

Research Article

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On soliton solutions of the Wu-Zhang system

DOI 10.1515/phys-2016-0004

Received November 04, 2015; accepted December 09, 2015

Abstract: In this paper, the extended tanh and Hirota methods are used to construct soliton solutions for the Wu-Zhang (WZ) system. Singular solitary wave, periodic and multi soliton solutions of the WZ system are obtained.

Keywords: extended tanh method; Hirota method; Wu-Zhang system; singular solitary wave solution; multi soliton solution

PACS: 02.30.Jr; 05.45.Yv; 02.70.Wz

1 Introduction

The mathematical modeling of scientific systems is generally expressed by nonlinear evolution equations. Therefore, it is crucial to find general solutions of these corresponding nonlinear equations. The general solutions of these equations can provide valuable information about the character and the physical structure of the equations. These are very important for physicians and engineers. Solitons are of general physical interest since the soliton approach is universal in different fields of modern non-linear science. Solitons are defined as localized waves

that propagate without change of their shape and velocity properties. they are also stable against mutual collisions [1]. There are various solutions to solitons that are obtained by incorporating techniques of integrability. Most of these methods use the wave variable transformation to reduce the nonlinear partial differential equation (NPDE) to an ordinary differential equation (ODE) in order to acquire the soliton solution. Some of these methods are G'/G -expansion [2], Jacobi elliptic function [3], inverse scattering [4], Hirota bilinear [5], exp-function [6], and the first integral [7] methods. There are also effective studies for acquiring traveling wave solutions of NPDE's [8–17]. It is very important to determine the propagation behaviour of traveling waves and the wave's stability in the presence of perturbations to comprehend the complex dynamics underlying coupled NLPDEs. In recent years, one of these coupled NLPDEs the WZ system has been studied by many researchers. These studies include the first integral method [18] and the ansatz method [19]. WZ systems with time dependent coefficients has been studied by Baleanu *et al.* [20]. We have used the WZ system as

$$p_t + pp_x + q_x = 0, \quad (1)$$

$$q_t + (pq)_x + \frac{1}{3}p_{xxx} = 0, \quad (2)$$

where q is the elevation of the water and p is the surface velocity of water. The system of Equation (1) and Equation (2) is indeed completely integrable because they have Lax pairs [21].

For simplicity, rearrange the WZ system, by using a Miura type transformation, in single form. Take $p = w_x$ and hence $q = -w_t - \frac{1}{2}w_x^2$. Therefore the single form of the WZ system is

$$w_{tt} + 2w_x w_{xt} + w_t w_{xx} + \frac{3}{2}w_x^2 w_{xx} - \frac{1}{3}w_{xxxx} = 0. \quad (3)$$

The organization of the manuscript is given as follow:

In the first section, we have presented the introduction. In second section, we provide mathematical formulation of the generalized extended tanh method and acquire some new optical soliton solutions of the WZ system. In the third section, we apply the Hirota method to acquire multi soliton solutions of the WZ system. In the last section, we present some conclusions about obtained solutions.

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2 The generalized extended tanh method

Sirendaoreji [22] presented an algebraic method to determine different forms of solitary wave solutions of PDE's. Then Xueqin [23] extended Sirendaoreji's method and obtained different explicit and exact solutions from the Sirendaoreji's method with the aid of an auxiliary ordinary equation:

$$\left(\frac{\partial \varphi}{\partial \xi}\right)^2 = A\varphi^2 + B\varphi^3 + C\varphi^4, \tag{4}$$

where A, B and C are constants. It has the following solutions,

$$\varphi_1(\xi) = \frac{2A \operatorname{sech}[\sqrt{A}\xi]}{\sqrt{\Delta} - B \operatorname{sech}[\sqrt{A}\xi]}, \tag{5}$$

$$\Delta = B^2 - 4AC > 0, A > 0.$$

$$\varphi_2(\xi) = -\frac{2A \operatorname{csc} h[\sqrt{A}\xi]}{\sqrt{-\Delta} + B \operatorname{sech}[\sqrt{A}\xi]}, \tag{6}$$

$$\Delta < 0, A > 0.$$

$$\varphi_3(\xi) = -\frac{2A \operatorname{sech}[\sqrt{A}\xi]}{B \operatorname{sech}[\sqrt{A}\xi] + \sqrt{4A^2 - \Delta} \tanh[\sqrt{A}\xi] - 2A}, \tag{7}$$

$$\Delta - 4A^2 < 0, A > 0.$$

$$\varphi_4(\xi) = -\frac{2A \operatorname{csch}[\sqrt{A}\xi]}{B \operatorname{csch}[\sqrt{A}\xi] - \sqrt{\Delta - 4A^2} \coth[\sqrt{A}\xi] + 2A}, \tag{8}$$

$$\Delta - 4A^2 < 0, A > 0.$$

Here, our aim is to transform the WZ system to the form of Equation (3) and obtain the different solutions of the WZ system from the obtained ones in literature with the solutions (5)-(8). Equation (3) transforms to the following ODE by using the wave variable transformation $\xi = k(x + \lambda t)$:

$$\lambda^2 w_{\xi\xi} + 3k\lambda w_{\xi} w_{\xi\xi} + \frac{3}{2}k^2 w_{\xi}^2 w_{\xi\xi} - \frac{1}{3}k^2 w_{\xi\xi\xi\xi} = 0. \tag{9}$$

