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RESEARCH PAPER

MONOTONE ITERATIVE METHOD FOR A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

Guotao Wang [1], Dumitru Baleanu [2], Lihong Zhang [1]

Abstract

By applying the monotone iterative technique and the method of lower and upper solutions, this paper investigates the existence of extremal solutions for a class of nonlinear fractional differential equations, which involve the Riemann-Liouville fractional derivative $D^q x(t)$. A new comparison theorem is also build. At last, an example is given to illustrate our main results.

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Key Words and Phrases: monotone iterative technique, nonlinear fractional differential equation, comparison theorem, extremal solutions

1. Introduction and preliminaries

Despite of the fact that the fractional calculus is as old as the classical one it started to be recognized as a powerful tool for analyzing the dynamics of the complex or hypercomplex systems (for details see [12],[8],[13],[9],[7],[3]). The beauty of this type of calculus is that it captures the properties of the nonlocal dynamics better that the other existing classical methods and techniques. The range of applications of the fractional calculus is very large at this moment and it is expected to increase in the nearest future (see for example [14],[11],[3],[5],[2],[1] and the references therein).

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Many open problems remain to be investigated in the area of nonlinear fractional differential equations and for this reason new methods and techniques should be invented and applied for systems from the real world applications.

Having all above mentioned in mind, in this paper we study the following nonlinear fractional differential equations

$$\begin{cases} (D^q x(t))' = f(t, x(t), D^q x(t)), \\ D^q x(0) = x_0^*, \ t^{1-q} x(t)|_{t=0} = x_0, \end{cases} \tag{1.1}$$

where $t \in J = [0, T]$ $(T > 0)$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $x_0^*, x_0 \in \mathbb{R}$ and D^q is the Riemann-Liouville fractional derivative of x, and q is such that $0 < q < 1$.

Definition **1.1**. The Riemann-Liouville (R-L) fractional derivative of order $\delta > 0$ for a function $f(t)$ is defined by

$$D^\delta f(t) = \frac{1}{\Gamma(n-\delta)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\delta-1} f(s) ds, \quad n = [\delta] + 1,$$

provided the right hand side is defined pointwise on $(0, \infty)$ and $[\delta]$ denoted the integer part of the order δ.

Definition **1.2**. The Riemann-Liouville (R-L) fractional integral of order δ for a function f is defined as

$$I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds, \quad \delta > 0,$$

provided that such integral exists.

The rest of the paper is organized as follows.

In Section **2**, we list several lemmas and a new comparison principle, playing an important role in the proof of the main results. Further, we consider the existence of the extremal solutions for a class of first order nonlinear differential equation involving the R-L fractional integral operator. In Section **3**, we formulate sufficient conditions which guarantee that problem (1.1) has extremal solutions. A one-sided Lipschitz condition is imposed. In Section **4**, an example is given to illustrate our main results.

2. Results for a differential equation involving R-L fractional integral operator

In this section, we study the following initial value problem for first order differential equation involving R-L fractional integral operator

$$\begin{cases} u'(t) = f(t, \mathcal{B}u(t), u(t)), \\ u(0) = x_0^*, \end{cases} \tag{2.1}$$

where f, x_0^* are the same as (1.1), and $\mathcal{B}u(t) = x_0 t^{q-1} + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} u(s)ds$.

LEMMA **2.1**. (Comparison result) *Let $p \in C^1(J, \mathbb{R})$ satisfy*

$$\begin{cases} p'(t) \geq -M(t)\mathcal{L}p(t) - N(t)p(t), \\ p(0) \geq 0, \end{cases} \tag{2.2}$$

where $M(t), N(t)$ are nonnegative integral functions and

$$\mathcal{L}p(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)ds.$$

If

$$\int_0^T [\frac{M(t)t^q}{\Gamma(q+1)} + N(t)]dt \leq 1, \tag{2.3}$$

then $p(t) \geq 0, \ \forall t \in J$.

