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Research Article

A Jacobi Collocation Method for Solving Nonlinear Burgers-Type Equations

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We solve three versions of nonlinear time-dependent Burgers-type equations. The Jacobi-Gauss-Lobatto points are used as collocation nodes for spatial derivatives. This approach has the advantage of obtaining the solution in terms of the Jacobi parameters α and β. In addition, the problem is reduced to the solution of the system of ordinary differential equations (SODEs) in time. This system may be solved by any standard numerical techniques. Numerical solutions obtained by this method when compared with the exact solutions reveal that the obtained solutions produce high-accurate results. Numerical results show that the proposed method is of high accuracy and is efficient to solve the Burgers-type equation. Also the results demonstrate that the proposed method is a powerful algorithm to solve the nonlinear partial differential equations.

1. Introduction

Spectral methods (see, e.g., [1–3] and the references therein) are techniques used in applied mathematics and scientific computing to numerically solve linear and nonlinear differential equations. There are three well-known versions of spectral methods, namely, Galerkin, tau, and collocation methods. Spectral collocation method is characterized by the fact of providing highly accurate solutions to nonlinear differential equations [3–6]; also it has become increasingly popular for solving fractional differential equations [7–9]. Bhrawy et al. [5] proposed a new Bernoulli matrix method for solving high-order Fredholm integro-differential equations with piecewise intervals. Saadatmandi and Dehghan [10] developed the Sinc-collocation approach for solving multipoint boundary value problems; in this approach the computation of solution of such problems is reduced to solve some algebraic equations. Bhrawy and Alofi [4] proposed the spectral-shifted Jacobi-Gauss collocation method to find an accurate solution of the Lane-Emden-type equation. Moreover, Doha et al. [11] developed the shifted Jacobi-Gauss collocation method to solve nonlinear high-order multipoint boundary value problems. To the best of our knowledge, there are no results on Jacobi-Gauss-Lobatto collocation method for solving Burgers-type equations arising in mathematical physics. This partially motivated our interest in such method.

For time-dependent partial differential equations, spectral methods have been studied in some articles for several decades. In [12], Ierley et al. investigated spectral methods to numerically solve time-dependent class of parabolic partial differential equations subject to periodic boundary conditions. Tal-Ezer [13, 14] introduced spectral methods using polynomial approximation of the evolution operator in the Chebyshev Least-Squares sense for time-dependent parabolic and hyperbolic equations, respectively. Moreover, Coutsias et al. [15] developed spectral integration method to solve some time-dependent partial differential equations. Zhang [16] applied the Fourier spectral scheme in spatial together with the Legendre spectral method to solve time-dependent partial differential equations and gave error estimates of the method. Tang and Ma [17] introduced the Legendre spectral method together with the Fourier approximation in spatial for time-dependent first-order hyperbolic equations with periodic boundary conditions. Recently, the author of [18] proposed an accurate numerical algorithm to solve the generalized Fitzhugh-Nagumo equation with time-dependent coefficients.

In [20], Bateman introduced the one-dimensional quasi-linear parabolic partial differential equation, while Burgers [21] developed it as mathematical modeling of turbulence, and it is referred as one-dimensional Burgers' equation. Many authors gave different solutions for Burgers' equation by using various methods. Kadalbajoo and Awasthi [22] and Gülsu [23] used a finite-difference approach method to find solutions of one-dimensional Burgers' equation. Crank-Nicolson scheme for Burgers' equation is developed by Kim, [24]. Nguyen and Reynen [25, 26], Gardner et al. [27, 28] and Kutluay et al. [29] used methods based on the Petrov-Galerkin, Least-Squares finite-elements, and B-spline finite element methods to solve Burgers' equation. A method based on collocation of modified cubic B-splines over finite elements has been investigated by Mittal and Jain in [30].

In this work, we propose a J-GL-C method to numerically solve the following three nonlinear time-dependent Burgers'-type equations:

(1) time-dependent 1D Burgers' equation:

$$u_t + \nu u u_x - \mu u_{xx} = 0; \quad (x,t) \in [A,B] \times [0,T], \tag{1}$$

(2) time-dependent 1D generalized Burger-Fisher equation:

$$u_t = u_{xx} - \nu u^{\delta} u_x + \gamma u \left(1 - u^{\delta}\right); \quad (x,t) \in [A,B] \times [0,T], \tag{2}$$

(3) time-dependent 1D generalized Burgers-Huxley equation:

$$u_t + \nu u^{\delta} u_x - u_{xx} - \eta u \left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right) = 0; \quad (x,t) \in [A,B] \times [0,T]. \tag{3}$$

In order to obtain the solution in terms of the Jacobi parameters α and β, the use of the Jacobi polynomials for solving differential equations has gained increasing popularity in recent years (see, [31–35]). The main concern of this paper is to extend the application of J-GL-C method to solve the three nonlinear time-dependent Burgers-type equations. It would be very useful to carry out a systematic study on J-GL-C method with general indexes $(\alpha, \beta > -1)$. The nonlinear time-dependent Burgers'-type equation is collocated only for the space variable at $(N-1)$ points, and for suitable collocation points, we use the $(N-1)$ nodes of the Jacobi-Gauss-Lobatto interpolation which depends upon the two general parameters $(\alpha, \beta > -1)$; these equations together with the two-point boundary conditions constitute the system of $(N+1)$ ordinary differential equations (ODEs) in time. This system can be solved by one of the possible methods of numerical analysis such as the Euler method, Midpoint method, and the Runge-Kutta method. Finally, the accuracy of the proposed method is demonstrated by test problems.

The remainder of the paper is organized as follows. In the next section, we introduce some properties of the Jacobi polynomials. In Section 3, the way of constructing the Gauss-Lobatto collocation technique for nonlinear time-dependent Burgers-type equations is described using the Jacobi polynomials, and in Section 4 the proposed method is applied to three problems of nonlinear time-dependent Burgers-type equations. Finally, some concluding remarks are given in Section 5.

