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Bulent Kilic*, Mustafa Inc, and Dumitru Baleanu

On combined optical solitons of the one-dimensional Schrödinger's equation with time dependent coefficients

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Abstract: This paper integrates dispersive optical solitons in special optical metamaterials with a time dependent coefficient. We obtained some optical solitons of the aforementioned equation. It is shown that the examined dependent coefficients are affected by the velocity of the wave. The first integral method (FIM) and ansatz method are applied to reach the optical soliton solutions of the one-dimensional nonlinear Schrödinger's equation (NLSE) with time dependent coefficients.

Keywords: FIM; metamaterial; optical soliton

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1 Introduction

The dynamics of optical solitons propagating through optical fibers for trans-continental and trans-oceanic distances is governed by the nonlinear Schrödinger's equation (NLSE). This NLSE is derived from Maxwell's equation in electromagnetic by the aid of multiple-scale perturbation analysis. The NLSE appears, in the literature of optical solitons, with several forms of nonlinearity that depends on the context where it is studied. The best known mathematical modeling of optical systems generally is expressed by types of NLSE. The details of NLSE are given in the studies on nonlinear optics [1–8].

It is crucial to reach general solutions of these corresponding nonlinear equations. Thus, the general solutions of these equations provide much information about the character and the structure of the equations for researchers. Many effective methods have been improved to provide much information for physicians and engineers. Some of these methods are Tanh [9], G'/G-expansion [10], Jacobi elliptic function [11], functional variable [12], Hirota bilinear [13], exp-function [14], and first integral methods [15]. All of these methods are effective methods for acquiring traveling wave solutions for NPDE.

The FIM initially has been presented to the literature by solving the Burgers-KdV equation by Feng [15]. This method has been successfully implemented to NPDE and some fractional differential equations, which are a new type of equations. In recent years, many studies on this method have been made. Raslan [16] has used this method for the Fisher equation. Tascan and Bekir [17] have used this method for the Cahn-Allen equation. Abbasbandy and Shirzadi [18] have investigated the Benjamin Bona-Mohany equation by this method. Jafari *et al.* [19] and Hosseini *et al.* [20] have researched w.r.t. the Biswas–Milovic equation, the KP equation, and so on [21–24].

For this paper, we present the governing equation for metamaterials in Section 2. The FIM is described and applied in Section 3. In order to construct the combined soliton solutions, an ansatz approach is applied in Section 4. Lastly, we give some conclusions in the last section.

2 Governing equation

Soliton pulse propagation properties in complex materials with simultaneous negative real dielectric permittivity and magnetic permeability, also known as double negative materials, have attracted much attention in recent research. These types of materials are not found in nature, but rather need to be fabricated through material processed engineering. Therefore, these materials are called metamaterials [25]. In recent years, the model equation that describes the propagation of solitons and other waves through these metamaterial waveguides has been studied by many researchers. One of these studies is by Ebadi and co-workers: the tanh function method [26, 27]. We have used the aforementioned equation with additional terms that account for

***Corresponding Author: Bulent Kilic:** Firat University, Science Faculty, Department of Mathematics, 23119 Elazig, Turkey, E-mail: bulentkilic@firat.edu.tr
Mustafa Inc: Firat University, Science Faculty, Department of Mathematics, 23119 Elazig, Turkey, E-mail: minc@firat.edu.tr
Dumitru Baleanu: Department of Mathematics, Çankaya University, Ogretmenler Cad. 14 06530, Balgat, Ankara,Turkey & Institute of Space Sciences, Magurele - Bucharest, Romania, E-mail: dumitru@cankaya.edu.tr

the metamaterials as

$$
\begin{aligned}
&i h_t + a(t) h_{xx} + b(t) |h|^2 h = \\
&\left(i\alpha(t) h_x + i\lambda(t) \left(|h|^2 h \right)_x + i\mu(t) \left(|h|^2 \right)_x h \right. , \\
&\left. + \phi_1 \left(|h|^2 h \right)_{xx} + \phi_2 |h|^2 h_{xx} + \phi_3 h^2 h_{xx}^{\star} \right.
\end{aligned}
\tag{1}
$$

where $a, b, \alpha, \lambda, \mu$ and ϕ_j are the group velocity dispersion, Kerr law nonlinearity, coefficient of intermodal dispersion, coefficient of self-steepening, nonlinear dispersion, and real-valued constants that account for specific metamaterials which were introduced earlier and reported in [26].

