ON LOCAL FRACTIONAL OPERATORS VIEW OF COMPUTATIONAL COMPLEXITY
Diffusion and Relaxation Defined on Cantor Sets

by

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This paper treats the description of non-differentiable dynamics occurring in complex systems governed by local fractional partial differential equations. The exact solutions of diffusion and relaxation equations with Mittag-Leffler and exponential decay defined on Cantor sets are calculated. Comparative results with other versions of the local fractional derivatives are discussed.

Key words: *complex systems, diffusion equation, relaxation equation, local fractional derivative, Cantor set*

Introduction

Fractional calculus has potential applications in physics and several studies addressed those topics during the last fifty years [1-4]. By means of several types of fractional derivatives (FD), namely, the Riemann-Liouville, Weyl, Riesz, Grunwald, and Caputo FD [5-9], the anomalous diffusion on different domains was adopted to characterize the mean square displacement with linear Brownian dependence on time [10-14]. The diffusion of fractional order via the Riemann-Liouville fractional operator and its exact solution were presented in [15]. The diffusive behaviors of fractional order in the perspective of the Caputo [16, 17], Riesz and Weyl fractional operators [18], the relaxation phenomena of fractional order via the Caputo [16, 19], and the Riemann-Liouville [20] FD were also discussed.

Fractal surfaces are a widely-studied field of modern physics in order to describe real world problems [21-23]. The connection between fractals and fractional operators emerged in some advanced results [24-28]. The fractal operator [29] and the fractional operator in non-integer dimensional space [30-33] deal with the phenomena of continuity and differentiability. However, in nature surfaces occur both with discontinuity and non-differentiability. The general Lebesgure outer measure in the interval [μ, η] is written in the form $L_\varepsilon([\mu,\eta]) = (\eta-\mu)^\varepsilon$ [34, 35]

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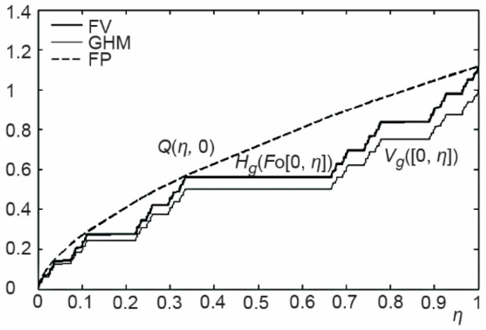

Figure 1. Comparison of the GHM, FV, and FP for $\mu = 0$

and the generalized Hausdorff measure (GHM) in the interval $[\mu, \eta]$ is $H_\varepsilon(\Xi \cap [\mu,\eta]) = (\eta - \mu)^\varepsilon$, where ε is the fractal dimension of the set Ξ [34-37]. Meanwhile, the fractal volume (FV) in the interval $[\mu, \eta]$ is written as $V_\varepsilon([\mu,\eta]) = (\eta - \mu)^\varepsilon/\Gamma(1+\varepsilon)$, where $\Gamma(1+\varepsilon)$ is the Gamma function [34, 35]. The version of fractional power (FP) expressed in natural co-ordinates in a fractional space is $Q(\eta,\mu) = (\eta - \mu)^\varepsilon/\Gamma(1+\varepsilon)$ [38, 39]. The GHM [36, 37], $H_\varepsilon(F \cap [0,\eta]) = \eta^\varepsilon$, the FV, $V_\varepsilon([0,\eta]) = \eta^\varepsilon/\Gamma(1+\varepsilon)$, and the FP, $Q(\eta,0) = \eta^\varepsilon/\Gamma(1+\varepsilon)$, are shown in fig. 1 when $\mu = 0$.

For a brief review of the literature we can mention the following studies. The local fractional operators (LFO) of Kolwankar and Gangal [40-41], Carpinteri and Sapora [42], Carpinteri and Cornetti [43], Carpinteri *et al.* [44], Jumarie [45, 46], Roul [47], Chen [48], Chen *et al.* [49, 50], Balankin and Elizarraraz [51], Balankin [52, 53], He [54], He *et al.* [55], Uchaikin [56], and other authors [34, 35, 57-61] were proposed. A brief review of the literature reveals the recent progress in this area. The diffusion via local fractional derivative (LFD) was considered to describe the dynamics on Cantor sets [42]. The diffusion equation (DE) with Hausdorff measure derivative was addressed in [48, 49]. The relaxation equation (RE) with Hausdorff measure derivative was investigated in [50]. The relaxation [54] and diffusion [61] with the smallest measure were presented as a potential tool for engineering application. The diffusion [58] and relaxation [61] defined Cantor sets involving LFD were also discussed.

