Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 371274, 10 pages
http://dx.doi.org/10.1155/2014/371274

Research Article

Certain Inequalities Involving the Fractional q-Integral Operators

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Received 31 December 2013; Accepted 10 March 2014; Published 23 April 2014

Academic Editor: Soheil Salahshour

We establish some inequalities involving Saigo fractional q-integral operator in the theory of quantum calculus by using the two parameters of deformation, q_1 and q_2, whose special cases are shown to yield corresponding inequalities associated with Riemann-Liouville and Kober fractional q-integral operators, respectively. Furthermore, we also consider their relevance with other related known results.

1. Introduction and Preliminaries

The fractional q-calculus is the q-extension of the ordinary fractional calculus (see, e.g., [1] and the references therein). The theory of q-calculus operators in the recent past has been applied in the areas like ordinary fractional calculus, optimal control problems, solutions of the q-difference (differential) and q-integral equations, q-fractional integral inequalities, q-transform analysis, and many more.

Fractional and q-fractional integral inequalities have proved to be one of the most powerful and far-reaching tools for the development of many branches of pure and applied mathematics. These inequalities have gained considerable popularity and importance during the past few decades due to their distinguished applications in numerical quadrature, transform theory, probability, and statistical problems, but the most useful ones are in establishing uniqueness of solutions in fractional boundary value problems and in fractional partial differential equations. A detailed account of such fractional integral inequalities along with their applications can be found in the research contributions by many authors (see, e.g., [2–12] and [13]; for a very recent work, see also [14]).

In a recent paper, Brahim and Taf [15] investigated certain fractional integral inequalities in quantum calculus. Here we aim at establishing certain (presumable) new q-integral inequalities associated with Saigo fractional q-integral operator, introduced by Garg and Chanchlani [16]. Results due to Brahim and Taf [15] and Sroysang [17] follow as special cases of our results, respectively.

For our purpose, we also recall the following definitions (see, e.g., [12, Section 6]) and some earlier works.

The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n := \begin{cases} 1, & (n=0), \\ \prod_{k=0}^{n-1}\left(1-aq^k\right), & (n\in\mathbb{N}), \end{cases} \tag{1}$$

where $a, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$.

The q-shifted factorial for negative subscript is defined by

$$(a;q)_{-n} := \frac{1}{\left(1-aq^{-1}\right)\left(1-aq^{-2}\right)\cdots\left(1-aq^{-n}\right)}, \quad (n\in\mathbb{N}_0). \tag{2}$$

We also write

$$(a;q)_\infty := \prod_{k=0}^{\infty}\left(1-aq^k\right), \quad (a,q\in\mathbb{C};|q|<1). \tag{3}$$

It follows from (1), (2), and (3) that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad (n\in\mathbb{Z}), \tag{4}$$

which can be extended to $n=\alpha\in\mathbb{C}$ as follows:

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \quad (\alpha\in\mathbb{C};|q|<1), \tag{5}$$

where the principal value of q^α is taken.

We begin by noting that Jackson was the first to develop q-calculus in a systematic way. The q-derivative of a function $f(t)$ is defined by

$$D_q\{f(t)\} := \frac{d_q}{d_qt}\{f(t)\} = \frac{f(qt)-f(t)}{(q-1)t}. \tag{6}$$

It is noted that

$$\lim_{q\to 1} D_q\{f(t)\} = \frac{d}{dt}\{f(t)\}, \tag{7}$$

if $f(t)$ is differentiable.

