Research Article

A Modified Generalized Laguerre-Gauss Collocation Method for Fractional Neutral Functional-Differential Equations on the Half-Line

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The modified generalized Laguerre-Gauss collocation (MGLC) method is applied to obtain an approximate solution of fractional neutral functional-differential equations with proportional delays on the half-line. The proposed technique is based on modified generalized Laguerre polynomials and Gauss quadrature integration of such polynomials. The main advantage of the present method is to reduce the solution of fractional neutral functional-differential equations into a system of algebraic equations. Reasonable numerical results are achieved by choosing few modified generalized Laguerre-Gauss collocation points. Numerical results demonstrate the accuracy, efficiency, and versatility of the proposed method on the half-line.

1. Introduction

The fractional calculus which means the calculus of derivatives and integral of any order started to be used as a powerful tool in various disciplines from science and engineering [1, 2]. The models involving fractional derivatives and integrals have memory; therefore it has proven to be very suitable for the description of memory and hereditary properties of various processes [3–11]. Interested readers can also see [12–21].

Neutral functional-differential equations play an important role in the mathematical modeling of several phenomena. It is well known that most of delay differential equations cannot be solved exactly. Therefore, numerical methods would be presented and developed to get approximate solutions of these equations. The rational approximation scheme has been proposed by Ishiwata and Muroya [22], for approximating the solution of delay differential equations. Moreover, the authors of [23] investigated the variational iteration method to find a closed form analytical solution of a class of the neutral functional-differential equations. Recently,

the authors of [24] proposed the Chebyshev cardinal functions in combination of operational matrices for numerical solution of delay differential equations which arise in electrodynamics. Very recently, Tohidi et al. [25] proposed and developed an efficient collocation method which depends on Bernoulli operational matrix for numerical solution of a class of delay differential equations. Meanwhile, Doha et al. [26] proposed an accurate approximate solution of generalized pantograph equations on the half-line using Jacobi rational-Gauss collocation method. In this paper we propose an approximate solution of a class of delay differential equations, namely, fractional neutral functional-differential equations (FNFDEs) with proportional delay.

In the last decade or so, comprehensive research has been accomplished on the development of numerical algorithms which are numerically stable for both linear and nonlinear FDEs. Tripathi et al. [27] presented a new operational matrix of hat functions to solve linear FDEs. The spectral tau method was proposed in [28] to achieve an accurate solution of linear and nonlinear FDEs subject to multipoint conditions. In [29],

Ma and Huang developed spectral collocation method for solving linear fractional integrodifferential equations. Yang and Huang [30] analyzed and developed the Jacobi collocation scheme for pantograph integrodifferential equations with fractional orders in finite interval. In [31], Yin et al. proposed a new fractional-order Legendre function with spectral method to solve partial FDEs; based on the operational matrix of these functions, the same authors developed their approach in combination with variational iteration formula to solve a class of FDEs; see [32]. More recently, the Jacobi Galerkin method was extended in [33] to solve stochastic FDEs, for fractional differential equations in unbounded domains. The operation matrices of fractional derivatives and fractional integrals of generalized Laguerre polynomials were investigated for solving multiterm FDEs on a semi-infinite interval; see [11, 12]. The generalized Laguerre spectral tau and collocation techniques were given in [12] to solve linear and nonlinear FDEs on the half- line. These spectral techniques were developed and generalized by using the modified generalized Laguerre polynomials in [34–36].

Our fundamental goal of this paper is to develop a suitable way to approximate the neutral fractional functional-differential equations with proportional delays on the interval $(0, \infty)$ using the modified generalized Laguerre polynomials; we propose the spectral modified generalized Laguerre-Gauss collocation (MGLC) method to find the solution $u_N(x)$. The modified generalized Laguerre spectral collocation (MGLC) approximation, which is more reliable, is employed to obtain approximate solution of neutral fractional functional-differential equations with proportional delays of order ν ($m - 1 < \nu < m$) and m initial conditions. For suitable collocation points we use the $(N - m + 1)$ nodes of the modified generalized Laguerre-Gauss interpolation on $(0, \infty)$. These equations together with initial conditions generate $(N + 1)$ algebraic equations which can be solved. Finally, the accuracy of the proposed methods is demonstrated by test problems; numerical results are presented in which the usual exponential convergence behavior of spectral approximations is exhibited.