By integrating Equation (9) and using the transformation $W = w_{\xi}$ we have:

$$\lambda^2 W + \frac{3}{2}k\lambda W^2 + \frac{1}{2}k^2 W^3 - \frac{1}{3}k^2 W_{\xi\xi} = 0. \tag{10}$$

If we subtract Equation (10) with the respect to W_{ξ} and then integrate once again, it reduces to the desired form of Equation (4)

$$(W_{\xi})^2 = \frac{3\lambda^2}{k^2} W^2 + \frac{3\lambda}{k} W^3 + \frac{3}{8} W^4. \tag{11}$$

With the aid of the solutions of (5)-(8), the following solutions of Equation (11) arise:

$$W_1 = \frac{\frac{6\lambda^2}{k^2} \operatorname{sech}[\sqrt{3}|\frac{\lambda}{k}|\xi]}{\frac{3\sqrt{2}}{2}|\frac{\lambda}{k}| - \frac{3\lambda}{k} \operatorname{sech}[\sqrt{3}|\frac{\lambda}{k}|\xi]}, \lambda^2 > 0, \frac{\lambda^2}{k^2} > 0. \tag{12}$$

$$W_2 = \frac{-\frac{6\lambda^2}{k^2} \operatorname{csch}[\sqrt{3}|\frac{\lambda}{k}|\xi]}{\frac{3\sqrt{2}}{2}|\frac{-\lambda}{k}| + \frac{3\lambda}{k} \operatorname{sech}[\sqrt{3}|\frac{\lambda}{k}|\xi]}, \lambda^2 < 0, \frac{\lambda^2}{k^2} > 0. \tag{13}$$

$$W_3 = \frac{-\frac{6\lambda^2}{k^2} \operatorname{sech}[\sqrt{3}|\frac{\lambda}{k}|\xi]}{\frac{3\lambda}{k} \operatorname{sech}[\sqrt{3}|\frac{\lambda}{k}|\xi] + |\frac{\lambda}{k}| \sqrt{\frac{15}{2}} \tanh[\sqrt{3}|\frac{\lambda}{k}|\xi] - \frac{6\lambda^2}{k^2}}, \tag{14}$$

$$-\frac{15\lambda^2}{2k^2} < 0, \frac{\lambda^2}{k^2} > 0.$$

$$W_4 = \frac{-\frac{6\lambda^2}{k^2} \operatorname{csch}[\sqrt{3}|\frac{\lambda}{k}|\xi]}{\frac{3\lambda}{k} \operatorname{csch}[\sqrt{3}|\frac{\lambda}{k}|\xi] - i|\frac{\lambda}{k}| \sqrt{\frac{15}{2}} \coth[\sqrt{3}|\frac{\lambda}{k}|\xi] + \frac{6\lambda^2}{k^2}}, \tag{15}$$

$$-\frac{15\lambda^2}{2k^2} < 0, \frac{\lambda^2}{k^2} > 0.$$

Several transformations were used to reach the solutions (12)-(15). For solution (12), we use the transformation $W = w_{\xi}$, and thus we have:

$$w = -4\sqrt{\frac{2}{3}} \operatorname{arctanh} \left[\frac{2\lambda + \sqrt{2}k|\frac{\lambda}{k}|}{\sqrt{2}k|\frac{\lambda}{k}|} \tanh \left[\frac{\sqrt{3}}{2} \left| \frac{\lambda}{k} \right| \xi \right] \right]. \tag{16}$$

Then from the transformations $p = w_x, q = -w_t - \frac{1}{2}w_x^2$ and $\xi = k(x + \lambda t)$ respectively, the solutions of the WZ system are as follows

$$p = \frac{2\lambda(2\lambda + \sqrt{2}k|\frac{\lambda}{k}|) \operatorname{sech}^2 \left[\frac{\sqrt{3}}{2} \left| \frac{\lambda}{k} \right| k(x + \lambda t) \right]}{-\lambda + (3\lambda + 2\sqrt{2}k|\frac{\lambda}{k}|) \tanh^2 \left[\frac{\sqrt{3}}{2} \left| \frac{\lambda}{k} \right| k(x + \lambda t) \right]}, \tag{17}$$

$$q = -\frac{2\lambda^3(4\lambda + 3\sqrt{2}k|\frac{\lambda}{k}|) \cosh \left[\frac{\sqrt{3}}{2} \left| \frac{\lambda}{k} \right| k(x + \lambda t) \right]}{\left([\lambda - (3\lambda + 2\sqrt{2}k|\frac{\lambda}{k}|)] \tanh^2 \left[\frac{\sqrt{3}}{2} \left| \frac{\lambda}{k} \right| k(x + \lambda t) \right] \right)^2} \tag{18}$$

$$\operatorname{sech}^4 \left[\frac{\sqrt{3}}{2} \left| \frac{\lambda}{k} \right| k(x + \lambda t) \right].$$