2. Some Properties of Jacobi Polynomials

The standard Jacobi polynomials of degree k ($P_k^{(\alpha,\beta)}(x)$, $k = 0, 1, \ldots$) with the parameters $\alpha > -1$, $\beta > -1$ are satisfying the following relations:

$$P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\alpha,\beta)}(x),$$
$$P_k^{(\alpha,\beta)}(-1) = \frac{(-1)^k \Gamma(k+\beta+1)}{k!\,\Gamma(\beta+1)}, \tag{4}$$
$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)}.$$

Let $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$; then we define the weighted space $L^2_{w^{(\alpha,\beta)}}$ as usual, equipped with the following inner product and norm:

$$(u,v)_{w^{(\alpha,\beta)}} = \int_{-1}^{1} u(x)\, v(x)\, w^{(\alpha,\beta)}(x)\, dx,$$
$$\|u\|_{w^{(\alpha,\beta)}} = (u,u)^{1/2}_{w^{(\alpha,\beta)}}. \tag{5}$$

The set the of Jacobi polynomials forms a complete $L^2_{w^{(\alpha,\beta)}}$-orthogonal system, and

$$\left\|P_k^{(\alpha,\beta)}\right\|_{w^{(\alpha,\beta)}} = h_k = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}. \tag{6}$$

Let $S_N(-1,1)$ be the set of polynomials of degree at most N, and due to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\phi \in S_{2N+1}(-1,1)$,

$$\int_{-1}^{1} w^{(\alpha,\beta)}(x)\,\phi(x)\,dx = \sum_{j=0}^{N} \varpi_{N,j}^{(\alpha,\beta)} \phi\left(x_{N,j}^{(\alpha,\beta)}\right), \tag{7}$$

where $x_{N,j}^{(\alpha,\beta)}$ $(0 \le j \le N)$ and $\varpi_{N,j}^{(\alpha,\beta)}$ $(0 \le j \le N)$ are the nodes and the corresponding Christoffel numbers of the Jacobi-Gauss-quadrature formula on the interval $(-1,1)$, respectively. Now, we introduce the following discrete inner product and norm:

$$(u,v)_{w^{(\alpha,\beta)}} = \sum_{j=0}^{N} u\left(x_{N,j}^{(\alpha,\beta)}\right) v\left(x_{N,j}^{(\alpha,\beta)}\right) \varpi_{N,j}^{(\alpha,\beta)},$$
$$\|u\|_{w^{(\alpha,\beta)}} = (u,u)_{w^{(\alpha,\beta)}}^{1/2}. \tag{8}$$

For $\alpha = \beta$, one recovers the ultraspherical polynomials (symmetric Jacobi polynomials) and for $\alpha = \beta = \mp 1/2$, $\alpha = \beta = 0$, the Chebyshev of the first and second kinds and the Legendre polynomials, respectively; and for the nonsymmetric Jacobi polynomials, the two important special cases $\alpha = -\beta = \pm 1/2$ (the Chebyshev polynomials of the third and fourth kinds) are also recovered.

3. Jacobi Spectral Collocation Method

Since the collocation method approximates the differential equations in physical space, it is very easy to implement and be adaptable to various problems, including variable coefficient and nonlinear differential equations (see, for instance [4, 6]). In this section, we develop the J-GL-C method to numerically solve the Burgers-type equations.

3.1. (1 + 1)-Dimensional Burgers' Equation. In 1939, Burgers has simplified the Navier-Stokes equation by dropping the pressure term to obtain his one-dimensional Burgers' equation. This equation has many applications in applied mathematics, such as modeling of gas dynamics [36, 37], modeling of fluid dynamics, turbulence, boundary layer behavior, shock wave formation, and traffic flow [38]. In this subsection, we derive a J-GL-C method to solve numerically the (1 + 1)-dimensional Burgers' model problem:

$$u_t + \nu u u_x - \mu u_{xx} = 0; \quad (x,t) \in D \times [0,T], \tag{9}$$

where

$$D = \{x : -1 \le x \le 1\}, \tag{10}$$

subject to the boundary conditions

$$u(-1,t) = g_1(t), \quad u(1,t) = g_2(t), \quad t \in [0,T], \tag{11}$$

and the initial condition

$$u(x,0) = f(x), \quad x \in D. \tag{12}$$

Now we assume that

$$u(x,t) = \sum_{j=0}^{N} a_j(t) P_j^{(\alpha,\beta)}(x), \tag{13}$$

and if we make use of (6)–(8), then we find

$$a_j(t) = \frac{1}{h_j} \sum_{i=0}^{N} P_j^{(\alpha,\beta)}\left(x_{N,i}^{(\alpha,\beta)}\right) \varpi_{N,i}^{(\alpha,\beta)} u\left(x_{N,i}^{(\alpha,\beta)}, t\right), \tag{14}$$

and accordingly, (14) takes the form

$$u(x,t) = \sum_{j=0}^{N} \left(\frac{1}{h_j} \sum_{i=0}^{N} P_j^{(\alpha,\beta)}\left(x_{N,i}^{(\alpha,\beta)}\right) \varpi_{N,i}^{(\alpha,\beta)} u\left(x_{N,i}^{(\alpha,\beta)}, t\right) \right) P_j^{(\alpha,\beta)}(x), \tag{15}$$

or equivalently takes the form

$$u(x,t) = \sum_{i=0}^{N} \left(\sum_{j=0}^{N} \frac{1}{h_j} P_j^{(\alpha,\beta)}\left(x_{N,i}^{(\alpha,\beta)}\right) P_j^{(\alpha,\beta)}(x) \varpi_{N,i}^{(\alpha,\beta)} \right) u\left(x_{N,i}^{(\alpha,\beta)}, t\right). \tag{16}$$

The spatial partial derivatives with respect to x in (9) can be computed at the J-GL-C points to give

$$u_x\left(x_{N,n}^{(\alpha,\beta)}, t\right) = \sum_{i=0}^{N} \left(\sum_{j=0}^{N} \frac{1}{h_j} P_j^{(\alpha,\beta)}\left(x_{N,i}^{(\alpha,\beta)}\right) \left(P_j^{(\alpha,\beta)}\left(x_{N,n}^{(\alpha,\beta)}\right)\right)' \varpi_{N,i}^{(\alpha,\beta)} \right) u\left(x_{N,i}^{(\alpha,\beta)}, t\right)$$
$$= \sum_{i=0}^{N} A_{ni} u\left(x_{N,i}^{(\alpha,\beta)}, t\right), \quad n = 0, 1, \ldots, N,$$
$$u_{xx}\left(x_{N,n}^{(\alpha,\beta)}, t\right) = \sum_{i=0}^{N} \left(\sum_{j=0}^{N} \frac{1}{h_j} P_j^{(\alpha,\beta)}\left(x_{N,i}^{(\alpha,\beta)}\right) \left(P_j^{(\alpha,\beta)}\left(x_{N,n}^{(\alpha,\beta)}\right)\right)'' \varpi_{N,i}^{(\alpha,\beta)} \right) u\left(x_{N,i}^{(\alpha,\beta)}, t\right)$$
$$= \sum_{i=0}^{N} B_{ni} u\left(x_{N,i}^{(\alpha,\beta)}, t\right), \tag{17}$$

where

$$A_{ni} = \sum_{j=0}^{N} \frac{1}{h_j} P_j^{(\alpha,\beta)}\left(x_{N,i}^{(\alpha,\beta)}\right) \left(P_j^{(\alpha,\beta)}\left(x_{N,n}^{(\alpha,\beta)}\right)\right)' \varpi_{N,i}^{(\alpha,\beta)},$$
$$B_{ni} = \sum_{j=0}^{N} \frac{1}{h_j} P_j^{(\alpha,\beta)}\left(x_{N,i}^{(\alpha,\beta)}\right) \left(P_j^{(\alpha,\beta)}\left(x_{N,n}^{(\alpha,\beta)}\right)\right)'' \varpi_{N,i}^{(\alpha,\beta)}. \tag{18}$$

