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Analysis of time-fractional Hunter-Saxton equation: a model of neumatic liquid crystal

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Abstract: In this work, a theoretical study of diffusion of neumatic liquid crystals was done using the concept of fractional order derivative. This version of fractional derivative is very easy to handle and obey to almost all the properties satisfied by the conventional Newtonian concept of derivative. The mathematical equation underpinning this physical phenomenon was solved analytically via the so-called homotopy decomposition method. In order to show the accuracy of this iteration method, we constructed a Hilbert space in which we proved its stability for the time-fractional Hunder-Saxton equation.

Keywords: neumatic liquid crystal; fractional derivative; stability analysis; special solution

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1 Introduction

Liquid crystals are material in a state that has properties between those of conventional and those of solid crystal. Examples of liquid crystals can be found both in the natural world and in technological applications. Furthermost contemporaneous electronic displays use liquid crystals. For instance, many proteins and cell membranes are liquid crystals. Other distinguished examples of liquid crystals are solutions of soap and various related detergents, as well as the tobacco mosaic virus. It is perhaps to recall that, the history of liquid crystals can be traced back to the work done in 1888 by Austrian botanical physiologist Friedrich Reinitzer, who examined the physico-chemical properties of various derivatives of cholesterol which nowadays belongs to the class of materials known as cholesteric liquid crystals [1–4]. However, one of the most common LC phases is the neumatic. Since every natural occurring can be measured or at least be converted into mathematical equation allowing us to predict the future. Therefore in mathematical community, the Hunter-Saxton equation is an integrable partial differential equation that arises in the theoretical study of neumatic liquid crystals. If the molecules in the liquid crystal are initially all aligned, and some of them are waggled slightly, this interruption in orientation will propagate through the crystal, and the Hunder-Saxton equation describes certain of such orientation waves. The equation under consideration here is given as

$$u_{tx} + u_x^2 + uu_x = \frac{1}{2}u_x^2. \tag{1}$$

2 Time-fractional Hunder-Saxton equation

However, when searching in the literature, we find out that, the orientation of research in modeling real world problems is shifting toward the use of fractional order derivative. Due to the fact that in many cases, the results have been in agreement with the observed facts. Therefore due to the great rewards of this concept, we shall in this work consider reverting the conventional Hunter-Saxton equation to the scope of fractional order derivative, this will be achieved by replacing the derivative respect to time to the fractional order derivative. The reason of using the fractional derivative in modelling real problems has been documented in [5]. Our new equation will throughout the test be

$${}_0^A D_t^\alpha[u_{tx}] + u_x^2 + uu_x = \frac{1}{2}u_x^2 \quad 0 < \alpha \le 1. \tag{2}$$

We inform that the fractional derivative chosen here is chosen in the way that any time alpha is integer we recover the conventional Hunder-Saxton equation. Also we

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chose this version that can satisfy the maximum properties of the conventional derivatives. The fractional derivative used here is referred to the Atangana's fractional derivative defined as

$$ {}_0^A D_t^{\alpha}[f(t)] = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon \left(t + \frac{1}{\Gamma(\alpha)}\right)^{1-\alpha}\right) - f(t)}{\varepsilon}. \quad (3) $$

The above definition has many interesting properties see for example:

Theorem 1. *Assuming that $f : [a, \infty) \to R$, be a functions such that, f is differentiable and also α–differentiable. Let g be a function defined in the range of f and also differentiable, then we have the following rule [6]*

$$ {}_0^A D_x^{\beta}(gof(x)) = f(x) {}_0^A D_x^{\beta}(g(f(x))). \quad (4) $$

Theorem 2. *[6] Assuming that, $g \neq 0$ and f are two functions β–differentiable with $\beta \in (0, 1]$ then, the following relations can be satisfied*

1. ${}_0^A D_x^{\alpha}\left(af(x) + bg(x)\right) = a {}_0^A D_x^{\alpha} f(x) + b {}_0^A D_x^{\alpha} g(x)$ *for all a and b real number.*
2. ${}_0^A D_x^{\alpha}(c) = 0$ *for c any given constant.*
3. ${}_0^A D_x^{\alpha}\left(f(x).g(x)\right) = g(x) {}_0^A D_x^{\alpha} f(x) + f(x) {}_0^A D_x^{\alpha} g(x)$
4. ${}_0^A D_x^{\alpha}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) {}_0^A D_x^{\alpha} f(x) - f(x) {}_0^A D_x^{\alpha} g(x)}{g^2(x)}$.