P r o o f. Suppose $p(t) \geq 0$ is not true, then there exists a $t_0 \in (0, T]$ such that $p(t_0) < 0$. Let $\max\{p(t) : 0 \leq t \leq t_0\} = \lambda$, then $\lambda \geq 0$.

Case 1: if $\lambda = 0$, then $p(t) \leq 0, \forall t \in [0, t_0]$. Thus, by (2.2), we have $p'(t) \geq 0, \forall t \in [0, t_0]$. So, we have $p(t_0) \geq p(0) \geq 0$, which contradicts with $p(t_0) < 0$.

Case 2: if $\lambda > 0$, then there exists a $t_1 \in [0, t_0]$ such that $p(t_1) = \lambda > 0$. From (2.2), we have

$$p'(t) \geq -\lambda[\frac{M(t)t^q}{\Gamma(q+1)} + N(t)], \ \forall t \in [0, t_0].$$

Thus, we have that

$$\begin{aligned} p(t_0) &= p(t_1) + \int_{t_1}^{t_0} p'(t)dt \\ &\geq \lambda - \lambda \int_0^T [\frac{M(t)t^q}{\Gamma(q+1)} + N(t)]dt \\ &= \lambda\Big\{1 - \int_0^T [\frac{M(t)t^q}{\Gamma(q+1)} + N(t)]dt\Big\}. \end{aligned}$$

Then, by $p(t_0) < 0$, we get $\int_0^T [\frac{M(t)t^q}{\Gamma(q+1)} + N(t)]dt > 1$, which contradicts with (2.3).

Hence $p(t) \geq 0, \forall t \in J$. This completes the proof. □

To study the nonlinear problem (2.1), we first consider the associated linear problem:

$$\begin{cases} u'(t) = \sigma(t) - M(t)\mathcal{B}u(t) - N(t)u(t), \\ u(0) = x_0^*, \end{cases} \tag{2.4}$$

where $\sigma \in C(J, \mathbb{R})$, $M(t), N(t)$ are nonnegative integral functions and $\mathcal{B}$ is defined as (2.1).

LEMMA **2.2**. *If (2.3) holds, then (2.4) has a unique solution* $u^* \in C^1(J, \mathbb{R})$.

P r o o f. Obviously, $u(t) \in C^1(J, \mathbb{R})$ is a solution of (2.4) if and only if $u(t) \in C(J, \mathbb{R})$ is a solution of the following integral equation

$$\begin{aligned} u(t) =& x_0^* e^{-\int_0^t N(\tau)d\tau} + \int_0^t e^{-\int_s^t N(\tau)d\tau}[\sigma(s) - M(s)\mathcal{B}u(s)]ds \\ \triangleq& Qu(t). \end{aligned} \tag{2.5}$$

For any $u, v \in C(J, \mathbb{R})$, by (2.5) we have

$$\begin{aligned} |Qu(t) - Qv(t)| &= \Big| \int_0^t M(s) e^{-\int_s^t N(\tau)d\tau} \mathcal{B}(u - v)(s) ds \Big| \\ &\le \int_0^T \frac{M(s)s^q}{\Gamma(q+1)} ds \|u - v\|_C. \end{aligned}$$

Therefore, from (2.3), it follows that

$$\|Qu - Qv\|_C < \|u - v\|_C.$$

Hence, T is a contraction operator on $C(J, \mathbb{R})$. Consequently, the Banach fixed point theorem implies that Q has a unique fixed point $u^* \in C(J, \mathbb{R})$, that is, u^* is a unique solution of (2.4). □

THEOREM **2.1**. *Assume that:*

(H_1) *There exist* $u_0, v_0 \in C(J, \mathbb{R})$ *satisfying* $u_0(t) \le v_0(t), \forall t \in J$,

$$\begin{cases} u_0'(t) \le f(t, \mathcal{B}u_0(t), u_0(t)), \\ u_0(0) \le x_0^*, \end{cases} \tag{2.6}$$

and v_0 *satisfies inverse inequalities of (2.6).*

(H_2) *The following condition holds*

$$f(t, u, v) - f(t, \overline{u}, \overline{v}) \ge -M(t)(u - \overline{u}) - N(t)(v - \overline{v}),$$

where $\mathcal{B}u_0(t) \le \overline{u} \le u < \mathcal{B}v_0(t)$, $u_0(t) \le \overline{v} \le v \le v_0(t), \forall t \in J$, *and* $M(t), N(t)$ *are as in Lemma 2.1 and satisfy (2.3).*