In this manuscript, some applications of LFO to dimensionless DE and RE defined on Cantor sets are investigated.

The investigations of LFO

In this section, we present the main aspects of LFO and the differences between them are discussed.

The LFD of the function $\Pi(\eta)$ of order $\nu\,(0 < \nu < 1)$ via the Riemann-Liouville FD, with the Kolwankar and Gangal's view proposed in [40-44], is defined by:

$$D^{(\nu)}\Pi(\eta) = \frac{\mathrm{d}^\nu \Pi(\eta)}{\mathrm{d}\eta^\nu}\Big|_{\zeta=\zeta_0} = \lim_{\mu\to\mu_0} \frac{\mathrm{d}^\nu[\Pi(\eta) - \Pi(\eta_0)]}{[\mathrm{d}(\eta-\eta_0)]^\nu} \tag{1}$$

where the term $\mathrm{d}^\nu[\Pi(\eta)]/[\mathrm{d}(\eta-\eta_0)]^\nu$ is the Riemann-Liouville FD of order ν of $\Pi(\eta)$.

From eq. (1) it follows that [27, 36, 41, 43]:

$$D^{(\nu)}A(\eta) = D^{(\nu)}S(\eta) = 1 \tag{2}$$

and

$$D^{(\nu)}B(\eta) = D^{(\nu)}[S(\eta)]^n = n[S(\eta)]^{n-1} \tag{3}$$

where $n \in \mathbb{N}$, $A(\eta) = S(\eta) \sim \eta^\nu/\Gamma(1+\nu)$, and $B(\eta) = [S(\eta)]^n$ are staircase functions defined on Cantor sets with fractal dimension ν. Following eq. (2), the operator, eq. (1), gives the fractional measure defined on Cantor sets $D^{(-\nu)}1 = S(1) - S(0) = 1/\Gamma(1+\nu)$ [27, 43]. The numerical method for LFD equation was discussed in [43] and, with more details, in [62-64].

The LFD equation with eq. (1) can be formulated:

$$D^{(\nu)}\Theta(\eta)=\Theta(\eta) \tag{4}$$

The model leads to the non-differentiable stretched solution:

$$\Theta(\eta)=\exp[S(\eta)] \tag{5}$$

which is a Lebesgue-Canto-like function. When $\nu=1$, we get back to the typical solution $\Theta(\eta)=\exp(\eta)$.

The LFD of the function of order $\chi(0<\chi<1)$, with the view proposed in [45-47], is defined by:

$$D^{(\chi)}\phi(\mu)=\frac{\mathrm{d}^{\chi}\phi(\mu)}{\mathrm{d}\mu^{\chi}}\Big|_{\mu=\mu_0}=\lim_{h\to 0}\frac{\Delta^{\chi}\phi(\mu)}{h^{\chi}} \tag{6}$$

where

$$\Delta^{\chi}\phi(\mu)=\sum_{k=0}^{\infty}(-1)^k\binom{\chi}{k}\phi[\mu+(\chi-k)h] \tag{7}$$

In view of eq. (5), we present [45, 46]:

$$D^{(\chi)}C(\mu)=D^{(\chi)}\frac{\mu^{\chi}}{\Gamma(1+\chi)}=1 \tag{8}$$

and

$$D^{(\chi)}D(\mu)=D^{(\chi)}\mu=\frac{\mu^{1-\chi}}{\Gamma(2-\chi)} \tag{9}$$

where $C(\mu)=\mu^{\chi}/[\Gamma(1+\chi)]$ and $D(\mu)=\mu$.

Adopting eq. (6), the LFD equation is [46]:

$$D^{(\chi)}\phi(\mu)=\phi(\mu) \tag{10}$$

This expression leads to a stretched Mittag-Leffler solution:

$$\phi(\mu)=E_{\chi}(\mu^{\chi}) \tag{11}$$

which is in agreement with the one in sense of Caputo FD, see [16, 50].

