional differential equations, Abstr. Appl. Anal. (2010) ID 865139, 8 pages.
[10] D. Baleanu, O. Mustafa, R. Agarwal, An existence result for a superlinear fractional differential equation, Appl. Math. Lett. 23 (9) (2010) 1129–1132.
[11] M. Belmekki, J.J. Nieto, R.R. Lopez, Existence of periodic solution for a nonlinear fractional differential equation, Bound. Value Probl. (2009) doi:10.1155/2009/324561.
[12] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applcations, Gordon and Breach, New York, 1993.
[13] E. Zeidler, Nonlinear Functional Analysis and its Applications, I: Fixed-Point Theorems, Springer-Verlag New York Inc., 1986.
[14] J.A. Pennline, Constructive existence and uniqueness for some nonlinear two-point boundary value problems, J. Math. Anal. Appl. 96 (1983) 584–598.