The function $F(t)$ is a q-antiderivative of $f(t)$ if $D_q\{F(t)\}=f(t)$. It is denoted by

$$\int f(t)\,d_qt. \tag{8}$$

The *Jackson integral* of $f(t)$ is thus defined, *formally*, by

$$\int f(t)\,d_qt := (1-q)\,t\sum_{j=0}^{\infty}q^j f\left(q^jt\right), \tag{9}$$

which can be easily generalized as follows:

$$\int f(t)\,d_qg(t) = \sum_{j=0}^{\infty} f\left(q^jt\right)\left(g\left(q^jt\right)-g\left(q^{j+1}t\right)\right). \tag{10}$$

Suppose that $0<a<b$. The definite q-integral is defined as follows:

$$\int_0^b f(t)\,d_qt := (1-q)\,b\sum_{j=0}^{\infty}q^j f\left(q^jb\right), \tag{11}$$

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt. \tag{12}$$

A more general version of (11) is given by

$$\int_0^b f(t)\,d_qg(t) = \sum_{j=0}^{\infty} f\left(q^jb\right)\left(g\left(q^jb\right)-g\left(q^{j+1}b\right)\right). \tag{13}$$

The classical Gamma function $\Gamma(z)$ (see, e.g., [12, Section 1.1]) was found by Euler while he was trying to extend the factorial $n! = \Gamma(n+1)$ $(n\in\mathbb{N}_0)$ to real numbers. The q-factorial function $[n]_q!$ $(n\in\mathbb{N}_0)$ of $n!$ defined by

$$[n]_q! := \begin{cases}1, & \text{if } n=0,\\ [n]_q[n-1]_q\cdots[2]_q[1]_q, & \text{if } n\in\mathbb{N},\end{cases} \tag{14}$$

can be rewritten as follows:

$$\begin{aligned}(1-q)^{-n}\prod_{k=0}^{\infty}\frac{\left(1-q^{k+1}\right)}{\left(1-q^{k+1+n}\right)} &= \frac{(q;q)_\infty}{(q^{n+1};q)_\infty}(1-q)^{-n} := \Gamma_q(n+1), \quad (0<q<1).\end{aligned} \tag{15}$$

Replacing n by $a-1$ in (15), Jackson [18] defined the q-Gamma function $\Gamma_q(a)$ by

$$\Gamma_q(a) := \frac{(q;q)_\infty}{(q^a;q)_\infty}(1-q)^{1-a}, \quad (0<q<1). \tag{16}$$

The q-analogue of $(t-a)^n$ is defined by the polynomial

$$\begin{aligned}(t-a)_q^n &:= \begin{cases}1, & (n=0),\\ (t-a)(t-qa)\cdots\left(t-q^{n-1}a\right), & (n\in\mathbb{N}_0).\end{cases}\\ &= t^n\left(\frac{a}{t};q\right)_n \quad (n\in\mathbb{N}).\end{aligned} \tag{17}$$

Definition 1. Let $\Re(\alpha)>0$, β, and η be real or complex numbers. Then a q-analogue of Saigo's fractional integral $I_q^{\alpha,\beta,\eta}$ is given for $|\tau/t|<1$ by (see [16, page 172, equation (2.1)])

$$\begin{aligned}I_q^{\alpha,\beta,\eta}\{f(t)\} &:= \frac{t^{-\beta-1}}{\Gamma_q(\alpha)}\\ &\quad\times\int_0^t\left(q\frac{\tau}{t};q\right)_{\alpha-1}\sum_{m=0}^{\infty}\frac{\left(q^{\alpha+\beta};q\right)_m\left(q^{-\eta};q\right)_m}{(q^\alpha;q)_m(q;q)_m}\\ &\quad\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\\ &\quad\times\left(\frac{\tau}{t}-1\right)_q^m f(\tau)\,d_q\tau.\end{aligned} \tag{18}$$

The integral operator $I_q^{\alpha,\beta,\eta}$ includes both the q-analogues of the Riemann-Liouville and Erdélyi-Kober fractional integral operators given by the following definitions.

Definition 2. A q-analogue of Riemann-Liouville fractional integral operator of a function $f(t)$ of an order α is given by (see [1])

$$I_q^{\alpha}\{f(t)\} : \left(= I_q^{\alpha,-\alpha,\eta}\{f(t)\}\right) = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t \left(q\frac{\tau}{t}; q\right)_{\alpha-1} f(\tau)\, d_q\tau, \quad (\alpha > 0;\ 0 < q < 1), \tag{19}$$

where $(a;q)_\alpha$ is given by (5).