The LFD of the function $\Phi(\tau)$ of order $\kappa(0<\kappa<1)$, with the Chen's view proposed in [48-50], is defined by:

$$D^{(\kappa)}\Phi(\tau)=\frac{\mathrm{d}^{\kappa}\Phi(\tau)}{\mathrm{d}\tau^{\kappa}}\Big|_{\tau=\tau_0}=\lim_{\tau\to\tau_0}\frac{\Phi(\tau)-\Phi(\tau_0)}{\tau^{\kappa}-\tau_0^{\kappa}} \tag{12}$$

where μ^{κ} is a fractal measure.

With eq. (12), the LFD equation can be written:

$$D^{(\kappa)}\Phi(\tau)=\Phi(\tau), \tag{13}$$

This expression leads to a stretched exponential solution [50]:

$$\Phi(\tau) = \exp(\tau^{\kappa}) \tag{14}$$

which is compared with the one in sense of Caputo FD, see [50].

From eq. (14), it follows that:

$$D^{(\kappa)}\tau^{\kappa} = 1 \tag{15}$$

$$D^{(\kappa)}\tau^{2\kappa} = 2\tau^{\kappa} \tag{16}$$

$$D^{(\kappa)}\tau^{n\kappa} = n\tau^{(n-1)\kappa}, \qquad (n > 1)\,. \tag{17}$$

Making use of eq. (17), we easily obtain eq. (14). Here, the operator (12) may be defined on the Hausdorff measure and follows that there is:

$$H_{\varepsilon}(\Xi \cap [0,\eta_2]) - H_{\varepsilon}(\Xi \cap [0,\eta_1]) = H_{\varepsilon}(\Xi \cap [\eta_1,\eta_2])$$

The LFD of the function $\Omega(\mu)$ of order $l\,(0 < l < 1)$, with the view proposed in [54-56], is defined by:

$$D^{(l)}\Omega(\mu) = \frac{\mathrm{d}\Omega(\mu)}{\mathrm{d}\mu^{l}}\Big|_{\mu=\mu_0} = \frac{\mathrm{d}\Omega(\mu)}{\mathrm{d}\sigma} = \lim_{\Delta\mu\to\mu_0} \frac{\Omega(\mu_B) - \Omega(\mu_A)}{\Upsilon\eta_0^{l}} \tag{18}$$

where $\mathrm{d}\sigma = \Upsilon\eta_0^{l}$ with geometrical parameter Υ and measure scale η_0.

Equation (18) leads to the LFD equation:

$$D^{(l)}\Omega(\mu) = \Omega(\mu) \tag{19}$$

This expression leads to a stretched exponential solution [54]:

$$\Omega(\mu) = \exp(\mu\Upsilon\eta_0^{l-1}) \tag{20}$$

Following eq. (18), we easily obtain:

$$D^{(l)}[\mu\Upsilon\eta_0^{l-1}] = 1 \tag{21}$$

$$D^{(l)}[\mu\Upsilon\eta_0^{l-1}]^{n} = n[\mu\Upsilon\eta_0^{l-1}]^{n-1}, \qquad n > 1 \tag{22}$$

Adopting eq. (22) yields a stretched exponential solution eq. (20).

The LFD of the function $\Theta(\tau)$ of order $\iota\,(0 < \iota < 1)$, with the view proposed in [51-53], is defined by:

$$D^{(\iota)}\Theta(\tau) = \left(\frac{\tau}{L_0} + 1\right)^{1-\iota} D^{(1)}\Theta(\tau) \tag{23}$$

With eq. (23), we have the LFD equation:

$$D^{(\iota)}\Phi(\tau) = \Phi(\tau) \tag{24}$$

that yields a stretched exponential solution [65]:

$$\Theta(\tau) = \exp\left[\frac{L_0}{\iota}\left(\frac{\tau}{L_0}+1\right)^{\iota}\right] \tag{25}$$

Similarly, from eq. (23) we have that:

$$D^{(\iota)}\left[\frac{L_0}{\iota}\left(\frac{\tau}{L_0}+1\right)^{\iota-1}\right] = 1 \tag{26}$$

$$D^{(\iota)}\left\{\left[\frac{L_0}{\iota}\left(\frac{\tau}{L_0}+1\right)^{\iota}\right]^{n}\right\} = n\left[\frac{L_0}{\iota}\left(\frac{\tau}{L_0}+1\right)^{\iota}\right]^{n-1}, \qquad n>1 \tag{27}$$

Equation (27) leads to a stretched exponential solution eq. (25).