3 Analytical solution

One of the difficult parts in the field of non-linear equation is to provide their solutions. The proof of existence of a solution does not show us the solution, but the derivation of an exact solution shows both the existence and the solution. More importantly, we shall mention that in the field of modelling real world problems, the proof of existence of solution is very useless because we need the solution of this problem to predict the future. Because of the usefulness of these solutions for real world problems, several scholars have devoted their attention to proposed methods to solve approximately or exactly these problems. We can name some of them for instance the homotopy perturbation method by He [7, 8]. The variational iteration method by Cheng [9], the Laplace perturbation method by Khan [10], the Adomian decomposition method [11] and the homotopy decomposition method [12, 13] and many others [14–16]. We shall make use of the recently proposed one that used the double Laplace transform and the Lagrange Multiplier. According to the methodology of this version, we shall first apply the inverse operator of beta -fractional derivative on both sides of Equation (2) to obtain

$$ u_x(x,t) - u_x(x,0) = {}_0^A I_t^{\alpha}\left[\frac{1}{2}u_x^2 - u_x^2 - uu_x\right], \quad (5) $$

where

$$ {}_0^A I_t^{\alpha}[f(t)] = \int_0^t \left(v + \frac{1}{\Gamma(\alpha)}\right)^{\alpha-1} f(v)dv \quad (6) $$

and is referred as beta-fractional integral. Again applying the integral in respect to x in both sides of Equation (5) we obtained

$$ u(x,t) = A(x,t) + \int_0^x \left({}_0^A I_t^{\alpha}\left[-\frac{1}{2}u_v^2 - uu_v\right]\right) dv. \quad (7) $$

The next step is to assume that, the solution of our equation can be obtained in series form as follows

$$ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \quad (8) $$

Therefore replacing the above expression in (7) and making use of the homotopy idea which allows us to include an embedding parameter says for example $p \in (01]$. Then after evaluating terms of same power of the embedding parameter, we obtain the following recurrence formulas

$$ u_0(x,t) = A(x,t) \quad (9) $$

$$ u_1(x,t) = \int_0^x \left({}_0^A I_t^{\alpha}\left[\frac{1}{2}\left(u_v^2\right)_0 - \left(u_v^2\right)_0 - u_0\left(u_v\right)_0\right]\right) dv, $$

$$ u_n(x,t) = \int_0^x \left({}_0^A I_t^{\alpha}\left[-\frac{1}{2}H_n - B_n\right]\right) dv, $$

with

$$ H_n = \sum_{j=0}^{n-1} \frac{1}{2}(u_v)_j (u_v)_{n-j-1}, \quad B_n = \sum_{j=0}^{n-1} (u_v)_j (u)_{n-j-1}. \quad (10) $$

Note that as soon as we provide the first term, which is the combination of the initial conditions, we can by using the following algorithm obtain the remaining terms.

Algorithm 3. – *Input: $I(x,t) = A(x,t)$ as early boarder,*
– *k–number terms in the series calculation*
– *Output: $u_{part}(x,t)$, the special solution*
Step 1 : put $u_0(x,t) = I(x,y,t)$ and $u_{part}(x,t) = u_0(x,t)$,
Step 2 : for $k = 0$ to $n-1$ do step 3, step 4 and step 5
Step 3 : compute accordingly

$$ H_n = \sum_{j=0}^{n-1} \frac{1}{2}(u_v)_j (u_v)_{n-j-1} $$

$$ B_n = \sum_{j=0}^{n-1} (u_v)_j (u)_{n-j-1}, $$

$$b_n = u_n(x,t) = \int_0^x \left({}_0^A I_t^\alpha \left[-\frac{1}{2} H_n - B_n \right] \right) dv, \quad n \geq 1,$$

Step 4 : Compute

$$u_{n+1}(x,t) = b_n + u_n,$$

Step 5 : Compute

$$u_{part}(x,t) = u_{part}(x,t) + u_{n+1}(x,t),$$

Stop.

We shall use the above algorithm to derive the solution of Equation (2).

4 Stability and convergence analysis

In this section we shall present first the stability analysis of the iteration method used to derive the special solution of Equation (2). To achieved this we propose the following theorem

Theorem 4. *Taking into account the initial conditions associate to the fractional Hunder-Saxton equation, the homotopy decomposition method provide a special solution of Equation* (2).