This paper is organized as follows. In Section 2 we give an overview of modified generalized Laguerre polynomials and their relevant properties needed hereafter, and in Section 3, the way of constructing the collocation technique for neutral fractional functional-differential equations with proportional delays is described using the modified generalized Laguerre polynomials. In Section 4, we present some numerical results exhibiting the accuracy and efficiency of our numerical algorithms. The last section offers some obtained conclusions.

2. Some Basic Preliminaries

We give some definitions and properties of fractional derivatives and modified generalized Laguerre polynomials.

2.1. The Fractional Derivative in the Caputo Sense. The two most commonly used definitions are the Riemann-Liouville operator and the Caputo operator.

Definition 1. The Riemann-Liouville fractional integral operator of order ν ($\nu > 0$) is defined as

$$J^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu - 1} f(t) \, dt, \quad \nu > 0, \ x > 0,$$

$$J^0 f(x) = f(x). \tag{1}$$

Definition 2. The Caputo fractional derivatives of order ν are defined as

$$D^\nu f(x) = J^{m-\nu} D^m f(x)$$

$$= \frac{1}{\Gamma(m - \nu)} \int_0^x (x - t)^{m-\nu-1} \frac{d^m}{dt^m} f(t) \, dt, \tag{2}$$

$$m - 1 < \nu < m, \ x > 0,$$

where D^m is the classical differential operator of order m.

For the Caputo derivative we have

$$D^\nu C = 0, \quad (C \text{ is a constant}), \tag{3}$$

$$D^\nu x^\beta = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \nu)} x^{\beta - \nu}, & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil \nu \rceil \\ & \text{or } \beta \notin N \text{ and } \beta > \lfloor \nu \rfloor, \end{cases} \tag{4}$$

where $\lceil \nu \rceil$ and $\lfloor \nu \rfloor$ are the ceiling and floor functions, respectively, while $N = \{1, 2, \ldots\}$ and $N_0 = \{0, 1, 2, \ldots\}$.

Caputo's fractional differentiation is a linear operation, similar to the integer-order differentiation

$$D^\nu (\lambda f(x) + \mu g(x)) = \lambda D^\nu f(x) + \mu D^\nu g(x), \tag{5}$$

where λ and μ are constants.

2.2. Properties of Modified Generalized Laguerre Polynomials.
Let $\Lambda = (0, \infty)$ and let $w^{(\alpha, \beta)}(x) = x^\alpha e^{-\beta x}$ be a weight function on Λ in the usual sense. Define

$$L^2_{w^{(\alpha,\beta)}}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{w^{(\alpha,\beta)}} < \infty\}, \tag{6}$$

equipped with the following inner product and norm:

$$(u, v)_{w^{(\alpha,\beta)}} = \int_\Lambda u(x) v(x) w^{(\alpha,\beta)}(x) \, dx,$$

$$\|v\|_{w^{(\alpha,\beta)}} = (u, v)^{1/2}_{w^{(\alpha,\beta)}}. \tag{7}$$

Next, let $L_i^{(\alpha,\beta)}(x)$ be the modified generalized Laguerre polynomial of degree i for $\alpha > -1$ and $\beta > 0$ is defined by

$$L_i^{(\alpha,\beta)}(x) = \frac{1}{i} x^{-\alpha} e^{\beta x} \partial_x^i \left(x^{i+\alpha} e^{-\beta x} \right), \quad i = 1, 2, \ldots. \tag{8}$$

Moreover, for $\alpha > -1$ and $\beta > 0$, we have

$$\partial_x L_i^{(\alpha,\beta)}(x) = -\beta L_{i-1}^{(\alpha+1,\beta)}(x),$$

$$L_{i+1}^{(\alpha,\beta)}(x) = \frac{1}{i+1}\left[(2i+\alpha+1-\beta x)L_i^{(\alpha,\beta)}(x)\right.$$
$$\left. -(i+\alpha)L_{i-1}^{(\alpha,\beta)}(x)\right], \quad i = 1, 2, \ldots, \tag{9}$$

where $L_0^{(\alpha,\beta)}(x) = 1$ and $L_1^{(\alpha,\beta)}(x) = -\beta x + \Gamma(\alpha+2)/\Gamma(\alpha+1)$.