The LFD of the function $\psi(\eta)$ of order $\varepsilon\,(0<\varepsilon<1)$, with the view proposed in [34, 35, 57-61, 65], is defined by:

$$D^{(\varepsilon)}\psi(\eta) = \frac{\mathrm{d}^{\varepsilon}\psi(\eta)}{\mathrm{d}\eta^{\varepsilon}}\Big|_{\eta=\eta_0} = \lim_{\eta\to\eta_0}\frac{\Delta^{\varepsilon}[\psi(\eta)-\psi(\eta_0)]}{(\eta-\eta_0)^{\varepsilon}} \tag{28}$$

where

$$\Delta^{\varepsilon}[\psi(\eta)-\psi(\eta_0)] \cong \Gamma(1+\varepsilon)\Delta[\psi(\eta)-\psi(\eta_0)] \tag{29}$$

For each $\xi>0$ there exists for $\delta>0$ such that [34, 35, 57-61]:

$$|\psi(\eta)-\psi(\eta_0)| < \xi^{\varepsilon} \tag{30}$$

whenever $0<|\eta-\eta_0|<\delta.$

Equation (29) was derived from fractal geometry [34, 35].

In order to interpret eq. (30), we recall the new series expansion [61]:

$$(\varphi+\phi)^{k\varepsilon} = \sum_{i=0}^{\infty}\binom{k\varepsilon}{i\varepsilon}\varphi^{(k-i)\varepsilon}\phi^{i\varepsilon} = \sum_{i=0}^{\infty}\binom{k\varepsilon}{i\varepsilon}\varphi^{i\varepsilon}\phi^{(k-i)\varepsilon} \tag{31}$$

where

$$\binom{k\varepsilon}{i\varepsilon} = \frac{\Gamma(1+k\varepsilon)}{\Gamma(1+i\varepsilon)\Gamma[1+(k-i)\varepsilon]} \tag{32}$$

for $k,\ i\in\mathbb{N}$ and $0<\varepsilon<1.$

The characteristics of the series expansion are [61]:

(M1) $(\varphi+\phi)^{k\varepsilon} = 1,$ when $k = 0$;

(M2) $(\varphi+\phi)^{k\varepsilon} = \varphi^{\varepsilon}+\phi^{\varepsilon},$ when $k = 1$;

(M3) $$(\varphi+\phi)^{\varepsilon}=(2\phi)^{\varepsilon}=(2\varphi)^{\varepsilon},\ \text{when}\ \varphi=\phi.$$

With the help of eq. (31), the non-differential difference (30) takes the form:

$$\Delta^{\varepsilon}[\psi(\eta)-\psi(\eta_0)]=\Gamma(1+\varepsilon)\Delta^{\varepsilon}\psi(\eta_0)\cong\Gamma(1+\varepsilon)[\psi(\eta)-\psi(\eta_0)] \tag{33}$$

where

$$\Delta^{\varepsilon}\psi(\eta_0)=\sum_{i=0}^{\infty}(-1)^i\begin{pmatrix}\varepsilon\\ i\varepsilon\end{pmatrix}\psi(\eta-i\rho) \tag{34}$$

with $\rho=\eta-\eta_0$.

In this case, we present two examples:

$$\frac{\mathrm{d}^{\varepsilon}}{\mathrm{d}\eta^{\varepsilon}}\frac{\eta^{\varepsilon}}{\Gamma(1+\varepsilon)}=\lim_{\Delta\mu\to 0}\frac{1}{\Gamma(1+\varepsilon)}\frac{\Gamma(1+\varepsilon)[(\eta+\Delta\eta)^{\varepsilon}-\eta^{\varepsilon}]}{(\Delta\eta)^{\varepsilon}}=1 \tag{35}$$