3 The first integral method

The principal structures of the FIM are as follows:
Step 1. Taking into account the usual NPDE as:

$$W(h, h_t, h_x, h_{xt}, h_{tt}, h_{xx}, \ldots) = 0 \tag{2}$$

then Equation (2) transforms the ODE as

$$L(H, H', H'', H''', \ldots) = 0 \tag{3}$$

such that $\xi = x \mp ct$ and $H' = \partial H(\xi)/\partial\xi$.
Step 2. The following can be taken in ODE (3):

$$h(x, t) = h(\xi). \tag{4}$$

Step 3. A new independent variable is produced by

$$H(\xi) = h(\xi),\; G(\xi) = \partial H(\xi)/\partial\xi \tag{5}$$

which produces a new system of ODEs:

$$
\begin{aligned}
\partial H(\xi)/\partial\xi &= G(\xi), \\
\partial F(\xi)/\partial\xi &= P(H(\xi), G(\xi)).
\end{aligned}
\tag{6}
$$

Step 4. In accordance with the qualitative theory of ODEs [28], if it is possible to find the integrals for system (6), the solutions of system (6) can be obtained immediately. On account of the particular independent plane system, there does not exist any approximation that can guide how to reach its first integrals. The Division Theorem (DT) [29] presented us an idea how to reach the first integrals.

3.1 Application

Equation (1) turns into the following ODEs by using the wave variable $h = H(\xi) e^{i[-\kappa x + wt]}$, where $\xi = \beta(x - vt)$. The real and imaginary parts yield the following pair of relations

$$
\begin{aligned}
&(\beta v + 2a\beta\kappa + \alpha\beta) H_\xi + (3\lambda\beta + 2\beta\mu - 2\beta\kappa(3\phi_1 \\
&\quad + \phi_2 - \phi_3)) H^2 H_\xi = 0,
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&a\beta^2 H_{\xi\xi} - \left(w + a\kappa^2 + \alpha\kappa \right) H + \\
&\left(b - \lambda\kappa + \kappa^2 (\phi_1 + \phi_2 + \phi_3) \right) H^3 - \\
&-\beta^2 (3\phi_1 + \phi_2 + \phi_3) H^2 H_{\xi\xi} - 6\beta^2 \phi_1 H H_\xi^2 = 0.
\end{aligned}
\tag{8}
$$

If we differentiate (7) once by ξ, we get

$$
\begin{aligned}
H_{\xi\xi} = -&\left(\frac{(3\lambda\beta + 2\beta\mu - 2\beta\kappa(3\phi_1 + \phi_2 - \phi_3))}{(\beta v + 2a\beta\kappa + \alpha\beta)} \right), \\
&\left(H^2 H_{\xi\xi} + 2 H H_\xi^2 \right).
\end{aligned}
\tag{9}
$$

Then by equating the right side of (9) to $-\beta^2 (3\phi_1 + \phi_2 + \phi_3) H^2 H_{\xi\xi} - 6\beta^2 \phi_1 H H_\xi^2$ in (8), we have the following constraint:

$$\phi_2 = -\phi_3,\; \beta\phi_1 \neq 0,\; 3\lambda - 6\kappa\phi_1 + 4\kappa\phi_3 + 2\mu \neq 0, \tag{10}$$

$$v = \frac{3\lambda - 3\alpha\beta^2\phi_1 - 6\kappa\phi_1 - 6a\beta^2\kappa\phi_1 + 4\kappa\phi_3 + 2\mu}{3\beta^2\phi_1}. \tag{11}$$

In (10) and (11), $H_{\xi\xi}$ can be replaced in (8) instead of $-\beta^2 (3\phi_1 + \phi_2 + \phi_3) H^2 H_{\xi\xi} - 6\beta^2 \phi_1 H H_\xi^2$. Then we have

$$
\begin{aligned}
&\left(1 + \alpha\beta^2 \right) H_{\xi\xi} - \left(w + a\kappa^2 + \alpha\kappa \right) H \\
&+ \left(b - \lambda\kappa + \kappa^2 (\phi_1 + \phi_2 + \phi_3) \right) H^3 = 0.
\end{aligned}
\tag{12}
$$

Then with another transformation $H_\xi = G$, we have

$$
\begin{aligned}
H_\xi &= G, \\
G_\xi &= \frac{\left(w + a\kappa^2 + \alpha\kappa \right)}{(1 + \alpha\beta^2)} H \\
&\quad - \frac{\left(b - \lambda\kappa + \kappa^2 (\phi_1 + \phi_2 + \phi_3) \right)}{(1 + \alpha\beta^2)} H^3.
\end{aligned}
\tag{13}
$$