The set of modified generalized Laguerre polynomials is the $L_{w^{(\alpha,\beta)}}^2(\Lambda)$-orthogonal system; namely,

$$\int_0^\infty L_j^{(\alpha,\beta)}(x) L_k^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x)\, dx = h_k \delta_{jk}, \tag{10}$$

where δ_{jk} is the Kronecker function and $h_k = \Gamma(k + \alpha + 1)/\beta^{\alpha+1}k!$.

The modified generalized Laguerre polynomials of degree i on the interval Λ are given by

$$L_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{i}(-1)^k \frac{\Gamma(i+\alpha+1)\beta^k}{\Gamma(k+\alpha+1)(i-k)!k!}x^k, \tag{11}$$
$$i = 0, 1, \ldots,$$

where $L_i^{(\alpha,\beta)}(0) = \Gamma(i+\alpha+1)/\Gamma(\alpha+1)\Gamma(i+1)$.

The special value

$$D^q L_i^{(\alpha,\beta)}(0) = \frac{(-1)^q \beta^q \Gamma(i+\alpha+1)}{(i-q)!\Gamma(q+\alpha+1)}, \quad i \geqslant q, \tag{12}$$

will be of important use later.

Since the analytic form of the modified generalized Laguerre polynomials $L_i^{(\alpha,\beta)}(x)$ of degree i is given by (11), by using (4), (5), and (11) we get

$$D^\nu L_i^{(\alpha,\beta)}(x)$$
$$= \sum_{k=0}^{i}(-1)^k \frac{\beta^k \Gamma(i+\alpha+1)}{(i-k)!k!\Gamma(k+\alpha+1)}D^\nu x^k \tag{13}$$
$$= \sum_{k=\lceil\nu\rceil}^{i}(-1)^k \frac{\beta^k \Gamma(i+\alpha+1)}{(i-k)!\Gamma(k-\nu+1)\Gamma(k+\alpha+1)}x^{k-\nu},$$
$$i = \lceil\nu\rceil, \ldots, N.$$

Now, approximating $x^{k-\nu}$ by $N + 1$ terms of modified generalized Laguerre series, we have

$$x^{k-\nu} = \sum_{j=0}^{N} b_j L_j^{(\alpha,\beta)}(x), \tag{14}$$

where b_j is given from (18) with $u(x) = x^{k-\nu}$, and

$$b_j = \sum_{\ell=0}^{j}(-1)^\ell \frac{\beta^{-k+\nu}j!\,\Gamma(k-\nu+\alpha+\ell+1)}{(j-\ell)!\,(\ell)!\,\Gamma(\ell+\alpha+1)}. \tag{15}$$

Employing (13)–(15) we get

$$D^\nu L_i^{(\alpha,\beta)}(x) = \sum_{j=0}^{N}\Psi_\nu(i,j)L_j^{(\alpha,\beta)}(x), \quad i = \lceil\nu\rceil, \ldots, N, \tag{16}$$

where $\Psi_\nu(i,j) = \sum_{k=\lceil\nu\rceil}^{i}\theta_{ijk}$, and

$$\theta_{ijk} = \sum_{\ell=0}^{j}\left((-1)^{k+\ell}\beta^\nu j!\,\Gamma(i+\alpha+1)\Gamma(k-\nu+\alpha+\ell+1)\right)$$
$$\times\left((i-k)!\,(j-\ell)!\,\ell!\,\Gamma(k-\nu+1)\right.$$
$$\left.\times\Gamma(k+\alpha+1)\Gamma(\alpha+\ell+1)\right)^{-1}. \tag{17}$$