$$\begin{aligned}\frac{\mathrm{d}^{\varepsilon}}{\mathrm{d}\eta^{\varepsilon}}\frac{\eta^{\gamma\varepsilon}}{\Gamma(1+\gamma\varepsilon)}&=\lim_{\Delta\eta\to 0}\left\{\frac{\Gamma(1+\varepsilon)}{\Gamma(1+\gamma\varepsilon)}\frac{(\eta+\Delta\eta)^{\gamma\varepsilon}-\eta^{\gamma\varepsilon}}{(\Delta\eta)^{\varepsilon}}\right\}\\&=\lim_{\Delta\eta\to 0}\left\{\frac{\Gamma(1+\varepsilon)}{\Gamma(1+\gamma\varepsilon)}\frac{\eta^{\gamma\varepsilon}+\dfrac{\Gamma(1+\gamma\varepsilon)}{\Gamma(1+\varepsilon)\Gamma[1+(\gamma-1)\varepsilon]}\eta^{(\gamma-1)\varepsilon}(\Delta\eta)^{\varepsilon}+..+-\eta^{\gamma\varepsilon}}{(\Delta\eta)^{\varepsilon}}\right\}\\&=\lim_{\Delta\eta\to 0}\left\{\frac{\Gamma(1+\varepsilon)}{\Gamma(1+\gamma\varepsilon)}\frac{\dfrac{\Gamma(1+\gamma\varepsilon)}{\Gamma(1+\varepsilon)\Gamma[1+(\gamma-1)\varepsilon]}\eta^{(\gamma-1)\varepsilon}(\Delta\eta)^{\varepsilon}}{(\Delta\eta)^{\varepsilon}}\right\}\\&=\frac{\eta^{(\gamma-1)\varepsilon}}{\Gamma[1+(\gamma-1)\varepsilon]}\end{aligned} \tag{36}$$

The integration functions in eqs. (35) and (36) are staircase functions defined on Cantor sets with fractal dimension ε.

It follows that:

$$\frac{\mathrm{d}^{\varepsilon}}{\mathrm{d}\eta^{\varepsilon}}E_{\varepsilon}(\eta^{\varepsilon})=\frac{\mathrm{d}^{\varepsilon}}{\mathrm{d}\eta^{\varepsilon}}\sum_{\gamma=0}^{\infty}\frac{\eta^{\gamma\varepsilon}}{\Gamma(1+\gamma\varepsilon)}=\sum_{\gamma=1}^{\infty}\frac{\eta^{(\gamma-1)\varepsilon}}{\Gamma[1+(\gamma-1)\varepsilon]}=\sum_{\gamma=0}^{\infty}\frac{\eta^{\gamma\varepsilon}}{\Gamma(1+\gamma\varepsilon)}=E_{\varepsilon}(\eta^{\varepsilon}) \tag{37}$$

where

$$E_{\varepsilon}(\eta^{\varepsilon})=\sum_{\gamma=0}^{\infty}\frac{\eta^{\gamma\varepsilon}}{\Gamma(1+\gamma\varepsilon)}$$

is the Mittag-Leffler function of Lebesgue-Canto-like type.

Hence, the LFD equation [34, 35, 61] is given by:

$$D^{(\varepsilon)}\psi(\eta)=\psi(\eta) \tag{38}$$

This model leads to a stretched solution in the term of Mittag-Leffler function of Lebesgue-Cantor-like type:

$$\psi(\eta) = E_{\varepsilon}(\eta^{\varepsilon}) \tag{39}$$

which is not in agreement with eq. (11) because the latter is a Mittag-Leffler function defined on Cantor sets. When $\varepsilon = 1$, we get back to the typical solution of eq. (1).

The LFD of a constant, with different LFO, is zero. So, for the stretched solutions of the LFD equation, we have:

– (T1) with eqs. (1) and (28) are Lebesgue-Cantor-like functions of exponential and Mittag-Leffler defined on Cantor sets, respectively;
– (T2) with eqs. (6) and (12) are Mittag-Leffler and exponential functions of power functions which are continuous functions;
– (T3) with eqs. (18) and (23) are exponential functions related to fractal geometry.

Remark that we can easily write the equation of a generalized Cantor set Ξ with Hausdorff measure ε in the form [66]:

$$\sum_{i=1}^{\infty} \Theta_i^{\varepsilon} = 1 \tag{40}$$

From eq. (31), it yields:

$$\sum_{i=1}^{\infty} \Theta_i^{\varepsilon} = \sum_{i=1}^{\infty} H_{\varepsilon}(\Xi \cap [0,1]) = 1 \tag{41}$$

where $\Theta_i (i \in \mathbb{N})$ represents the contraction ratios.

Equation (41) shows that a generalized Cantor set is defined on the interval [0, 1] when its diameter is equal to 1. In this case, (M2) can be used to interpolation of $(\varphi + \phi)^{\varepsilon} = \varphi^{\varepsilon} + \phi^{\varepsilon}$, see [65].

Diffusions defined on Cantor sets

In this section, the exact solutions for the DE defined on Cantor sets with LFD are investigated.