Proof. Let us consider the following operator

$$M(u) = {}_0^A D_t^\alpha [u_{tx}] = -\frac{1}{2} u_x^2 - u u_x. \tag{11}$$

Furthermore let us consider the following the Hilbert space $H = L^2((\eta, \lambda) \times [O, T])$, defined as

$$H = \left\{ (u, v) : (\eta, \lambda) \times [O, T] \text{ with, } \int uv d\imath dk < \infty \right\}.$$

Then by definition of operator (11) we have the following

$$\begin{aligned} M(u) - M(v) &= -\frac{1}{2} u_x^2 - u u_x + \frac{1}{2} v_x^2 + v v_x \qquad (12) \\ &= -\frac{1}{2} \left[\left(u_x^2 - v_x^2 \right) + \left(\left(u^2 \right)_x - \left(v^2 \right)_x \right) \right]. \end{aligned}$$

Let us consider the inner product and the following properties in fact; the explanation on Hilbert space accommodates plentiful designs of the inner product spaces where the metric constructed by the inner product precedes a comprehensive metric space. Inner product spaces have an unconsciously defined norm based on the inner product of the space itself that does not the parallelogram equality [17, 18]:

$$\|x\| = \sqrt{(x,x)}. \tag{13}$$

It is a well defined by the no negativity axiom of the definition of the inner product space. The following properties can be observed

$$|(x,y)| \leq \|x\| \, \|y\|. \tag{14}$$

The above is the well-known Cauchy-Schwarz inequality. Also the following can be obtained

$$\|a.x\| = |a| \,.\, \|x\|. \tag{15}$$

The above is called homogeneity. The last interesting one for this paper will be given as:

$$\|x+y\| \leq \|x\| + \|y\|. \tag{16}$$

The above is called triangular inequality. Therefore with the above properties in mind, we shall evaluate

$$\begin{aligned} &(M(u) - M(v), u - v) = \\ &\left(\frac{1}{2} \left[\left(v_x^2 - u_x^2 \right) + \left(\left(v^2 \right)_x - \left(u^2 \right)_x \right) \right], u - v \right). \end{aligned} \tag{17}$$

Making use of the inner product properties we have

$$\begin{aligned} &(M(u) - M(v), u - v) = \\ &-\frac{1}{2} \left[\left(u_x^2 - v_x^2, u - v \right) + \left(\left(u^2 \right)_x - \left(v^2 \right)_x, u - v \right) \right]. \end{aligned} \tag{18}$$

Making use of the Cauchy-Schwartz inequality and also the above properties we obtain the following inequalities under the condition that $(u,u), (v,v) \leq N^2$

$$\left(\left(u^2 \right)_x - \left(v^2 \right)_x, u - v \right) \leq 2N^2 \delta_1 \|u - v\|^2. \tag{19}$$

If in addition we have the following condition $(u_x, u_x), (v_x, v_x) \leq D^2$ then we can obtain the following inequality

$$\left(\left(u^2 \right)_x - \left(v^2 \right)_x, u - v \right) \leq 2D^2 \delta_2 \|u - v\|^2. \tag{20}$$

Therefore replacing Equation (20) and (19) into Equation (17) we obtain

$$(M(u) - M(v), u - v) \geq \left(N^2 \delta_1 + D^2 \delta_2 \right) \|u - v\|^2. \tag{21}$$

Then, for all $u, v \in H$ we can find a positive constant g such that

$$(M(u) - M(v), u - v) \geq g \|u - v\|^2, g = \left(N^2 \delta_1 + D^2 \delta_2 \right). \tag{22}$$

We shall follow our proof by evaluating

$$\begin{aligned} &(M(u) - M(v), w) = \\ &-\frac{1}{2} \left[\left(u_x^2 - v_x^2, w \right) + \left(\left(u^2 \right)_x - \left(v^2 \right)_x, w \right) \right]. \end{aligned} \tag{23}$$

However, using derivation as earlier, we have the following inequality

$$\left(M(u) - M(v), w\right) \geq \left(N^2\delta_1 + D^2\delta_2\right) \|u - v\| \, \|w\| . \quad (24)$$

Thus, for all $u, v, w \in H$, we can find a positive constant g such that

$$\left(M(u) - M(v), w\right) \geq g \|u - v\| \, \|w\| . \quad (25)$$

With the Equation (25) and (22) satisfied, the proof of stability is completed. □