In the same manner, similar calculations can be done for the other solutions (13), (14) and (15).

3 Hirota method

In this section, we illustrate the Hirota method [24] to acquire another type of solutions-multi soliton solutions of the WZ system. Firstly, we have to apply the following Miura type transformation to obtain the WZ system's reduced form's operator form

$$w(x, t) = 2 \frac{\partial \ln f(x, t)}{\partial x} = 2 \frac{f_x}{f}. \tag{19}$$

When the transformation (19) is applied to the WZ system, the following partial derivatives of the reduced form of the WZ system are generated.

$$\begin{aligned} w_x &= 2 \left(\frac{f_{xx}f - f_x^2}{f^2} \right), \quad w_t = 2 \left(\frac{f_{xt}f - f_x f_t}{f^2} \right), \\ w_{tt} &= 2 \left(\frac{f_{xtt}}{f} - 2 \frac{f_{xt}f_t}{f^2} - \frac{f_x f_{tt}}{f^2} + 2 \frac{f_x f_t^2}{f^3} \right), \\ w_{xt} &= 2 \left(\frac{f_{xxt}f - f_{xx}f_t}{f^2} - 2 \frac{f_x f_{xt}}{f^2} + 2 \frac{f_x^2 f_t}{f^3} \right), \\ w_{xx} &= 2 \left(\frac{f_{xxx}}{f} - 3 \frac{f_x f_{xx}}{f^2} + 2 \frac{f_x^3}{f^3} \right), \\ w_{xxx} &= 2 \left(\frac{f_{5x}}{f} - 5 \frac{f_y f_{4x}}{f^2} - 10 \frac{f_{2x} f_{3x}}{f^2} + 20 \frac{f_x^2 f_{3x}}{f^3} \right. \\ &\quad \left. + 30 \frac{f_x f_{2x}^2}{f^3} - 60 \frac{f_x^3 f_{2x}}{f^4} + 24 \frac{f_x^5}{f^5} \right). \end{aligned} \tag{20}$$

If we substitute the expressions in (20) into the Equation (3), we obtain the bilinear form of the WZ system in the operator form.

D is the characteristic linear operator of the Hirota method and $f(x, t)$ is the auxiliary function. The Hirota bilinear form of Equation (3) is as follows:

$$P(D)\{f \cdot f\} = \left(D_x D_t^2 + \frac{1}{3} D_x^5 \right) \{f \cdot f\} = 0. \tag{21}$$

For the Hirota method, the auxiliary function $f(x, t)$ in (21) has a perturbation expansion. Its perturbation expansion is given by:

$$f(x, t) = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x, t), \tag{22}$$

such that ε is a small parameter.

To reach the multi soliton solution or N -soliton solution we have,

$$f_1 = \sum_{i=1}^N \exp(\Phi_i), \tag{23}$$

such that

$$\Phi_i = k_i x + l_i t. \tag{24}$$

We find the dispersion as

$$l_{i1} = \pm \frac{1}{\sqrt{3}} i k_j^2, \tag{25}$$

and substituting the dispersion relation term (25) into (24) yields:

$$\Phi_i = k_i x \pm \frac{1}{\sqrt{3}} i k_j^2 t, \tag{26}$$

where

$$f_1 = \exp(\Phi_1) = \exp(k_1 x + \frac{1}{\sqrt{3}} i k_1^2 t). \tag{27}$$

Therefore, we have the following auxiliary function $f(x, t)$ for $N = 1$ in (23)

$$f = 1 + \exp(\Phi_1) = 1 + \exp(k_1 x + \frac{1}{\sqrt{3}} i k_1^2 t). \tag{28}$$

Consequently, 1-soliton solution of the WZ system is obtained as

$$w(x, t) = \frac{2k_1 e^{k_1 x + \frac{1}{\sqrt{3}} i k_1^2 t}}{1 + e^{k_1 x + \frac{1}{\sqrt{3}} i k_1^2 t}}. \tag{29}$$