Definition 3. A q-analogue of the Erdélyi-Kober fractional integral operator for $\alpha > 0$, $\eta \in \mathbb{R}$, and $0 < q < 1$ is given by (see [1])

$$I_q^{\eta,\alpha}\{f(t)\} : \left(= I_q^{\alpha,0,\eta}\{f(t)\}\right) = \frac{t^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^t \left(q\ \frac{\tau}{t}; q\right)_{\alpha-1} \tau^{\eta} f(\tau)\, d_q\tau, \quad (\alpha > 0;\ 0 < q < 1). \tag{20}$$

2. Saigo Fractional q-Integral Inequalities

In this section, we establish certain fractional q-integral inequalities, some of which are (presumably) new ones. For our purpose, we begin with providing comparison properties for the fractional q-integral operators asserted by the following lemma.

Lemma 4 (Choi and Agarwal [13]). *Let $0 < q < 1$ and $f : [0,\infty) \to \mathbb{R}$ be a continuous function with $f(t) \geq 0$ for all $t \in [0,\infty)$. Then one has the following inequalities:*

(i) *the Saigo fractional q-integral operator of the function $f(t)$ in* (18)*:*

$$I_q^{\alpha,\beta,\eta}\{f(t)\} \geq 0, \tag{21}$$

for all $\alpha > 0$ and $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta > 0$ and $\eta < 0$;

(ii) *the q-analogue of Riemann-Liouville fractional integral operator of the function $f(t)$ of an order α in* (19)*:*

$$I_q^{\alpha}\{f(t)\} \geq 0, \tag{22}$$

for all $\alpha > 0$;

(iii) *the q-analogue of Erdélyi-Kober fractional integral operator of the function $f(t)$ in* (20)*:*

$$I_q^{\eta,\alpha}\{f(t)\} \geq 0, \tag{23}$$

for all $\alpha > 0$ and $\eta \in \mathbb{R}$.

Proof. Applying (11) to the q-integral in (18), we have

$$I_q^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\beta-1}}{\Gamma_q(\alpha)} \times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q^{\alpha};q)_m (q;q)_m} \times q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}} \cdot (1-q)\, t \times \sum_{j=0}^{\infty} q^j \left\{ \left(q^{j+1};q\right)_{\alpha-1} \left(q^j - 1\right)_q^m f\left(q^j\ t\right) \right\}. \tag{24}$$

It is easy to see that

$$\Gamma_q(\alpha) > 0; \qquad (q^{\alpha};q)_m > 0; \qquad \left(q^{j+1};q\right)_{\alpha-1} = \frac{\left(q^{j+1};q\right)_\infty}{\left(q^{\alpha+j};q\right)_\infty} > 0, \tag{25}$$

for all $\alpha > 0$ and $j, m \in \mathbb{N}_0$. Next, for simplicity,

$$h(j,m;q) := (-1)^m \left(q^j - 1\right)_q^m \qquad (j, m \in \mathbb{N}_0). \tag{26}$$

Then we claim that $h(j,m;q) \geq 0$ for all $j, m \in \mathbb{N}_0$. We find from (17) that

$$h(j,m;q) = (-1)^m q^{jm} \left(\frac{1}{q^j}; q\right)_m = q^{jm} (-1)^m \prod_{k=0}^{m-1} \left(1 - q^{k-j}\right). \tag{27}$$

It is easy to see that, if $m > j$, then $h(j,m;q) = 0$. On the other hand, if $m \leq j$, then we have

$$h(j,m;q) = q^{jm} \prod_{k=0}^{m-1} \left(q^{k-j} - 1\right) > 0, \tag{28}$$

since $k - j < 0$ for all k with $0 \leq k \leq m-1 < m \leq j$. Finally we find that, under the given conditions, each term in the double series of (24) is nonnegative. This completes the proof of (21). The other two inequalities in (22) and (23) may be easily proved. □