5 Special solution

In this section, we shall make use of Algorithm 3 to derive the special solution of the time-fractional Hunder-Saxton equation. If we choose for instance the initial condition to be

$$A(x, t) = x. \quad (26)$$

Then we shall have

$$u_0(x, t) = x$$

$$u_1(x, t) = \frac{x^2\left(-\left(t + \frac{1}{\Gamma[\alpha]}\right)^{\alpha} + \left(\frac{1}{\Gamma[\alpha]}\right)^{\alpha}\right)}{2\alpha}$$

$$u_2(x, t) = \frac{x^3\left(\frac{1}{\Gamma[\alpha]}\right)^{2\alpha}\left(-1 + (1 + t\Gamma[\alpha])^{\alpha}\right)^2}{4\alpha^2}$$

$$u_3(x, t) = -\frac{x^4\left(\frac{1}{\Gamma[\alpha]}\right)^{3\alpha}\left(-1 + (1 + t\Gamma[\alpha])^{\alpha}\right)^3}{8\alpha^3}$$

$$u_4(x, t) = \frac{x^5\left(\frac{1}{\Gamma[\alpha]}\right)^{4\alpha}\left(-1 + (1 + t\Gamma[\alpha])^{\alpha}\right)^4}{16\alpha^4}$$

$$u_5(x, t) = -\frac{x^6\left(\frac{1}{\Gamma[\alpha]}\right)^{5\alpha}\left(-1 + (1 + t\Gamma[\alpha])^{\alpha}\right)^5}{32\alpha^5}$$

$$u_6(x, t) = \frac{x^7\left(\frac{1}{\Gamma[\alpha]}\right)^{6\alpha}\left(-1 + (1 + t\Gamma[\alpha])^{\alpha}\right)^6}{64\alpha^6}.$$

One can notice that, the terms of the series solution follow a certain pattern; therefore we can conclude that for any $n \geq 1$ the nth term of the solution is given as

$$u_n(x, t) = (-1)^n \frac{x^{n+1}\left(\frac{1}{\Gamma[\alpha]}\right)^{n\alpha}\left(-1 + (1 + t\Gamma[\alpha])^{\alpha}\right)^n}{(2\alpha)^n}. \quad (27)$$

So that the solution of the time-fractional Hunder-Saxton equation with the chosen initial conditions can be expressed as

$$u(x, t) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}\left(\frac{1}{\Gamma[\alpha]}\right)^{n\alpha}\left(-1 + (1 + t\Gamma[\alpha])^{\alpha}\right)^n}{(2\alpha)^n}. \quad (28)$$

The special solution or the approximate solution is given in terms of certain m as

$$u(x, t, m) = \sum_{n=0}^{m}(-1)^n \frac{x^{n+1}\left(\frac{1}{\Gamma[\alpha]}\right)^{n\alpha}\left(-1 + (1 + t\Gamma[\alpha])^{\alpha}\right)^n}{(2\alpha)^n}.$$

6 Results and discussion

In the recent decades several physical problems that have been represented in mathematical formula with fractional derivatives have offered great results. For example in the field of groundwater flow and groundwater pollution, the results offered by the fractional derivatives were in good agreement of experimental data. Now let ${}_0^A D_t^{\alpha}[f(t)] = v(t)$ as a physical interpretation of the fractional derivative we have that when $0 < \beta < 1$, $v(t)$ is represented with an additional function $g_t(\beta) = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}$ describing the inhomogeneous time scale, which depends not only β, but also on the parameter t representing all the individual times measured of the moving object. This is in agreement with the current views in physic. In this point of view, $v(t)$ is almost the real velocity of the moving object at the real distance $f(t)$ whereas, when $\beta = 1$, $v(t)$ is the average velocity describing the ratio between the approximate distance travelled and the local time t which is considered as the flowing equably time. Therefore, the result obtained via the fractional derivative, represented the real orientation of the waves.

7 Conclusion

An integrable partial differential equation that arises in the theoretical study of neumatic liquid crystals was further extended to the concept of fractional derivative. The fractional derivative used here is referred to beta-fractional derivative. This fractional derivative has interesting properties that cannot be offered by the well-known fractional derivatives. We make use of the so-called homotopy decomposition method (HDM) to derive a special solution of the time-fractional Hunder-Saxton equation. The show the efficiency of the used method, we used the inner product present the .

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