A function $u(x) \in L_{w^{(\alpha,\beta)}}^2(\Lambda)$ may be expressed in terms of modified generalized Laguerre polynomials as

$$u(x) = \sum_{j=0}^{\infty}a_j L_j^{(\alpha,\beta)}(x), a_j$$
$$= \frac{1}{h_k}\int_0^\infty u(x)L_j^{(\alpha,\beta)}(x)w^{(\alpha,\beta)}(x)\,dx, \tag{18}$$
$$j = 0, 1, 2, \ldots.$$

In practice, only the first $(N+1)$ terms of modified generalized Laguerre polynomials are considered. Then we have

$$u_N(x) = \sum_{j=0}^{N}a_j L_j^{(\alpha,\beta)}(x) = C^T\phi(x). \tag{19}$$

2.3. Modified Generalized Laguerre-Gauss and Gauss-Radau Quadratures. Let $\xi_{G,i,j}^{(\alpha,\beta)}$ and $\xi_{R,i,j}^{(\alpha,\beta)}$, $0 \leq j \leq i$, be the zeros of $L_{i+1}^{(\alpha,\beta)}(x)$ and $x\partial_x L_{i+1}^{(\alpha,\beta)}(x)$, respectively. They are arranged in ascending order. Denote by $\omega_{Z,i,j}^{(\alpha,\beta)}$, $(0 \leq j \leq i)$, $Z = G, R$, the corresponding Christoffel numbers such that

$$\int_\Lambda \phi(x)w^{(\alpha,\beta)}(x)\,dx = \sum_{j=0}^{i}\phi\left(\xi_{Z,i,j}^{(\alpha,\beta)}\right)\omega_{Z,i,j}^{(\alpha,\beta)}, \tag{20}$$

where

$$\omega_{G,i,j}^{(\alpha,\beta)} = \frac{\Gamma(i+\alpha+2)}{\beta^\alpha\Gamma(i+2)}\frac{1}{\xi_{G,i,j}^{(\alpha,\beta)}\left[\partial_x L_{i+1}^{(\alpha,\beta)}\left(\xi_{G,i,j}^{(\alpha,\beta)}\right)\right]^2}, \quad 0 \leq j \leq i. \tag{21}$$

For the Gauss-Radau weights, we have

$$\omega_{R,i,j}^{(\alpha,\beta)} = \begin{cases} \dfrac{(\alpha+1)\Gamma^2(\alpha+1)\Gamma(i+1)}{\beta^{\alpha+1}\Gamma(i+\alpha+2)}, & j = 0, \\[2mm] \dfrac{\Gamma(i+\alpha+1)}{\beta^\alpha\Gamma(i+2)} \\[1mm] \quad\times\dfrac{1}{L_{i+1}^{(\alpha,\beta)}\left(\xi_{R,i,j}^{(\alpha,\beta)}\right)\partial_x L_i^{(\alpha,\beta)}\left(\xi_{R,i,j}^{(\alpha,\beta)}\right)}, & 1 \leq j \leq i. \end{cases} \tag{22}$$

Note that the earlier two types of quadratures have close relations:

$$\xi_{R,i,j}^{(\alpha,\beta)} = \xi_{G,i-1,j-1}^{(\alpha+1,\beta)},$$

$$x\omega_{R,i,j}^{(\alpha,\beta)} = \left(\xi_{R,i,j}^{(\alpha,\beta)}\right)^{-1} \omega_{G,i-1,j-1}^{(\alpha+1,\beta)}, \tag{23}$$

$$1 \le j \le i.$$

In the next section, we will extend the application of spectral methods (see [37–42]) by using the modified generalized Laguerre polynomials to solve fractional neutral functional-differential equations on a semi-infinite domain.