We start with the dimensionless DE defined on Cantor sets with eq. (1) [42]:

$$\frac{\partial \Theta(\eta,\sigma)}{\partial \sigma} = \frac{\partial^{2\nu} \Theta(\eta,\sigma)}{\partial \eta^{2\nu}} \tag{42}$$

subject to initial condition:

$$\Theta(\eta, \sigma = 0) = \delta(\eta) \tag{43}$$

According to the view of the evolution-semigroup [40], we easily observe that:

$$\Theta(\eta,\sigma) = \exp\left(\frac{\partial^{2\nu}}{\partial \eta^{2\nu}}\right)\Theta(\eta, \sigma = 0) \tag{44}$$

Using the Fourier transform, eq. (44) leads to:

$$\Theta(\eta,\sigma) = \frac{1}{\sqrt{\pi\sigma}} \exp\left[-\frac{S(\eta)^2}{\sigma}\right] \tag{45}$$

This solution of eq. (42) is written in the form:

$$\Theta(\eta,\sigma)=\frac{1}{\sqrt{\pi\sigma}}\exp\left[-\frac{S(\eta)^2}{\sigma}\right]\sim\frac{1}{\sqrt{\pi\sigma}}\exp\left(-\frac{\eta^{2\nu}}{\sigma}\right) \tag{46}$$

This solution is an exponential function of Lebesgue-Cantor-like type.

We consider the dimensionless DE defined on Cantor sets with (28) [42]:

$$\frac{\partial^{\varepsilon}\psi(\eta,\sigma)}{\partial\sigma^{\varepsilon}}=\frac{\partial^{2\varepsilon}\psi(\eta,\sigma)}{\partial\eta^{2\varepsilon}} \tag{47}$$

subject to initial condition:

$$\psi(\eta,\sigma=0)=\delta_{\varepsilon}(\eta) \tag{48}$$

With the result [58], we have the solution of non-differentiability:

$$\psi(\eta,\sigma)=\psi_0\sigma^{\beta\varepsilon}E_{\varepsilon}\left[-\frac{\eta^{2\varepsilon}}{(4\sigma)^{\varepsilon}}\right] \tag{49}$$

where β is a unknown constant, and ψ_0 – an initial value.

It follows that:

$$\psi(\eta,\sigma=0)=\lim_{\sigma\to 0}\psi_0\sigma^{\beta\varepsilon}E_{\varepsilon}\left[-\frac{\eta^{2\varepsilon}}{(4\sigma)^{\varepsilon}}\right] \tag{50}$$

According to [61], we have:

$$\delta_{\varepsilon}(\sigma)=\lim_{\sigma\to 0}\frac{1}{\dfrac{(4\pi\sigma)^{\frac{\varepsilon}{2}}}{\Gamma(1+\varepsilon)}}E_{\varepsilon}\left[-\frac{\eta^{2\varepsilon}}{(4\mu)^{\varepsilon}}\right] \tag{51}$$

such that:

$$\Phi_0=\frac{1}{\dfrac{(4\pi)^{\frac{\varepsilon}{2}}}{\Gamma(1+\varepsilon)}} \tag{52}$$

and

$$\beta=-\frac{1}{2} \tag{53}$$

Thus, we have:

$$\psi(\eta,\sigma)=\frac{1}{\dfrac{(4\pi)^{\frac{\varepsilon}{2}}}{\Gamma(1+\varepsilon)}}\sigma^{-\frac{\varepsilon}{2}}E_{\varepsilon}\left[-\frac{\eta^{2\varepsilon}}{(4\sigma)^{\varepsilon}}\right] \tag{54}$$

This is exact solution of DE with eq. (28) and is a Lebesgue-Cantor-like function of Mittag-Leffler defined on Cantor sets.

When $\nu = 1$ and $\varepsilon = 1$, we get back to the typical solution:

$$\psi(\eta,\sigma) = \frac{1}{(4\pi\sigma)^{1/2}}\exp\left(-\frac{\eta^2}{4\sigma}\right) \tag{55}$$

Relaxations defined on Cantor sets

In this section, the exact solutions for the RE defined on Cantor sets are discussed.

We begin with the RE defined on Cantor sets with:

$$\frac{\partial^\nu \Theta(\eta)}{\partial \eta^\nu} + \omega\Theta(\eta) = 0 \tag{56}$$

subject to initial condition:

$$\Theta(\eta = 0) = 1 \tag{57}$$

We easily obtain:

$$\Theta(\eta) = \exp[-\omega\Omega_c(\eta)] \tag{58}$$

where $\Omega_c(\eta) \sim \eta^\nu$ is a Lebesgue-Cantor function.