Making use of (17) and (18) enables one to rewrite (9) in the form:

$$\dot{u}_n(t) + \nu u_n(t) \sum_{i=0}^{N} A_{ni} u_i(t) - \mu \sum_{i=0}^{N} B_{ni} u_i(t) = 0, \quad n = 1, \ldots, N-1, \tag{19}$$

where

$$u_n(t) = u\left(x_{N,n}^{(\alpha,\beta)}, t\right). \tag{20}$$

Using Equation (19) and using the two-point boundary conditions (11) generate a system of $(N-1)$ ODEs in time:

$$\dot{u}_n(t) + \nu u_n(t) \sum_{i=1}^{N-1} A_{ni} u_i(t) - \mu \sum_{i=1}^{N-1} B_{ni} u_i(t) + \nu d_n(t) - \mu \widetilde{d_n}(t) = 0, \quad n = 1, \ldots, N-1, \tag{21}$$

where

$$\begin{aligned} d_n(t) &= A_{no} g_1(t) + A_{nN} g_2(t), \\ \widetilde{d_n}(t) &= B_{no} g_1(t) + B_{nN} g_2(t). \end{aligned} \tag{22}$$

Then the problem (9)–(12) transforms to the SODEs:

$$\begin{aligned} &\dot{u}_n(t) + \nu u_n(t) \sum_{i=1}^{N-1} A_{ni} u_i(t) - \mu \sum_{i=1}^{N-1} B_{ni} u_i(t) \\ &\quad + \nu d_n(t) - \mu \widetilde{d_n}(t) = 0, \quad n = 1, \ldots, N-1, \\ &u_n(0) = f\left(x_{N,n}^{(\alpha,\beta)}\right), \end{aligned} \tag{23}$$

which may be written in the following matrix form:

$$\begin{aligned} \dot{\mathbf{u}}(t) &= \mathbf{F}(t, u(t)), \\ \mathbf{u}(0) &= \mathbf{f}, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \dot{\mathbf{u}}(t) &= \left[\dot{u}_1(t), \dot{u}_2(t), \ldots, \dot{u}_{N-1}(t)\right]^T, \\ \mathbf{f} &= \left[f(x_{N,1}), f(x_{N,2}), \ldots, f(x_{N,N-1})\right]^T, \\ \mathbf{F}(t, u(t)) &= \left[F_1(t, u(t)), F_2(t, u(t)), \ldots, F_{N-1}(t, u(t))\right]^T, \\ F_n(t, u(t)) &= -\nu u_n(t) \sum_{i=1}^{N-1} A_{ni} u_i(t) + \mu \sum_{i=1}^{N-1} B_{ni} u_i(t) \\ &\quad - \nu d_n(t) + \mu \widetilde{d_n}(t), \quad n = 1, \ldots, N-1. \end{aligned} \tag{25}$$

The SODEs (24) in time may be solved using any standard technique, like the implicit Runge-Kutta method.

3.2. (1+1)-Dimensional Burger-Fisher Equation. The Burger-Fisher equation is a combined form of Fisher and Burgers' equations. The Fisher equation was firstly introduced by Fisher in [39] to describe the propagation of a mutant gene. This equation has a wide range of applications in a large number of the fields of chemical kinetics [40], logistic population growth [41], flame propagation [42], population in one-dimensional habitual [43], neutron population in a nuclear reaction [44], neurophysiology [45], autocatalytic chemical reactions [19], branching the Brownian motion processes [40], and nuclear reactor theory [46]. Moreover, the Burger-Fisher equation has a wide range of applications in various fields of financial mathematics, applied mathematics and physics applications, gas dynamic, and traffic flow. The Burger-Fisher equation can be written in the following form:

$$u_t = u_{xx} - \nu u u_x + \gamma u(1-u); \quad (x,t) \in D \times [0,T], \tag{26}$$

where

$$D = \{x : -1 < x < 1\}, \tag{27}$$

subject to the boundary conditions

$$u(-1,t) = g_1(t), \qquad u(1,t) = g_2(t), \tag{28}$$

and the initial condition

$$u(x,0) = f(x), \quad x \in D. \tag{29}$$

The same procedure of Section 3.1 can be used to reduce (26)–(29) to the system of nonlinear differential equations in the unknown expansion coefficients of the sought-for semianalytical solution. This system is solved by using the implicit Runge-Kutta method.

3.3. (1 + 1)-Dimensional Generalized Burgers-Huxley Equation. The Huxley equation is a nonlinear partial differential equation of second order of the form

$$u_t - u_{xx} - u(k-u)(u-1) = 0; \quad k \neq 0. \tag{30}$$

It is an evolution equation that describes the nerve propagation [47] in biology from which molecular CB properties can be calculated. It also gives a phenomenological description of the behavior of the myosin heads II. In addition to this nonlinear evolution equation, combined forms of this equation and Burgers' equation will be investigated. It is interesting to point out that this equation includes the convection term u_x and the dissipation term u_{xx} in addition to other terms. In this subsection, we derive J-GL-C method to solve numerically the (1+1)-dimensional generalized Burgers-Huxley equation:

$$u_t + \nu u^\delta u_x - u_{xx} - \eta u\left(1 - u^\delta\right)\left(u^\delta - \gamma\right) = 0, \quad (x,t) \in D \times [0,T], \tag{31}$$

where

$$D = \{x : -1 < x < 1\}, \tag{32}$$

subject to the boundary conditions:

$$u(-1,t) = g_1(t), \qquad u(1,t) = g_2(t), \tag{33}$$

and the initial condition:

$$u(x,0) = f(x), \quad x \in D. \tag{34}$$

The same procedure of Sections 3.1 and 3.2 is used to solve numerically (30)–(34).

4. Numerical Results

To illustrate the effectiveness of the proposed method in the present paper, three test examples are carried out in this section. The comparison of the results obtained by various choices of the Jacobi parameters α and β reveals that the present method is very effective and convenient for all choices of α and β. We consider the following three examples.