3. Fractional Neutral Functional-Differential Equations

In this section, we use the modified generalized Laguerre pseudospectral method based on the modified generalized Laguerre-Gauss quadrature points as the collocation nodes, to solve numerically the following model problem:

$$\left(u\left(x\right) + a\left(x\right)u\left(p_m x\right)\right)^{(\nu)}$$

$$= \beta u\left(x\right) + \sum_{n=0}^{m-1} b_n\left(x\right)D^{\gamma_n}u\left(p_n x\right) + f\left(x\right), \quad x \ge 0, \tag{24}$$

with the initial conditions

$$\sum_{n=0}^{m-1} c_{in}u^{(n)}\left(0\right) = \lambda_i, \quad i = 0, 1, \dots, m-1. \tag{25}$$

Here, a and b_n ($n = 0, 1, \dots, m-1$) are given analytical functions; $m-1 < \nu \le m$, $0 < \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \nu$, and β, p_n, c_{in}, λ_i denote given constants with $0 < p_n < 1$ ($n = 0, 1, \dots, m$). By using the modified generalized Laguerre-Gauss collocation method, we can approximate the fractional neutral functional-differential equations with proportional delays on a semi-infinite domain directly, without any artificial boundary and variable transformation. Let us first introduce some basic notation that will be used in the sequel. We set

$$S_N\left(0,\infty\right) = \text{span}\left\{L_0^{(\alpha,\beta)}\left(x\right), L_1^{(\alpha,\beta)}\left(x\right), \dots, L_N^{(\alpha,\beta)}\left(x\right)\right\}, \tag{26}$$

and we define the discrete inner product and norm as follows:

$$\left(u,v\right)_{w_N^{(\alpha,\beta)}} = \sum_{j=0}^{N} u\left(\xi_{Z,N,j}^{(\alpha,\beta)}\right)v\left(\xi_{Z,N,j}^{(\alpha,\beta)}\right)\omega_{Z,N,j}^{(\alpha,\beta)}, \tag{27}$$

$$\|u\|_{w_N^{(\alpha,\beta)}} = \sqrt{\left(u,u\right)_{w_N^{(\alpha,\beta)}}},$$

where $\xi_{Z,N,j}^{(\alpha,\beta)}$ and $\omega_{Z,N,j}^{(\alpha,\beta)}$ are the nodes and the corresponding weights of the modified generalized Laguerre-Gauss quadrature formula on the interval $(0,\infty)$, respectively. Obviously,

$$\left(u,v\right)_{w_N^{(\alpha,\beta)}} = \left(u,v\right)_{w^{(\alpha,\beta)}}, \quad \forall u,v \in S_{2N+1}. \tag{28}$$

Thus, for any $u \in S_N(0,\infty)$, the norms $\|u\|_{w_N^{(\alpha,\beta)}}$ and $\|u\|_{w^{(\alpha,\beta)}}$ coincide.

Associating with this quadrature rule, we denote by $I_N^{L_T^{(\alpha,\beta)}}$ the modified generalized Laguerre-Gauss interpolation:

$$I_N^{L_T^{(\alpha,\beta)}}u\left(\xi_{Z,N,k}^{(\alpha,\beta)}\right) = u\left(\xi_{Z,N,k}^{(\alpha,\beta)}\right), \quad 0 \le k \le N. \tag{29}$$

The modified generalized Laguerre-Gauss collocation method for solving (24) and (25) is to seek $u_N(x) \in S_N(0,\infty)$, such that

$$D^{\nu}\left(u\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right) + a\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right)u\left(p_m\xi_{Z,N-m,k}^{(\alpha,\beta)}\right)\right)$$

$$= \beta u\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right) + \sum_{n=0}^{m-1} b_n\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right)D^{\gamma_n}u\left(p_n\xi_{Z,N-m,k}^{(\alpha,\beta)}\right)$$

$$+ f\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right),$$

$$k = 0, 1, \dots, N-m,$$

$$\sum_{n=0}^{m-1} c_{in}u^{(n)}\left(0\right) = \lambda_i, \quad i = 0, 1, \dots, m-1. \tag{30}$$

We now derive the algorithm for solving (24) and (25). To do this, let

$$u_N\left(x\right) = \sum_{h=0}^{N} a_h L_h^{(\alpha,\beta)}\left(x\right),$$

$$\mathbf{a} = \left(a_0, a_1, \dots, a_N\right)^{T}. \tag{31}$$

We first approximate $D^{\nu}u(x)$ and $D^{\gamma_n}u(x)$, as (31). By substituting these approximations in (24), we get

$$\left(\sum_{h=0}^{N} a_h L_h^{(\alpha,\beta)}\left(x\right) + a\left(x\right)\sum_{h=0}^{N} a_h L_h^{(\alpha,\beta)}\left(p_m x\right)\right)^{(\nu)}$$

$$= \beta\sum_{h=0}^{N} a_h L_h^{(\alpha,\beta)}\left(x\right) \tag{32}$$

$$+ \sum_{n=0}^{m-1}\sum_{h=0}^{N} a_h b_n\left(x\right)D^{\gamma_n}L_h^{(\alpha,\beta)}\left(p_n x\right) + f\left(x\right).$$