Then there exist monotone iterative sequences $\{u_n\}, \{v_n\} \subset [u_0, v_0]$ *which converge uniformly to the extremal solutions* u^*, v^* *of (2.1), respectively, where* $\{u_n\}, \{v_n\}$ *are defined by*

$$u_n(t) = \int_0^t e^{-\int_s^t N(\tau)d\tau}[f(s, \mathcal{B}u_{n-1}(s), u_{n-1}(s)) - M(s)\mathcal{B}(u_n - u_{n-1})(s) + N(s)u_{n-1}(s)]ds + x_0^* e^{-\int_0^t N(\tau)d\tau},$$

$$v_n(t) = \int_0^t e^{-\int_s^t N(\tau)d\tau}[f(s, \mathcal{B}v_{n-1}(s), v_{n-1}(s)) - M(s)\mathcal{B}(v_n - v_{n-1})(s) + N(s)v_{n-1}(s)]ds + x_0^* e^{-\int_0^t N(\tau)d\tau},$$

and

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le u^* \le v^* \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.7}$$

P r o o f. For $\forall \eta \in [u_0, v_0]$, we consider

$$\begin{cases} u'(t) = f(t, \mathcal{B}\eta(t), \eta(t)) + M(t)\mathcal{B}\eta(t) + N(t)\eta(t) - M(t)\mathcal{B}u(t) - N(t)u(t), \\ u(0) = x_0^*. \end{cases} \tag{2.8}$$

By Lemma **2.2**, we know (2.8) has a unique solution $u \in C^1(J, \mathbb{R})$.

Denote an operator $Q : [u_0, v_0] \to C(J, \mathbb{R})$ by $u = Q\eta$, then

$$Q\eta = x_0^* e^{-\int_0^t N(\tau)d\tau} + \int_0^t e^{-\int_s^t N(\tau)d\tau}[f(s, \mathcal{B}\eta(s), \eta(s)) + M(s)\mathcal{B}\eta(s) + N(s)\eta(s) - M(s)\mathcal{B}u(s)]ds.$$

First, we shall show that the operator Q has the following two properties:

(1) $u_0 \le Qu_0,\ Qv_0 \le v_0$; (2) Q is nondecreasing.

To prove (1), let $u_1 = Qu_0,\ p = u_1 - u_0$. By (H_1), we can get that

$$\begin{cases} p'(t) \ge -M(t)\mathcal{B}p(t) - N(t)p(t), \\ p(0) \ge 0. \end{cases}$$

By Lemma **2.1**, we have $p(t) \ge 0, \forall t \in J$. That is, $u_0 \le Qu_0$. Similarly, we can show $Qv_0 \le v_0$. Then (1) holds.

To prove (2), let $u_1 = Qu_0,\ v_1 = Qv_0$ and $p = v_1 - u_1$. By (H_2), we have

$$\begin{cases} p'(t) = f(t, \mathcal{B}v_0(t), v_0(t)) - f(t, \mathcal{B}u_0(t), u_0(t)) + M(t)\mathcal{B}v_0(t) + N(t)v_0(t) \\ \qquad -M(t)\mathcal{B}v_1(t) - N(t)v_1(t) - M(t)\mathcal{B}u_0(t) - N(t)u_0(t) \\ \qquad +M(t)\mathcal{B}u_1(t) + N(t)u_1(t) \\ \qquad \ge -M(t)\mathcal{B}p(t)) - N(t)p(t), \\ p(0) \ge 0. \end{cases}$$

Hence, by Lemma **2.1**, we have $p(t) \geq 0, \forall t \in J$. That is, $Qu_0 \leq Qv_0$, which implies Q is nondecreasing.