Example 1. Consider the nonlinear time-dependent one-dimensional generalized Burgers-Huxley equation:

$$u_t = u_{xx} - \nu u^{\delta} u_x + \eta u \left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right); \quad (x,t) \in [A,B] \times [0,T], \tag{35}$$

subject to the boundary conditions:

$$\begin{aligned} u(A,t) &= \left[\frac{\gamma}{2} + \frac{\gamma}{2} \times \tanh\left[\gamma \frac{-\nu\delta + \delta\sqrt{\nu^2 + 4\eta(1+\delta)}}{4(\delta+1)} \times \left(A - \left(\nu\gamma(1+\delta-\gamma)\left(\sqrt{\nu^2+4\eta(1+\delta)} - \nu\right)\right) \times \left(2(\delta+1)^2\right)^{-1} t\right)\right]\right]^{1/\delta}, \\ u(B,t) &= \left[\frac{\gamma}{2} + \frac{\gamma}{2} \times \tanh\left[\gamma \frac{-\nu\delta + \delta\sqrt{\nu^2 + 4\eta(1+\delta)}}{4(\delta+1)} \times \left(B - \left(\nu\gamma(1+\delta-\gamma)\left(\sqrt{\nu^2+4\eta(1+\delta)} - \nu\right)\right) \times \left(2(\delta+1)^2\right)^{-1} t\right)\right]\right]^{1/\delta}, \end{aligned} \tag{36}$$

and the initial condition:

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[x\gamma \frac{-\nu\delta + \delta\sqrt{\nu^2 + 4\eta(1+\delta)}}{4(\delta+1)}\right]\right]^{1/\delta}, \quad x \in [A,B]. \tag{37}$$

The exact solution of (35) is

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2} \times \tanh\left[\gamma \frac{-\nu\delta + \delta\sqrt{\nu^2 + 4\eta(1+\delta)}}{4(\delta+1)} \times \left(x - \frac{\nu\gamma(1+\delta-\gamma)\left(\sqrt{\nu^2+4\eta(1+\delta)} - \nu\right)}{2(\delta+1)^2} t\right)\right]\right]^{1/\delta}. \tag{38}$$

The difference between the measured value of the approximate solution and its actual value (absolute error), given by

$$E(x,t) = \left|u(x,t) - \tilde{u}(x,t)\right|, \tag{39}$$

where $u(x,t)$ and $\tilde{u}(x,t)$, is the exact solution and the approximate solution at the point (x,t), respectively.

In the cases of $\gamma = 10^{-3}$, $\nu = \eta = \delta = 1$, and $N = 4$, Table 1 lists the comparison of absolute errors of problem (35) subject to (36) and (37) using the J-GL-C method for different choices of α and β with references [19], in the interval $[0,1]$. Moreover in Tables 2 and 3, the absolute errors of this problem with $\alpha = \beta = 1/2$ and various choices of x, t for $\delta = 1$ (3), in both intervals $[0,1]$ and $[-1,1]$, are given, respectively. In Table 4, maximum absolute errors with various choices of (α, β) for both values of $\delta = 1, 3$ are given where $\nu = \gamma = \eta = 0.001$, in both intervals $[0,1]$ and $[-1,1]$. Moreover, the absolute errors of problem (35) are shown in Figures 1, 2, and 3 for $\delta = 1, 2$, and 3 with values of parameters listed in their captions, respectively, while in Figure 4, we plotted the approximate solution of this problem where $\alpha = 0$, $\beta = 1$, $\nu = \eta = \gamma = 10^{-3}$, and $N = 12$ for $\delta = 1$. These figures demonstrate the good accuracy of this algorithm for all choices of α, β, and N and moreover in any interval.

Example 2. Consider the nonlinear time-dependent one dimensional Burgers-type equation:

$$u_t + \nu u u_x - \mu u_{xx} = 0; \quad (x,t) \in [A,B] \times [0,T], \tag{40}$$

Table 1: Comparison of absolute errors of Example 1 with results from different articles, where $N = 4$, $\gamma = 10^{-3}$, and $\nu = \eta = \delta = 1$.

(x, t)	ADM [19]	Our method for various values of (α, β) with $N = 4$ $(0, 0)$	$(-1/2, -1/2)$	$(1/2, 1/2)$	$(-1/2, 1/2)$	$(0, 1)$
(0.1, 0.05)	1.94×10^{-7}	2.99×10^{-8}	1.28×10^{-9}	2.32×10^{-8}	1.28×10^{-9}	1.34×10^{-8}
(0.1, 0.1)	3.87×10^{-7}	5.98×10^{-8}	2.56×10^{-9}	4.63×10^{-8}	2.56×10^{-9}	2.68×10^{-8}
(0.1, 1)	38.75×10^{-7}	5.97×10^{-7}	2.46×10^{-8}	4.63×10^{-7}	2.46×10^{-8}	2.68×10^{-7}
(0.5, 0.05)	1.94×10^{-7}	1.02×10^{-8}	5.26×10^{-8}	1.06×10^{-8}	5.26×10^{-8}	3.64×10^{-8}
(0.5, 0.1)	3.87×10^{-7}	2.03×10^{-8}	1.05×10^{-7}	2.13×10^{-8}	1.05×10^{-7}	7.28×10^{-8}
(0.5, 1)	38.57×10^{-7}	2.04×10^{-7}	10.54×10^{-7}	2.13×10^{-7}	10.54×10^{-7}	7.28×10^{-7}
(0.9, 0.05)	1.94×10^{-7}	7.93×10^{-9}	5.22×10^{-8}	4.49×10^{-9}	5.22×10^{-8}	3.07×10^{-8}
(0.9, 0.1)	3.87×10^{-7}	1.59×10^{-8}	1.04×10^{-7}	8.99×10^{-9}	1.04×10^{-7}	6.15×10^{-8}
(0.9, 1)	38.76×10^{-7}	1.59×10^{-7}	10.46×10^{-7}	9.01×10^{-8}	10.46×10^{-7}	6.14×10^{-7}

Table 2: Absolute errors with $\alpha = \beta = 1/2$, $\delta = 1$ and various choices of x, t for Example 1.

x	t	A	B	ν	γ	η	N	E	A	B	E
0.0	0.1	0	1	0.001	0.001	0.001	12	7.04×10^{-12}	−1	1	2.47×10^{-11}
0.1								5.61×10^{-11}			4.19×10^{-12}
0.2								7.29×10^{-11}			7.01×10^{-12}
0.3								8.26×10^{-12}			4.59×10^{-11}
0.4								4.37×10^{-11}			5.74×10^{-11}
0.5								2.47×10^{-11}			1.44×10^{-11}
0.6								7.01×10^{-12}			2.16×10^{-11}
0.7								5.74×10^{-11}			3.74×10^{-11}
0.8								2.15×10^{-11}			1.03×10^{-10}
0.9								1.03×10^{-10}			1.25×10^{-10}
1								4.64×10^{-12}			4.64×10^{-12}
0.0	0.2	0	1	0.001	0.001	0.001	12	1.33×10^{-11}	−1	1	4.92×10^{-11}
0.1								1.11×10^{-10}			8.33×10^{-12}
0.2								1.46×10^{-10}			1.42×10^{-11}
0.3								1.65×10^{-11}			9.20×10^{-11}
0.4								8.74×10^{-11}			1.15×10^{-10}
0.5								4.92×10^{-11}			2.87×10^{-11}
0.6								1.42×10^{-11}			4.30×10^{-11}
0.7								1.14×10^{-10}			7.49×10^{-11}
0.8								4.30×10^{-11}			2.05×10^{-10}
0.9								2.05×10^{-10}			2.51×10^{-10}
1								1.09×10^{-11}			1.09×10^{-11}

subject to the boundary conditions

$$u(A,t) = \frac{c}{\nu} - \frac{c}{\nu}\tanh\left[\frac{c}{2\mu}(A-ct)\right],$$
$$u(B,t) = \frac{c}{\nu} - \frac{c}{\nu}\tanh\left[\frac{c}{2\mu}(B-ct)\right], \tag{41}$$

and the initial condition

$$u(x,0) = \frac{c}{\nu} - \frac{c}{\nu}\tanh\left[\frac{c}{2\mu}x\right], \quad x \in [A,B]. \tag{42}$$