Making use of (16), we deduce that

$$\left(\sum_{h=0}^{N} a_h L_h^{(\alpha,\beta)}\left(x\right) + a\left(x\right)\sum_{h=0}^{N} a_h L_h^{(\alpha,\beta)}\left(p_m x\right)\right)^{(\nu)}$$

$$= \beta\sum_{h=0}^{N} a_h L_h^{(\alpha,\beta)}\left(x\right)$$

$$+ \sum_{n=0}^{m-1}\sum_{h=0}^{N}\sum_{f=0}^{M} a_h b_n\left(x\right)\Psi_{\gamma_n}\left(h,f\right)L_f^{(\alpha,\beta)}\left(p_n x\right) + f\left(x\right). \tag{33}$$

Also, by substituting (31) into (25) we obtain

$$\sum_{n=0}^{m-1}\sum_{f=0}^{M}a_{in}D^{(n)}L_f^{(\alpha,\beta)}(0) = \lambda_i. \tag{34}$$

Now, collocating (33) at the $(N - m + 1)$ modified generalized Laguerre-Gauss interpolation points yields

$$\left(\sum_{h=0}^{N}a_hL_h^{(\alpha,\beta)}\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right)\right.$$

$$\left. + a\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right)\sum_{h=0}^{N}a_hL_h^{(\alpha,\beta)}\left(p_m\xi_{Z,N-m,k}^{(\alpha,\beta)}\right)\right)^{(\nu)}$$

$$= \beta\sum_{h=0}^{N}a_hL_h^{(\alpha,\beta)}\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right) \tag{35}$$

$$+ \sum_{n=0}^{m-1}\sum_{h=0}^{N}\sum_{f=0}^{M}a_hb_n\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right)\Psi_{\gamma_n}(h,f)L_f^{(\alpha,\beta)}$$

$$\times\left(p_n\xi_{Z,N-m,k}^{(\alpha,\beta)}\right) + f\left(\xi_{Z,N-m,k}^{(\alpha,\beta)}\right).$$

Next (34), after using (12), can be written as

$$\sum_{n=0}^{m-1}\sum_{f=0}^{M}(-1)^q a_{in}\frac{\beta^q\Gamma(f+\alpha+1)}{(f-q)!\Gamma(q+\alpha+1)} = \lambda_i. \tag{36}$$

Finally, (35) with relation (36) generates $(N + 1)$ set of algebraic equations which can be solved for the unknown coefficients a_j, $j = 0, 1, 2, \ldots, N$, by using any standard solver technique.

4. Numerical Results

In this section, we implement the proposed method to solve two numerical examples from the fractional neutral functional-differential equations with proportional delays in semi-infinite intervals to demonstrate the accuracy and capability of the proposed algorithm. The absolute errors in the given tables are the values of $|u(x) - u_N(x)|$ at selected points.

Example 1. Consider the following fractional neutral functional-differential equation with proportional delay:

$$u^{1/2}(x) = -u(x) + \frac{1}{3}u\left(\frac{x}{4}\right) + \frac{1}{2}u^{1/2}\left(\frac{x}{4}\right) + g(x),$$
$$u(0) = 1, \tag{37}$$
$$x \in [0, 5],$$

where

$$g(x) = -\frac{1}{\Gamma(1/2)}\int_0^x(x-t)^{-1/2}\sin t\,dt + \cos x - \frac{1}{3}\cos\left(\frac{x}{4}\right)$$
$$+ \frac{1}{2\Gamma(1/2)}\int_0^x(x-t)^{-1/2}\sin\left(\frac{t}{4}\right)dt, \tag{38}$$

and the exact solution is given by $u(x) = \cos x$.