For convenience and simplicity, we define the following function $\mathscr{H}$ by

$$\mathscr{F}(\tau, t, u(\tau); \alpha, \beta, \eta; q) := \frac{t^{-\beta-1}}{\Gamma_q(\alpha)} \left(q\frac{\tau}{t};\ q\right)_{\alpha-1} \cdot \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q^{\alpha};q)_m (q;q)_m} \times q^{(\eta-\beta)m} (-1)^m q^{-\binom{m}{2}} \times \left(\frac{\tau}{t} - 1\right)_q^m u(\tau), \tag{29}$$

where $t > 0$, $0 \leq \tau \leq t$; $\alpha > 0$, $\beta, \eta \in \mathbb{R}$ with $\alpha + \beta > 0$ and $\eta < 0$; $0 < q < 1$; $u : [0, \infty) \to [0, \infty)$ is a continuous function. As in the process of Lemma 4, it is seen that

$$\mathscr{F}(\tau, t, u(\tau); \alpha, \beta, \eta; q) \geq 0, \tag{30}$$

under the conditions given in (29).

Here we present six q-integral inequalities involving the Saigo fractional q-integral operator (18) stated in Theorems 5 to 12 below.

Theorem 5. *Let $q_1, q_2 \in (0, t)$, f, g, and h be three continuous and synchronous functions on $[0, \infty)$ and $u : [0, \infty) \to [0, \infty)$ be a continuous function. Then the following inequality holds true. For $t \in (0, \infty)$,*

$$\begin{aligned} & I_{q_2}^{\gamma,\delta,\zeta}\{u(t) f(t) g(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t) f(t) g(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) h(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t) h(t)\} \\ &\geq I_{q_2}^{\gamma,\delta,\zeta}\{u(t) g(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t) f(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t) f(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t) h(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t) g(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) h(t)\}, \end{aligned} \tag{31}$$

for all $\alpha > 0$, $\gamma > 0$, and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha + \beta > 0$, $\gamma + \delta > 0$, $\eta < 0$, and $\zeta < 0$.

Proof. Let f, g, and h be three continuous and synchronous functions on $[0, \infty)$. Then, for all $\tau, \rho \geq 0$, we have

$$((f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho))) \geq 0, \tag{32}$$

which implies that

$$\begin{aligned} & f(\tau) g(\tau) h(\tau) + f(\rho) g(\rho) h(\rho) \\ &\quad + f(\tau) g(\tau) h(\rho) + f(\rho) g(\rho) h(\tau) \\ &\geq f(\tau) g(\rho) h(\tau) + f(\tau) g(\rho) h(\rho) \\ &\quad + f(\rho) g(\tau) h(\tau) + f(\rho) g(\tau) h(\rho). \end{aligned} \tag{33}$$

Multiplying both sides of (33) by $\mathscr{F}(\tau, t, u(\tau);\ \alpha, \beta, \eta; q_1)$ in (29) together with (30) and taking q-integration of the resulting inequality with respect to τ from 0 to t with the aid of Definition 1, we get

$$\begin{aligned} & I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t) h(t)\} \\ &\quad + h(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t)\} \\ &\quad + f(\rho) g(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t) h(t)\} \\ &\quad + f(\rho) g(\rho) h(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} \\ &\geq g(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) h(t)\} \\ &\quad + g(\rho) h(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t)\} \\ &\quad + f(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t) h(t)\} \\ &\quad + f(\rho) h(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t)\}. \end{aligned} \tag{34}$$

Next, multiplying both sides of (34) by $\mathscr{F}(\rho, t, u(\rho);\ \gamma, \delta, \zeta; q_2)$ in (29) together with (30), taking q-integration of the resulting inequality with respect to ρ from 0 to t, and using Definition 1, we are led to the desired result (31). This completes the proof of Theorem 5. ☐

Theorem 6. *Let $q_1, q_2 \in (0, t)$, f, g, and h be three continuous and synchronous functions on $[0, \infty)$ and $u, v : [0, \infty) \to [0, \infty)$ be a continuous function. Then the following inequality holds true. For $t \in (0, \infty)$,*

$$\begin{aligned} & I_{q_2}^{\gamma,\delta,\zeta}\{v(t) f(t) g(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t) f(t) g(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) h(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t) h(t)\} \\ &\geq I_{q_2}^{\gamma,\delta,\zeta}\{v(t) g(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t) f(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t) f(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t) h(t)\} \\ &\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t) g(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) h(t)\}, \end{aligned} \tag{35}$$

for all $\alpha > 0$, $\gamma > 0$, and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha + \beta > 0$, $\gamma + \delta > 0$, $\eta < 0$, and $\zeta < 0$.