In accordance with the FIM, it is supposed that $H(\xi)$ and $G(\xi)$ are non-trivial solutions of Equation (13) and $F(H, G) = \sum_{i=0}^{r} a_i(H) G^i$ is an irreducible function in the domain $C[H, G]$ such that

$$F(H(\xi), G(\xi)) = \sum_{i=0}^{r} a_i(H) G^i = 0, \tag{14}$$

where $a_i(H)$, $(i = 0, 1, 2, \ldots, r)$ are polynomials of H and $a_r(H) \neq 0$. Equation (12) is the first integral for system (13), owing to the DT, there exists $g(H) + f(H)G$ in $C[H, G]$ as:

$$dF/d\xi = \frac{dF}{dH}\frac{dH}{d\xi} + \frac{dF}{dG}\frac{dG}{d\xi} = [g(H) + f(H)G]\sum_{i=0}^{r} a_i(H)G^i. \tag{15}$$

Here, we only consider $r = 1$ in Equation (15).

If we equate the coefficients of $G^i (i = 0, 1, 2, \ldots, r)$ of Equation (15) for $r = 1$, we have

$$\dot{a}_1(H) = a_1(H)g(H) \tag{16}$$

$$\dot{a}_0(H) = a_1(H)g(H) + h(H)a_0(H) \tag{17}$$

$$a_0(H)g(H) = a_1(H)\left[\frac{(w + \alpha\kappa^2 + \alpha\kappa)}{(1 + \alpha\beta^2)}H - \frac{(b - \lambda\kappa + \kappa^2(\phi_1 + \phi_2 + \phi_3))}{(1 + \alpha\beta^2)}H^3\right] \tag{18}$$

Since $a_i(H)(i = 0, 1)$ is a polynomial of H, $a_1(H)$ is a constant and $h(H) = 0$ from (16). For convenience, let $a_1(H) = 1$, and equalizing the degrees of $g(H)$ and $a_0(H)$ we conclude the degree of $g(H)$ is equal to one. Then, we assume that $g(H) = A_1 + 2A_2H$, and we obtain the following from Equations (17) and (18):

$$a_0(H) = A_2H^2 + A_1H + A_0. \tag{19}$$

Replacing $a_0(H)$, $a_1(H)$ and $g(H)$ in Equation (18), to separate the common factors of the same terms, then equating the coefficients of H^i to zero, we have the following case:

$$A_1 = 0, A_2 = \pm\sqrt{\frac{(\lambda\kappa - b - \kappa^2(\phi_1 + \phi_2 + \phi_3))}{2(1 + \alpha\beta^2)}}, \quad A_0 = \pm\frac{(w + \alpha\kappa^2 + \alpha\kappa)}{\sqrt{2(1 + \alpha\beta^2)(\lambda\kappa - b - \kappa^2(\phi_1 + \phi_2 + \phi_3))}}. \tag{20}$$

Putting (20) into (14), we have

$$H_\xi = \pm\frac{(w + \alpha\kappa^2 + \alpha\kappa)}{\sqrt{2(1 + \alpha\beta^2)(\lambda\kappa - b - \kappa^2(\phi_1 + \phi_2 + \phi_3))}} \pm\sqrt{\frac{(\lambda\kappa - b - \kappa^2(\phi_1 + \phi_2 + \phi_3))}{2(1 + \alpha\beta^2)}}H^2(\xi). \tag{21}$$

If we solve the Equations (21), we have the following dark soliton solution

$$H = -\sqrt{\frac{w + \alpha\kappa^2 + \alpha\kappa}{\lambda\kappa - b - \kappa^2(\phi_1 + \phi_2 + \phi_3)}}\tanh\left[\sqrt{\frac{w + \alpha\kappa^2 + \alpha\kappa}{2(1 + \alpha\beta^2)}}\right], \tag{22}$$

and the original solution of Equation (2) is

$$h(x, t) = \left(-\sqrt{\frac{w + \alpha\kappa^2 + \alpha\kappa}{\lambda\kappa - b - \kappa^2(\phi_1 + \phi_2 + \phi_3)}}\tanh\left[\sqrt{\frac{w + \alpha\kappa^2 + \alpha\kappa}{2(1 + \alpha\beta^2)}}\beta(x - vt)\right]\right)e^{i[-\kappa x + wt]}. \tag{23}$$

4 The Ansatz approach

We use an ansatz approach to seek other types of soliton solutions of Equation (1).

First, Equation (12) will be integrated to reach the combined bright-dark [30] soliton solution of Equation (1). So we will seek a solution of the following form

$$H(\xi) = \theta_0 \operatorname{sech}[\xi] - i\theta_1 \tanh[\xi], \tag{24}$$

where θ_0 and θ_1 are amplitudes of the bright and dark solitons, respectively.