Next, let $u_n = Qu_{n-1},\ v_n = Qv_{n-1},\ n = 1, 2, \cdots$. Combining with the properties of the operator Q, we obtain that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{2.9}$$

Obviously, u_n, v_n $(n = 1, 2, \cdots)$ satisfy

$$\begin{cases} u_n'(t) = f(t, \mathcal{B}u_{n-1}(t), u_{n-1}(t)) - M(t)\mathcal{B}(u_n - u_{n-1})(t) - N(t)(u_n - u_{n-1})(t), \\ u_n(0) = x_0^*, \end{cases}$$

and

$$\begin{cases} v_n'(t) = f(t, \mathcal{B}v_{n-1}(t), v_{n-1}(t)) - M(t)\mathcal{B}(v_n - v_{n-1})(t) - N(t)(v_n - v_{n-1})(t), \\ v_n(0) = x_0^*, \end{cases}$$

Employing standard arguments, we find that $\lim\limits_{n\to\infty} u_n(t) = u^*(t)$ and $\lim\limits_{n\to\infty} v_n(t) = v^*(t)$ uniformly on J, and $u^*(t),\ v^*(t)$ are solutions of (2.1). Hence, (2.7) holds.

Finally, we prove that u^*, v^* are the extremal solutions of (2.1) in $[u_0, v_0]$. Let $w \in [u_0, v_0]$ be any solution of (2.1), then $Qw = w$. By $u_0 \leq w \leq v_0$ and the properties of Q, we have

$$u_n \leq w \leq v_n,\ n = 1, 2, \cdots. \tag{2.10}$$

Thus, taking limit in (2.10) as $n \to \infty$, we have $u^* \leq w \leq v^*$. That is, u^*, v^* are the extremal solutions of (2.1) in $[u_0, v_0]$.

This completes the proof. □

3. Main results for fractional differential equation (1.1)

In this section we prove the existence of extremal solutions of (1.1) by employing Theorem **2.1**.

Let $C_{1-q}(J, \mathbb{R}) = \{u \in C(0, T];\ t^{1-q}u \in C(J, \mathbb{R})\}$ and $DC_{1-q}(J, \mathbb{R}) = \{u \in C_{1-q}(J, \mathbb{R});\ D^q u \in C^1(J, \mathbb{R})\}$.

Theorem **3.1**. *Assume that:*

(H_1') *There exist $y_0, z_0 \in DC_{1-q}(J, \mathbb{R})$ satisfying $y_0(t) \leq z_0(t)$ and $D^q y_0(t) \leq D^q z_0(t), \forall t \in J$,*

$$\begin{cases} (D^q y_0(t))' \leq f(t, y_0(t), D^q y_0(t)), \\ D^q y_0(0) \leq x_0^*,\ t^{1-q} y_0(t)|_{t=0} = x_0, \end{cases} \tag{3.1}$$

and z_0 satisfies inverse inequalities of (3.1).

(H_2') *The following condition holds*

$$f(t, x, y) - f(t, \overline{x}, \overline{y}) \geq -M(t)(x - \overline{x}) - N(t)(y - \overline{y}),$$

where $y_0(t) \leq \overline{x} \leq x < z_0(t)$, $D^q y_0(t) \leq \overline{y} \leq y \leq D^q z_0(t), \forall t \in J$, *and* $M(t), N(t)$ *are as in Lemma 2.1 and satisfy (2.3).*

Then there exist iterative sequences $\{y_n\}, \{z_n\}$ *which converge uniformly to the extremal solutions* y^*, z^* *of (1.1), respectively.*

P r o o f. Let $D^q x(t) = u(t)$ in (1.1), then (1.1) is equivalent to the following equation

$$\begin{cases} D^q x(t) = u(t), \\ u'(t) = f(t, \mathcal{B}u(t), u(t)), \\ t^{1-q}x(t)|_{t=0} = x_0, \\ u(0) = x_0^*. \end{cases} \tag{3.2}$$

For any $u \in C(J, R)$, the system

$$\begin{cases} D^q x(t) = u(t), \\ t^{1-q}x(t)|_{t=0} = x_0, \end{cases} \tag{3.3}$$

has a unique solution

$$x(t) = x_0 t^{q-1} + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} u(s)ds. \tag{3.4}$$

Define an operator $x(t) = \mathcal{B}u(t)$, then $\mathcal{B}$ is continuous and nondecreasing.