In the similar way, we consider the RE defined on Cantor sets with:

$$\frac{\partial^\varepsilon \psi(\eta)}{\partial \eta^\varepsilon} + \omega\psi(\eta) = 0 \tag{59}$$

subject to initial condition:

$$\psi(\eta = 0) = 1 \tag{60}$$

We can see that:

$$\psi(\eta) = E_\varepsilon(-\omega\eta^\varepsilon) \tag{61}$$

which is a Lebesgue-Cantor-like function of Mittag-Leffler defined on Cantor sets. When $\nu = 1$ and $\varepsilon = 1$, we get back to the typical solution.

Discussion

In this section, the comparisons between DE and RE within LFO are presented in details.

The dimensionless DE with LFD [50, 61] is:

$$\frac{\partial^\kappa \Phi(\mu,\tau)}{\partial \tau^\kappa} = \frac{\partial^{2\kappa} \Phi(\mu,\tau)}{\partial \mu^{2\kappa}} \tag{62}$$

that has an exact solution in the form:

$$\Phi(\mu,\tau) = \frac{1}{\sqrt{4\pi\mu^\kappa}}\exp\left(-\frac{\sigma^{2\kappa}}{4\tau^\kappa}\right) \tag{63}$$

On the other hand, the RE within LFD:

$$D^{\kappa}\Phi(\tau)+\omega\Phi(\tau)=0 \tag{64}$$

has an exact solution [50, 61] given by:

$$\Phi(\tau)=\exp(-\omega\tau^{\kappa}) \tag{65}$$

where the LFO is considered in sense of the view in [48-50].

The solution of the dimensionless DE, with the view in [48-50], is in the perspective of differentiability while the dimensionless DE defined on Cantor sets are of non-differentiability. Comparative results of DE with LFO are shown in fig. 2. Similarily, the solution of the dimensionless RE, with the view proposed in [48-50], is of differentiability while the dimensionless DE defined on Cantor sets are of non-differentiability. Comparative results of REs with LFO are shown in fig. 3.

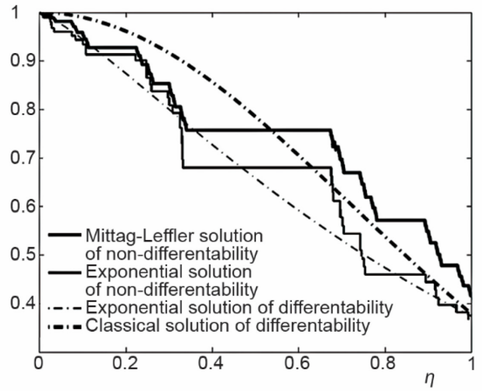

Figure 2. The exact solutions of DE with LFO

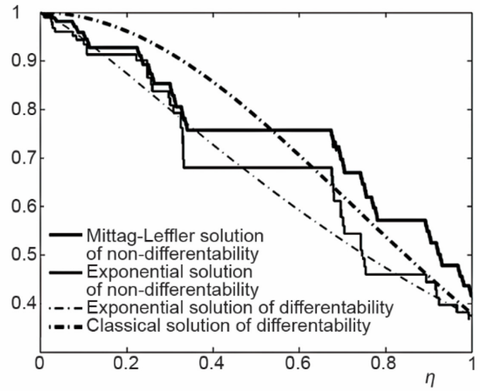

Figure 3. The exact solutions of RE with LFO

Conclusions

In the last decades the LFD were introduced mainly due to the need of developing more appropriate solutions of the DE and RE that occur in real world applications. This work addressed the LFO that are potential methods to describe complex systems governed by local fractional partial differential equation. The DE and RE with LFO are selected to better explain the complexity of dynamical system. The comparison of the exact solutions of the differentiability and the non-differentiability with Mittag-Leffler and exponential decay were discussed. The Mittag-Leffler solution of non-differentiability of DE and RE with our view, the exponential solution of non-differentiability of DE and RE, with the Kolwankar and Gangal's view, and the exponential solution of differentiability of DE and RE, with the Chen's view, are compared with the classical solutions. The results reveal the potential for the LFO in describing the complex dynamics defined on Cantor sets.

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Nomenclature

μ – space co-ordinate, [m]
κ – fractal dimension, [–]
τ – time, [s]

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