So, from the transformations $p = w_x$, $q = -w_t - \frac{1}{2} w_x^2$, the 1-soliton solutions of the WZ system are:

$$p = \frac{2k_1^2 e^{k_1 x + \frac{1}{\sqrt{3}} i k_1^2 t}}{\left(1 + e^{k_1 x + \frac{1}{\sqrt{3}} i k_1^2 t}\right)^2}, \tag{30}$$

$$q = \frac{k_1^4 e^{2k_1 x + \frac{2}{\sqrt{3}} i k_1^2 t}}{\left(1 + e^{k_1 x + \frac{1}{\sqrt{3}} i k_1^2 t}\right)^4}. \tag{31}$$

If we take $N = 2$ in Equation (22) for the 2- soliton solution of the WZ system, then we have the following expression:

$$f_1 = \exp(\Phi_1) + \exp(\Phi_2), \tag{32}$$

and it is achieved such that

$$f_2 = \sum_{1 \leq i < j \leq N} A(i, j) \exp(\Phi_i + \Phi_j), \tag{33}$$

where the generalized phase shift $A(i, j)$ is defined by

$$A(i, j) = -\frac{P(k_i - k_j)^2}{P(k_i + k_j)^2}, \quad 1 \leq i < j \leq 2. \tag{34}$$

Thus, from the Equations (32)-(34) it is acquired as

$$\begin{aligned} f &= 1 + e^{k_1 x + \frac{1}{\sqrt{3}} i k_1^2 t} + e^{k_2 x + \frac{1}{\sqrt{3}} i k_2^2 t} - \\ &\frac{(k_1 - k_2)(l_1 - l_2)^2 + \frac{1}{3}(k_1 - k_2)^5}{(k_1 + k_2)(l_1 + l_2)^2 + \frac{1}{3}(k_1 + k_2)^5} \\ &e^{(k_1 + k_2)x + \frac{1}{\sqrt{3}} i(k_1^2 + k_2^2)t}, \end{aligned} \tag{35}$$

$$\frac{(k_1 - k_3)(l_1 - l_3)^2 + \frac{1}{3}(k_1 - k_3)^5}{(k_1 + k_3)(l_1 + l_3)^2 + \frac{1}{3}(k_1 + k_3)^5}$$

$$e^{(k_1+k_3)x + \frac{1}{\sqrt{3}}i(k_1^2+k_3^2)t}$$

$$\frac{(k_2 - k_3)(l_2 - l_3)^2 + \frac{1}{3}(k_2 - k_3)^5}{(k_2 + k_3)(l_2 + l_3)^2 + \frac{1}{3}(k_2 + k_3)^5}$$

$$e^{(k_2+k_3)x + \frac{1}{\sqrt{3}}i(k_2^2+k_3^2)t}$$

$$\frac{(k_1 - k_2)(l_1 - l_2)^2 + \frac{1}{3}(k_1 - k_2)^5}{(k_1 + k_2)(l_1 + l_2)^2 + \frac{1}{3}(k_1 + k_2)^5}$$

$$\frac{(k_1 - k_3)(l_1 - l_3)^2 + \frac{1}{3}(k_1 - k_3)^5}{(k_1 + k_3)(l_1 + l_3)^2 + \frac{1}{3}(k_1 + k_3)^5}$$

$$\frac{(k_2 - k_3)(l_2 - l_3)^2 + \frac{1}{3}(k_2 - k_3)^5}{(k_2 + k_3)(l_2 + l_3)^2 + \frac{1}{3}(k_2 + k_3)^5}$$

$$e^{(k_1+k_2+k_3)x + \frac{1}{\sqrt{3}}i(k_1^2+k_2^2+k_3^2)t}$$

then in the same manner, from the transformations $p = w_x$, $q = -w_t - \frac{1}{2}w_x^2$, the solutions of the WZ system are obtained.

The higher level soliton solutions, for $N \geq 4$ can be obtained in a parallel manner. This confirms the fact that the WZ system is completely integrable and gives rise to multiple soliton solutions of any order.

4 Conclusion

In this work we studied the WZ system. We used the generalized extended tanh and Hirota methods to construct new exact soliton solutions of the WZ system. Some of the solutions are singular solitary wave and periodic solutions types. Furthermore, multi soliton solutions are obtained as well. Based on this study, it could be said that the generalized extended tanh method has some advantages about the variety of the solution types over the Hirota method. On the other hand Hirota method has advantages about the more general solution (solutions can be obtained dep-

ending on N). These two models are presented together in this work for the first time. These methods can be applied to nonlinear evolution equations. For the next study, we will discuss the WZ system for integral motions with the aid of He's variational principle.

Acknowledgement: This research project was supported by a grant from the "Research Center of the Center for Female Scientific and Medical Colleges", Deanship of Scientific Research, King Saud University. The authors are thankful to Visiting professor program at King Saud University for support.

Competing interests: The authors declare that they have no competing interests.

Authors' contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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