If we apply the generalized tanh method [48], then we find that the analytical solution of (40) is

$$u(x,t) = \frac{c}{\nu} - \frac{c}{\nu}\tanh\left[\frac{c}{2\mu}(x-ct)\right]. \tag{43}$$

In Table 5, the maximum absolute errors of (40) subject to (41) and (42) are introduced using the J-GL-C method, with various choices of (α, β) in both intervals $[0, 1]$ and $[-1, 1]$. Absolute errors between exact and numerical solutions of this problem are introduced in Table 6 using the J-GL-C method for $\alpha = \beta = 1/2$ with $N = 20$, and $\nu = 10$, $\mu = 0.1$ and $c = 0.1$ in both intervals $[0, 1]$ and $[-1, 1]$. In Figures 5, 6, and 7, we displayed the absolute errors of problem (40) for

TABLE 3: Absolute errors with $\alpha = \beta = 1/2$, $\delta = 3$ and various choices of x, t for Example 1.

x	t	A	B	ν	γ	η	N	E	A	B	E
0.0	0.1	0	1	0.001	0.001	0.001	12	5.62×10^{-8}	−1	1	8.32×10^{-9}
0.1								6.50×10^{-6}			2.99×10^{-6}
0.2								4.67×10^{-6}			1.65×10^{-6}
0.3								3.08×10^{-6}			2.44×10^{-6}
0.4								1.65×10^{-6}			3.09×10^{-6}
0.5								8.33×10^{-9}			1.97×10^{-6}
0.6								1.65×10^{-6}			4.67×10^{-6}
0.7								3.09×10^{-6}			3.79×10^{-6}
0.8								4.67×10^{-6}			6.53×10^{-6}
0.9								6.53×10^{-6}			35.32×10^{-6}
1								5.66×10^{-8}			5.66×10^{-8}
0.0	0.2	0	1	0.001	0.001	0.001	12	2.27×10^{-7}	−1	1	2.54×10^{-8}
0.1								12.98×10^{-6}			5.97×10^{-6}
0.2								9.32×10^{-6}			3.32×10^{-6}
0.3								6.16×10^{-6}			4.89×10^{-6}
0.4								3.30×10^{-6}			6.21×10^{-6}
0.5								2.54×10^{-8}			3.93×10^{-6}
0.6								3.32×10^{-6}			9.36×10^{-6}
0.7								6.21×10^{-6}			7.58×10^{-6}
0.8								9.36×10^{-6}			13.10×10^{-6}
0.9								13.10×10^{-6}			70.74×10^{-6}
1								2.27×10^{-7}			2.27×10^{-7}

TABLE 4: Maximum absolute errors with various choices of (α, β) for both values of $\delta = 1, 3$ Example 3.

α	β	A	B	ν	γ	η	δ	N	M_E	A	B	M_E
0	0	0	1	0.001	0.001	0.001	1	12	1.39×10^{-9}	−1	1	2.55×10^{-9}
1/2	1/2								6.38×10^{-10}			1.72×10^{-9}
−1/2	−1/2								3.83×10^{-9}			4.65×10^{-9}
−1/2	1/2								5.04×10^{-9}			4.58×10^{-9}
0	1								2.16×10^{-9}			2.42×10^{-9}
0	0	0	1	0.001	0.001	0.001	3	12	1.84×10^{-4}	−1	1	14.75×10^{-4}
1/2	1/2								2.23×10^{-5}			4.13×10^{-4}
−1/2	−1/2								33.44×10^{-4}			81.17×10^{-4}
−1/2	1/2								56.96×10^{-4}			90.97×10^{-4}
0	1								5.24×10^{-4}			16.46×10^{-4}

different numbers of collation points and different choices of α and β in interval $[0, 1]$ with values of parameters being listed in their captions. Moreover, in Figure 8, we see that, the approximate solution and the exact solution are almost coincided for different values of t (0, 0.5 and 0.9) of problem (40) where $\nu = 10$, $\mu = 0.1$, $c = 0.1$, $\alpha = \beta = -0.5$, and $N = 20$ in interval $[-1, 1]$.

Example 3. Consider the nonlinear time-dependent one-dimensional generalized Burger-Fisher-type equation:

$$u_t = u_{xx} - \nu u^{\delta} u_x + \gamma u \left(1 - u^{\delta}\right); \quad (x, t) \in [A, B] \times [0, T], \tag{44}$$

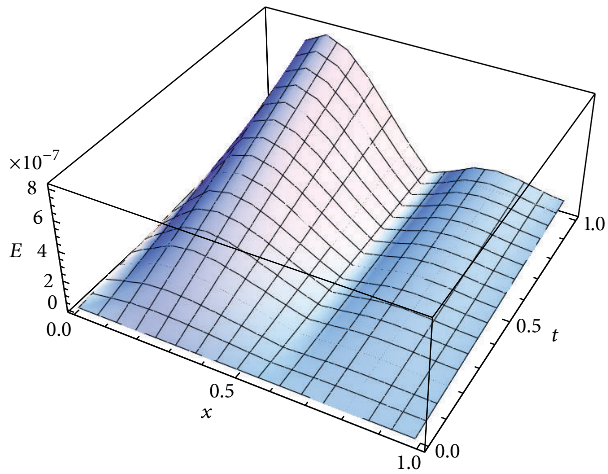

FIGURE 1: The absolute error of problem (35) where $\alpha = \beta = 0$, $\nu = \eta = \gamma = 10^{-3}$, and $N = 4$ for $\delta = 1$.

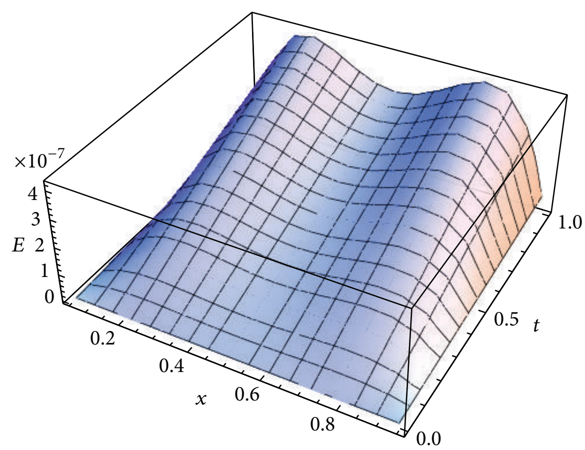

FIGURE 2: The absolute error of problem (35) where $\alpha = \beta = 1/2$, $\nu = \eta = \gamma = 10^{-3}$, and $N = 4$ for $\delta = 2$.