By substituting (22) into (12) and setting the coefficients of each term of $\operatorname{sech}^i[\xi]\tanh^j[\xi]$ $(i, j = 0, 1, 2)$ to zero we get the following relations:

$$\theta_0 = \theta_1 = \pm\sqrt{\frac{1 + \alpha\beta^2}{2(b - \lambda\kappa + \kappa^2(\phi_1 + \phi_2 + \phi_3))}}, \quad w = -\frac{1}{2}\left(1 + \alpha\beta^2 + 2\alpha\kappa + i\kappa^2\right). \tag{25}$$

From (25) the combined bright-dark soliton solution of Equation (1) is obtained:

$$h(x, t) = \pm\sqrt{\frac{1 + \alpha\beta^2}{2(b - \lambda\kappa + \kappa^2(\phi_1 + \phi_2 + \phi_3))}}(\operatorname{sech}[\beta(x - vt)] - i\tanh[\beta(x - vt)])\, e^{i\left[-\kappa x - \frac{1}{2}\left(1 + \alpha\beta^2 + 2\alpha\kappa + i\kappa^2\right)t\right]}. \tag{26}$$

Second, Equation (12) will be integrated to reach the combined-dark soliton solution of Equation (1). So we will seek a solution of the following form

$$H(\xi) = \theta_0 \tanh[\xi] - i\theta_1 \operatorname{sech}[\xi], \qquad (27)$$

where θ_0 and θ_1 ($\theta_0 > 0, \theta_1 > 0$) are the amplitudes of the dark and bright solitons, respectively.

By substituting (27) into (12) and setting the coefficients of each term of $\operatorname{sech}^i[\xi]\tanh^j[\xi]$ $(i, j = 0, 1, 2)$ to zero we get the following relations:

$$\theta_0 = \theta_1 = \pm\sqrt{\frac{-1-\alpha\beta^2}{2\left(b - \lambda\kappa + \kappa^2(\phi_1 + \phi_2 + \phi_3)\right)}}, \qquad (28)$$
$$w = -\frac{1}{2}\left(1 + \alpha\beta^2 + 2\alpha\kappa + a\kappa^2\right).$$

From (28) it is the combined-dark soliton solution of Equation (1) is obtained:

$$h(x,t) = \pm\sqrt{\frac{-1-\alpha\beta^2}{2\left(b - \lambda\kappa + \kappa^2(\phi_1 + \phi_2 + \phi_3)\right)}} \left(\tanh[\beta(x - vt)] - i\operatorname{sech}[\beta(x - vt)]\right) e^{i\left[-\kappa x - \frac{1}{2}\left(1+\alpha\beta^2+2\alpha\kappa+a\kappa^2\right)t\right]}. \qquad (29)$$

Third, Equation (12) will be integrated to reach the combined-bright soliton solution of Equation (1). So we will seek a solution of the following form

$$H(\xi) = \theta_0 \tanh[\xi] + i\theta_1 \operatorname{sech}[\xi], \qquad (30)$$

where θ_0 and θ_1 ($\theta_0 > 0, \theta_1 > 0$) are the amplitudes of the dark and bright solitons respectively.

By substituting (30) into (12) and setting the coefficients of each term of $\operatorname{sech}^i[\xi]\tanh^j[\xi]$ $(i, j = 0, 1, 2)$ to zero we get the same relations as (28) and the combined-bright soliton solution of Equation (1) is obtained:

$$h(x,t) = \pm\sqrt{\frac{-1-\alpha\beta^2}{2\left(b - \lambda\kappa + \kappa^2(\phi_1 + \phi_2 + \phi_3)\right)}} \left(\tanh[\beta(x - vt)] + i\operatorname{sech}[\beta(x - vt)]\right) e^{i\left[-\kappa x - \frac{1}{2}\left(1+\alpha\beta^2+2\alpha\kappa+a\kappa^2\right)t\right]}. \qquad (31)$$

5 Conclusion

We used the FIM and antsatz approaches for acquiring several new exact solutions of the one-dimensional NLSE with time dependent coefficients. We have acquired different types of exact solutions which are dark, combined-bright, and combined-dark optical solitons. These obtained solutions are new according to our research of the literature. It has been shown that the velocity function $w(t)$ is related to the group velocity term $a(t)$ in (28). Consequently, the FIM and ansatz approaches are crucial ones to construct different types of the exact solutions of the NPDE and systems.

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