Hence, from (3.2)-(3.4), (1.1) is transformed into first order integro-differential equation (2.1).

Let $u_0(t) = D^q y_0(t), v_0(t) = D^q z_0(t)$, we have $u_0(t) \leq v_0(t)$. By (H_1'), it follows

$$y_0(t) = x_0 t^{q-1} + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} u_0(s)ds$$

and

$$z_0(t) = x_0 t^{q-1} + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_0(s)ds$$

which implies that $y_0(t) = \mathcal{B}u_0(t), z_0(t) = \mathcal{B}v_0(t)$, and u_0, v_0 satisfy (H_1).

By (H_2'), it is easy to verify the condition (H_2) holds.

Therefore, by Theorem **2.1**, we obtain that (2.1) has extremal solutions $u^*, v^* \in C^1(J, \mathbb{R})$ in $[u_0, v_0]$.

Let $y^* = \mathcal{B}u^*$, then

$$y^*(t) = x_0 t^{q-1} + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} u^*(s)ds. \tag{3.5}$$

From (3.5), it follows by a simple computation that

$$\begin{cases} D^q y^*(t) = u^*(t), \\ t^{1-q}y^*(t)|_{t=0} = x_0, \end{cases} \tag{3.6}$$

Since u^* satisfies (2.1) and y^* satisfies (3.6), then y^* is a solution of (1.1). Similarly, we can show that z^* is a solution of (1.1).

It is easy to show that y^*, z^* are extremal solutions of (1.1). This completes the proof. □

4. Example

Consider the problem:

$$\begin{cases} (D^q x(t))' = \frac{1}{5}t^2(t-x(t))^3 - 2t^2 D^q x(t), \\ D^q x(0) = 0,\ t^{1-q}x(t)|_{t=0} = 0, \end{cases} \tag{4.1}$$

where $t \in J = [0,1], D^q$ is the standard Riemann-Liouville fractional derivative and $q = \dfrac{3}{4}$.

Take $y_0(t) = 0, z_0(t) = t^2$. It is not difficult to verify that u_0, v_0 are lower and upper solutions of (4.1), respectively. Thus, condition (H_1') holds.

Let

$$f(t,x,y) = \frac{1}{5}t^2(t-x)^3 - 2t^2 y,$$

then

$$f(t,x,y) - f(t,\overline{x},\overline{y}) \geq -\frac{3}{5}t^2(x-\overline{x}) - 2t^2(y-\overline{y}),$$

where $y_0(t) \leq \overline{x} \leq x < z_0(t),\ D^q y_0(t) \leq \overline{y} \leq y \leq D^q z_0(t), \forall t \in J$.

For $M(t) = \frac{3}{5}t^2, N(t) = 2t^2$, we see that condition (2.3) holds. Thus, all conditions of Theorem **3.1** are satisfied.

Therefore, by Theorem **3.1**, there exist iterative sequences $\{y_n\}, \{z_n\}$ which converge uniformly to the extremal solutions y^*, z^* of (4.1), respectively.

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[1] *School of Mathematics and Computer Science*
Shanxi Normal University
Linfen, Shanxi – 041004, People's Republic of CHINA
e-mail: wgt2512@163.com

[2] *Cankaya University, Department of Mathematics*
Faculty of Art and Sciences, Balgat
Ankara – 06530, TURKEY
e-mail: dumitru@cankaya.edu.tr *Received: December 1, 2011*
and: Institute of Space Sciences, Magurele – Bucharest, ROMANIA

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