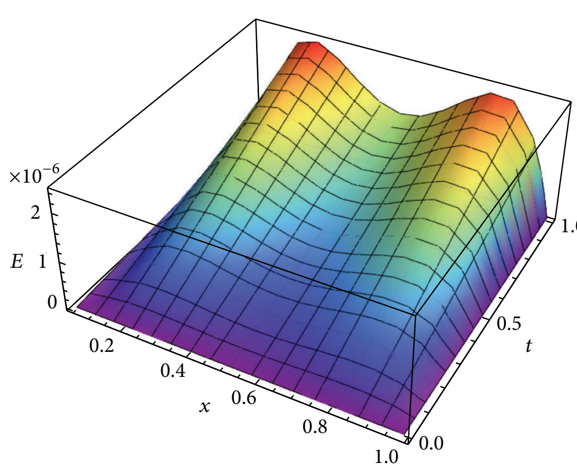

FIGURE 3: The absolute error of problem (35) where $-\alpha = \beta = 1/2$, $\nu = \eta = \gamma = 10^{-3}$, and $N = 4$ for $\delta = 3$.

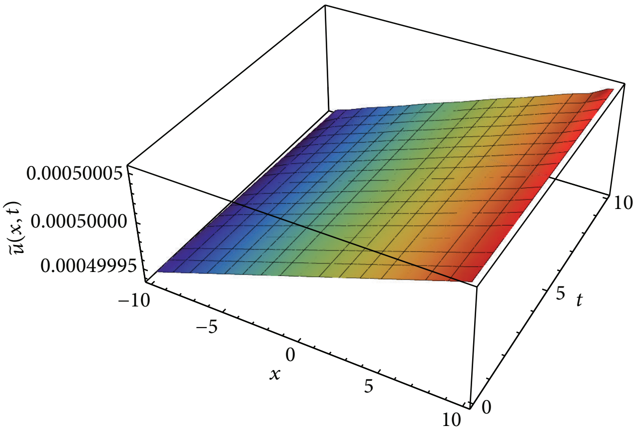

FIGURE 4: The approximate solution of problem (35) where $\alpha = 0$, $\beta = 1$, $\nu = \eta = \gamma = 10^{-3}$, and $N = 12$ for $\delta = 1$.

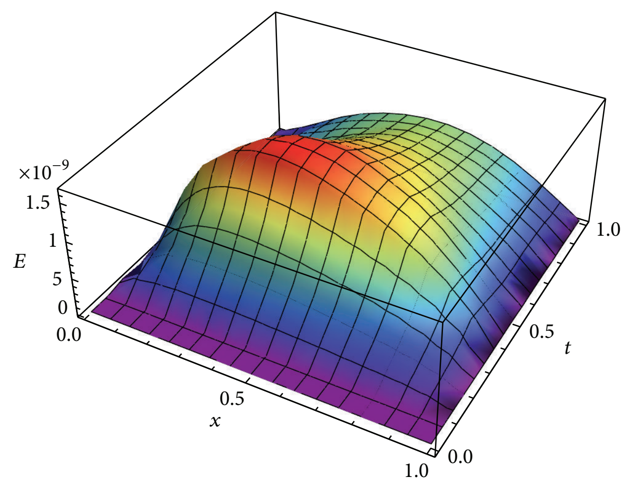

FIGURE 5: The absolute error of problem (40) where $\nu = 10$, $\mu = 0.1$, $c = 0.1$, $-\alpha = \beta = 1/2$, and $N = 12$.

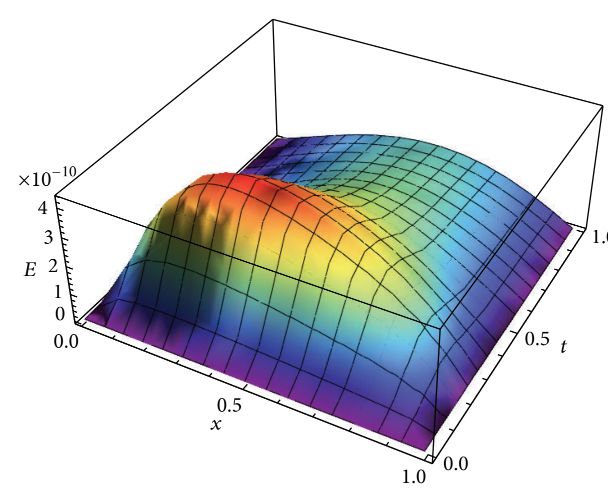

FIGURE 6: The absolute error of problem (40) where $\nu = 10$, $\mu = 0.1$, $c = 0.1$, $\alpha = \beta = 1/2$, and $N = 20$.

Table 5: Maximum absolute errors with various choices of (α, β) for Example 2.

α	β	A	B	μ	ν	N	M_E	A	B	M_E
0	0	0	1	0.1	10	4	1.49×10^{-6}	−1	1	5.62×10^{-7}
1/2	1/2						2.45×10^{-6}			8.25×10^{-7}
−1/2	−1/2						6.51×10^{-7}			6.51×10^{-7}
−1/2	1/2						3.84×10^{-6}			4.67×10^{-7}
1/2	−1/2						1.20×10^{-6}			1.20×10^{-6}
0	0	0	1	0.1	10	16	1.43×10^{-9}	−1	1	1.06×10^{-9}
1/2	1/2						1.62×10^{-9}			1.18×10^{-9}
−1/2	−1/2						6.70×10^{-10}			4.45×10^{-10}
−1/2	1/2						6.68×10^{-10}			4.49×10^{-10}
1/2	−1/2						6.67×10^{-10}			4.23×10^{-10}

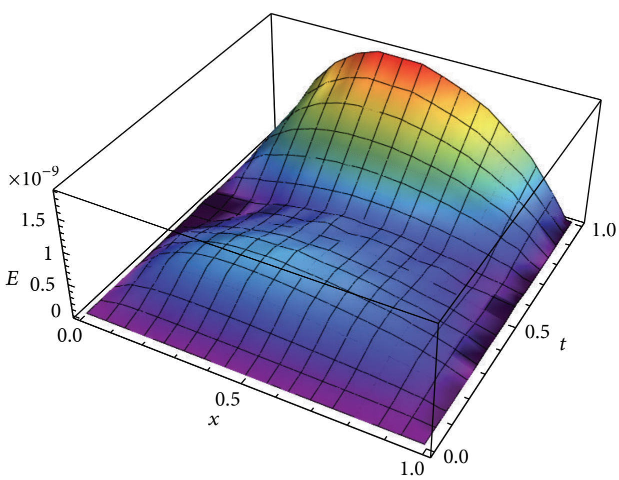

Figure 7: The absolute error of problem (40) where $\nu = 10$, $\mu = 0.1$, $c = 0.1$, $\alpha = \beta = -1/2$, and $N = 16$.