Table 1: Absolute errors using MGLC method at $N = 22$ for Example 1.

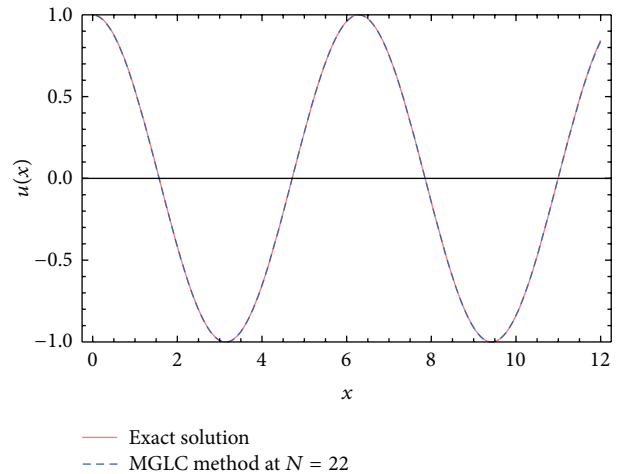| x	$\alpha = 1, \beta = 2$	$\alpha = 4, \beta = 3$	$\alpha = \beta = 5$
0.0	$7.993 \cdot 10^{-15}$	$0.000 \cdot 10^{-00}$	$2.220 \cdot 10^{-16}$
0.5	$5.164 \cdot 10^{-4}$	$7.041 \cdot 10^{-4}$	$1.746 \cdot 10^{-4}$
1.0	$5.066 \cdot 10^{-4}$	$5.763 \cdot 10^{-4}$	$1.416 \cdot 10^{-4}$
1.5	$3.521 \cdot 10^{-4}$	$4.936 \cdot 10^{-4}$	$1.257 \cdot 10^{-4}$
2.0	$2.793 \cdot 10^{-4}$	$4.268 \cdot 10^{-4}$	$1.059 \cdot 10^{-4}$
2.5	$4.480 \cdot 10^{-4}$	$3.862 \cdot 10^{-4}$	$8.856 \cdot 10^{-5}$
3.0	$2.269 \cdot 10^{-4}$	$3.437 \cdot 10^{-4}$	$8.429 \cdot 10^{-5}$
3.5	$1.998 \cdot 10^{-4}$	$3.113 \cdot 10^{-4}$	$7.249 \cdot 10^{-5}$
4.0	$5.164 \cdot 10^{-4}$	$3.037 \cdot 10^{-4}$	$6.271 \cdot 10^{-5}$
4.5	$1.141 \cdot 10^{-3}$	$2.841 \cdot 10^{-4}$	$1.159 \cdot 10^{-4}$
5.0	$1.154 \cdot 10^{-3}$	$2.268 \cdot 10^{-4}$	$5.200 \cdot 10^{-5}$

Figure 1: Graph of exact solution and approximate solution for $\alpha = 4$, $\beta = 2$ at $N = 22$ for Example 1.

--- Exact solution
--- MGLC method at $N = 22$

Table 1 lists the results obtained by the modified generalized Laguerre collocation method in terms of maximum absolute errors at $N = 22$ with $\alpha = 1, \beta = 2, \alpha = 4, \beta = 3$, and $\alpha = \beta = 5$. In the case of $\alpha = 4, \beta = 2$, the approximate solution by the presented method is shown in Figure 1, to make it easier to compare with the exact solution.

Example 2. Consider the following fractional neutral functional-differential equation with proportional delay:

$$u^{5/2}(x) = u(x) + u^{1/2}\left(\frac{x}{2}\right) + u^{3/2}\left(\frac{x}{3}\right) + \frac{1}{2}u^{5/2}\left(\frac{x}{4}\right)$$

$$+ \frac{\Gamma(4)}{\Gamma(3/2)}x^{1/2} - x^3 - \frac{\Gamma(4)}{\Gamma(7/2)}\left(\frac{x}{2}\right)^{1/2}$$