subject to the boundary conditions

$$u(A,t) = \left[\frac{1}{2} - \frac{1}{2} \times \tanh\left[\frac{\nu\delta}{2(\delta+1)}\left(A - \left(\frac{\nu}{\delta+1} + \frac{\gamma(\delta+1)}{\nu}\right)t\right)\right]\right]^{1/\delta},$$

$$u(B,t) = \left[\frac{1}{2} - \frac{1}{2} \times \tanh\left[\frac{\nu\delta}{2(\delta+1)}\left(B - \left(\frac{\nu}{\delta+1} + \frac{\gamma(\delta+1)}{\nu}\right)t\right)\right]\right]^{1/\delta}, \quad (45)$$

Table 6: Absolute errors with $\alpha = \beta = 1/2$ and various choices of x, t for Example 2.

x	t	A	B	μ	ν	N	E	A	B	E
0.0	0.1	0	1	0.1	10	20	1.34×10^{-12}	−1	1	9.21×10^{-11}
0.1							8.71×10^{-11}			8.70×10^{-11}
0.2							1.07×10^{-10}			8.16×10^{-11}
0.3							1.07×10^{-10}			7.61×10^{-11}
0.4							1.01×10^{-10}			7.01×10^{-11}
0.5							9.21×10^{-11}			6.38×10^{-11}
0.6							8.16×10^{-11}			5.70×10^{-11}
0.7							7.01×10^{-11}			4.88×10^{-11}
0.8							5.70×10^{-11}			3.85×10^{-11}
0.9							3.85×10^{-11}			2.30×10^{-11}
1							1.34×10^{-12}			1.34×10^{-12}
0.0	0.2	0	1	0.1	10	20	1.77×10^{-12}	−1	1	3.35×10^{-10}
0.1							2.71×10^{-10}			3.17×10^{-10}
0.2							3.61×10^{-10}			2.98×10^{-10}
0.3							3.78×10^{-10}			2.77×10^{-10}
0.4							3.64×10^{-10}			2.55×10^{-10}
0.5							3.34×10^{-10}			2.31×10^{-10}
0.6							2.98×10^{-10}			2.03×10^{-10}
0.7							2.55×10^{-10}			1.71×10^{-10}
0.8							2.02×10^{-10}			1.30×10^{-10}
0.9							1.30×10^{-10}			7.70×10^{-11}
1							1.77×10^{-12}			1.77×10^{-12}

and the initial condition

$$u(x,0) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\nu\delta}{2(\delta+1)}(x)\right]\right]^{1/\delta}, \quad x \in [A,B]. \quad (46)$$

The exact solution of (44) is

$$u(x,t) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left[\frac{\nu\delta}{2(\delta+1)}\left(x - \left(\frac{\nu}{\delta+1} + \frac{\gamma(\delta+1)}{\nu}\right)t\right)\right]\right]^{1/\delta}. \quad (47)$$

In Table 7, we listed a comparison of absolute errors of problem (44) subject to (45) and (46) using the J-GL-C method with [19]. Absolute errors between exact and numerical solutions of (44) subject to (45) and (46) are introduced in Table 8 using the J-GL-C method for $\alpha = \beta = 0$ with $N = 16$, respectively, and $\nu = \gamma = 10^{-2}$. In Figures 9 and 10, we displayed the absolute errors of problem (44) where $\nu = \gamma = 10^{-2}$ at $N = 20$ and ($\alpha = \beta = 0$ and $\alpha = \beta = -1/2$) in interval $[-1, 1]$, respectively. Moreover, in Figures 11 and 12, we see that, in interval $[-1, 1]$, the approximate solution and the exact solution are almost coincided for different values of t (0, 0.5 and 0.9) of problem (44) where $\nu = \gamma = 10^{-2}$ at $N = 20$ and ($\alpha = \beta = 0$ and $\alpha = \beta = -1/2$), respectively. This asserts that the obtained numerical results are accurate and can be compared favorably with the analytical solution.

Table 7: Comparison of absolute errors of Example 3 with results from [19], where $N = 4$, $\alpha = \beta = 0$, and various choices of x, t.

x	t	γ	ν	δ	[19]	E	x	t	γ	ν	δ	[19]	E
0.1	0.005	0.001	0.001	1	9.69×10^{-6}	1.85×10^{-6}	0.1	0.0005	1	1	2	1.40×10^{-3}	3.83×10^{-5}
	0.001				1.94×10^{-6}	3.72×10^{-7}		0.0001				2.80×10^{-4}	3.88×10^{-5}
	0.01				1.94×10^{-5}	3.72×10^{-6}		0.001				2.80×10^{-3}	3.76×10^{-5}
0.5	0.005				9.69×10^{-6}	7.04×10^{-6}	0.5	0.0005				1.35×10^{-3}	2.32×10^{-5}
	0.001				1.94×10^{-6}	1.41×10^{-6}		0.0001				2.69×10^{-4}	2.38×10^{-5}
	0.01				1.94×10^{-5}	1.41×10^{-5}		0.001				2.69×10^{-3}	2.25×10^{-5}
0.9	0.005				9.69×10^{-6}	3.21×10^{-6}	0.9	0.0005				1.28×10^{-3}	1.58×10^{-5}
	0.001				1.94×10^{-6}	6.42×10^{-7}		0.0001				2.55×10^{-4}	1.55×10^{-5}
	0.01				1.94×10^{-5}	6.42×10^{-6}		0.001				2.55×10^{-3}	1.61×10^{-5}

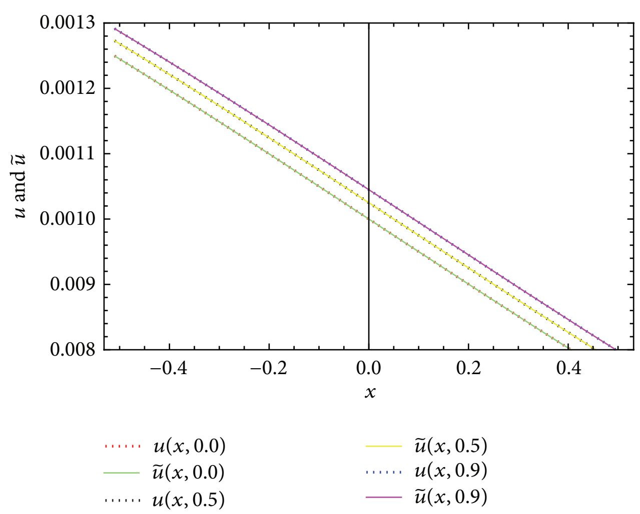

Figure 8: The approximate and exact solutions for different values of t (0, 0.5 and 0.9) of problem (40) where $\nu = 10$, $\mu = 0.1$, $c = 0.1$, $\alpha = \beta = -0.5$, and $N = 20$.

Table 8: Absolute errors with $\alpha = \beta = 0$, $\delta = 1$ and various choices of x, t for Example 3.

x	t	γ	ν	δ	N	E	x	t	γ	ν	δ	N	E
0.0	0.1	1	1	1	16	3.26×10^{-9}	0.0	0.2	1	1	1	16	3.57×10^{-11}
0.1						3.82×10^{-9}	0.1						7.75×10^{-11}
0.2						4.28×10^{-9}	0.2						1.96×10^{-10}
0.3						4.66×10^{-9}	0.3						3.52×10^{-10}
0.4						4.93×10^{-9}	0.4						5.62×10^{-10}
0.5						5.05×10^{-9}	0.5						7.31×10^{-10}
0.6						4.93×10^{-9}	0.6						7.87×10^{-10}
0.7						4.49×10^{-9}	0.7						8.23×10^{-10}
0.8						3.61×10^{-9}	0.8						8.17×10^{-10}
0.9						2.26×10^{-9}	0.9						8.05×10^{-10}
1.0						1.13×10^{-11}	1.0						1.09×10^{-11}

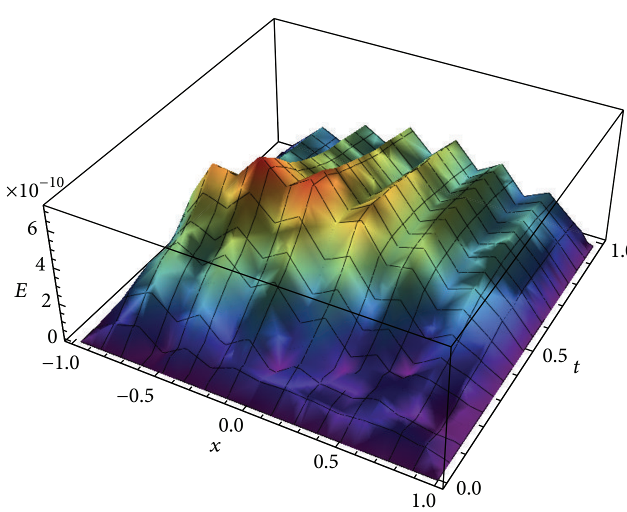

Figure 9: The absolute error of problem (44) where $\alpha = \beta = 0$ and $\nu = \gamma = 10^{-2}$ at $N = 20$.

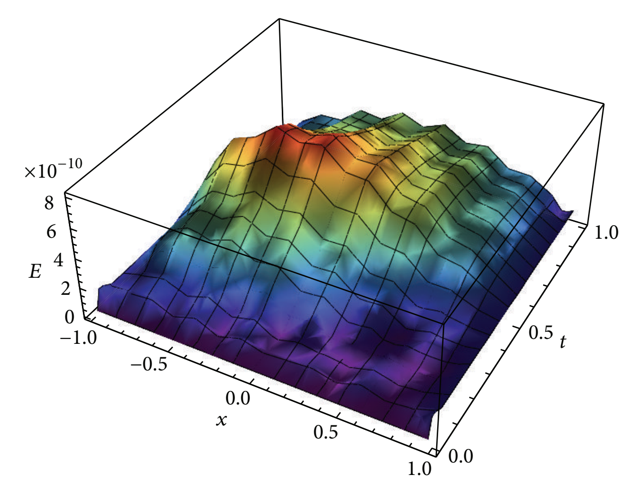

Figure 10: The absolute error of problem (44) where $\alpha = \beta = 1/2$ and $\nu = \gamma = 10^{-2}$ at $N = 20$.

5. Conclusion

An efficient and accurate numerical scheme based on the J-GL-C spectral method is proposed to solve nonlinear time-dependent Burgers-type equations. The problem is reduced to the solution of a SODEs in the expansion coefficient of the solution. Numerical examples were given to demonstrate

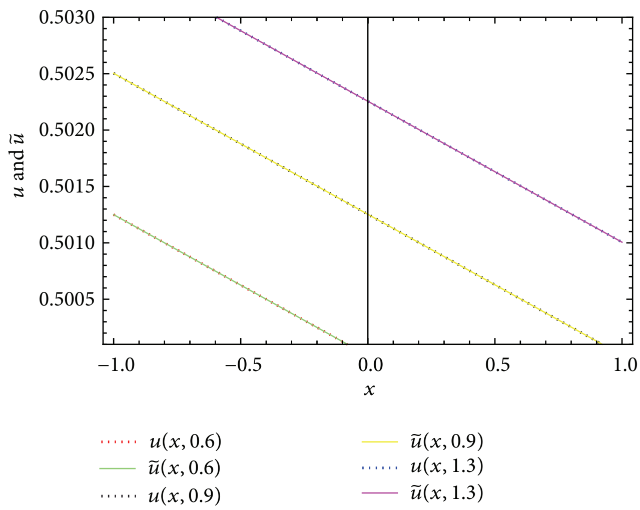

FIGURE 11: The approximate and exact solutions for different values of t (0, 0.5 and 0.9) of problem (44) where $\alpha = \beta = 0$ and $\nu = \gamma = 10^{-2}$ at $N = 20$.

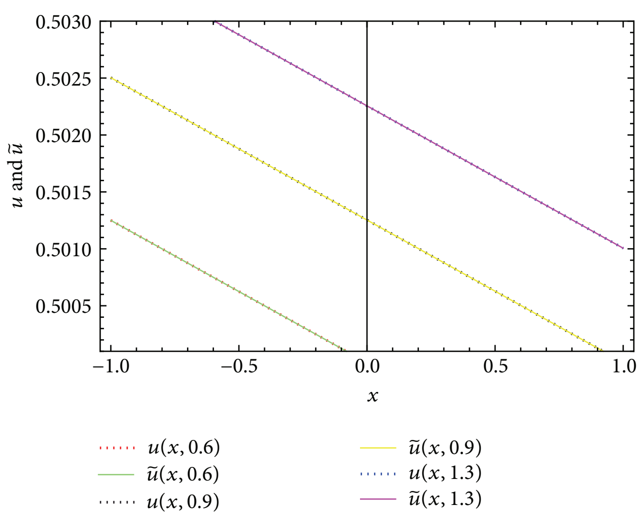

FIGURE 12: The approximate and exact solutions for different values of t (0, 0.5 and 0.9) of problem (44) where $\alpha = \beta = 1/2$ and $\nu = \gamma = 10^{-2}$ at $N = 20$.

the validity and applicability of the method. The results show that the J-GL-C method is simple and accurate. In fact by selecting few collocation points, excellent numerical results are obtained.

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