$$- \frac{\Gamma(4)}{\Gamma(5/2)}\left(\frac{x}{3}\right)^{3/2} - \frac{\Gamma(4)}{2\Gamma(3/2)}\left(\frac{x}{4}\right)^{1/2}, \quad x \in [0, 1], \tag{39}$$
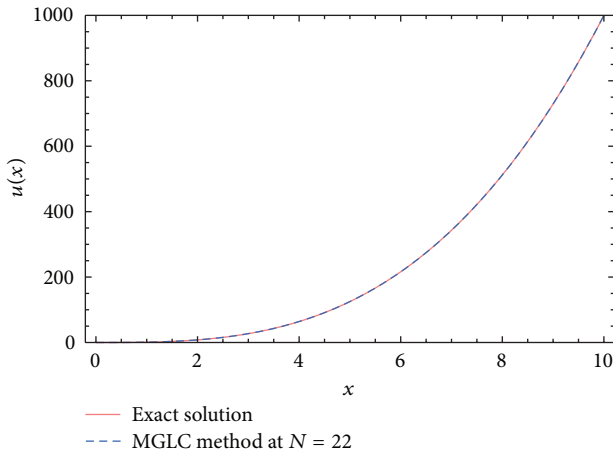
FIGURE 2: Graph of exact solution and approximate solution for $\alpha = 2$, $\beta = 1$ at $N = 22$ for Example 2.

TABLE 2: Absolute errors using MGLC method at $N = 22$ for Example 2.

x	$\alpha = 0$, $\beta = 1$	$\alpha = 3$, $\beta = 2$	$\alpha = \beta = 2$
0.0	$3.221 \cdot 10^{-11}$	$5.995 \cdot 10^{-15}$	$1.154 \cdot 10^{-14}$
0.1	$7.800 \cdot 10^{-5}$	$2.466 \cdot 10^{-5}$	$1.057 \cdot 10^{-5}$
0.2	$3.118 \cdot 10^{-4}$	$1.117 \cdot 10^{-4}$	$1.067 \cdot 10^{-5}$
0.3	$4.312 \cdot 10^{-4}$	$2.276 \cdot 10^{-4}$	$8.451 \cdot 10^{-5}$
0.4	$2.384 \cdot 10^{-4}$	$3.035 \cdot 10^{-4}$	$3.128 \cdot 10^{-4}$
0.5	$2.027 \cdot 10^{-4}$	$3.301 \cdot 10^{-4}$	$6.556 \cdot 10^{-4}$
0.6	$6.146 \cdot 10^{-4}$	$3.251 \cdot 10^{-4}$	$1.075 \cdot 10^{-3}$
0.7	$6.439 \cdot 10^{-4}$	$3.181 \cdot 10^{-4}$	$1.551 \cdot 10^{-3}$
0.8	$4.964 \cdot 10^{-6}$	$3.332 \cdot 10^{-4}$	$2.096 \cdot 10^{-3}$
0.9	$1.417 \cdot 10^{-3}$	$3.789 \cdot 10^{-4}$	$2.756 \cdot 10^{-3}$
1.0	$3.521 \cdot 10^{-3}$	$4.465 \cdot 10^{-4}$	$3.593 \cdot 10^{-3}$

subject to

$$u(0) = 0,$$
$$u'(0) = 0, \quad\quad (40)$$
$$u''(0) = 0,$$

with exact solution $u(x) = x^3$.

In Table 2, we list the absolute errors obtained by the modified generalized Laguerre collocation method, with several values of α, β and at $N = 22$. Meanwhile, Figure 2 presents the MGLC solution with $\alpha = 2$, $\beta = 1$ at $N = 22$ and exact solution, which are found to be in excellent agreement.

5. Conclusions

We have presented the modified generalized Laguerre-Gauss collocation scheme for approximating the solution of fractional neutral functional-differential equations with proportional delays on the half-line. The technique was based on modified generalized Laguerre polynomials and Gauss quadrature integration of such polynomials, which is defined on the half-line. In the previous section, two numerical results were given to demonstrate the accuracy, efficiency, and versatility of the proposed method on the half-line. Indeed, reasonable numerical results were achieved by choosing few modified generalized Laguerre-Gauss collocation points.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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