Proof. To prove the above result, multiplying both sides of (34) by $\mathscr{F}(\rho, t, v(\rho);\ \gamma, \delta, \zeta; q_2)$ in (29) together with (30) and taking the q-integration of the resulting inequality with respect to ρ from 0 to t with the aid of Definition 1, we get the desired result (35). ☐

Remark 7. It may be noted that the inequalities in (31) and (35) are reversed if functions f, g, and h are asynchronous. It is also easily seen that the special case $u = v$ of (35) in Theorem 6 reduces to that in Theorem 5.

Theorem 8. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h are be three continuous and synchronous functions on $[0,\infty)$, satisfying the following condition:*

$$\psi \le f(x) \le \Psi, \qquad \phi \le g(x) \le \Phi, \qquad \omega \le h(x) \le \Omega,$$
$$(\phi, \psi, \omega, \Phi, \Psi, \Omega \in \mathbb{R}; x \in [0,\infty)). \tag{36}$$

Then the following inequality holds true. For $t \in (0,\infty)$,

$$\begin{aligned}
&\Big| I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)g(t)h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)\} \\
&\quad + I_{q_1}^{\alpha,\beta,\eta}\{u(t)h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)f(t)g(t)\} \\
&\quad + I_{q_1}^{\alpha,\beta,\eta}\{u(t)g(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)f(t)h(t)\} \\
&\quad + I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{g(t)h(t)u(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)g(t)h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)f(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)g(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)g(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)h(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)f(t)g(t)h(t)\}\Big| \\
&\le I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)\}(\Psi-\psi)(\Phi-\phi)(\Omega-\omega),
\end{aligned} \tag{37}$$

for all $\alpha > 0$, $\gamma > 0$, and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha+\beta > 0$, $\gamma+\delta > 0$, $\eta < 0$, and $\zeta < 0$.

Proof. Let f, g, and h be three continuous and synchronous functions on $[0,\infty)$. Then, for all $\tau, \rho \ge 0$, we have from (36)

$$|f(\tau) - f(\rho)| \le (\Psi - \psi), \qquad |g(\tau) - g(\rho)| \le (\Phi - \phi),$$
$$|h(\tau) - h(\rho)| \le (\Omega - \omega), \tag{38}$$

which implies that

$$\begin{aligned}
&|(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) - h(\rho))| \\
&\quad \le (\Psi - \psi)(\Phi - \phi)(\Omega - \omega).
\end{aligned} \tag{39}$$

Let us define a function

$$\begin{aligned}
\mathscr{A}(\tau,\rho) &= f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\tau) \\
&\quad + f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\rho) \\
&\quad - f(\tau)g(\rho)h(\tau) - f(\rho)g(\rho)h(\rho) \\
&\quad - f(\tau)g(\tau)h(\rho) - f(\rho)g(\tau)h(\tau).
\end{aligned} \tag{40}$$

Multiplying both sides of (40) by $\mathscr{F}(\tau, t, u(\tau);\ \alpha, \beta, \eta; q_1)$ in (29) together with (30), taking q-integration of the resulting inequality with respect to τ from 0 to t, and using (18), we get

$$\begin{aligned}
&\int_0^t \mathscr{F}(\tau, t, u(\tau);\ \alpha,\beta,\eta;q_1)\mathscr{A}(\tau,\rho)\, d_{q_1}\tau \\
&\quad = I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)g(t)h(t)\} \\
&\qquad + f(\rho)g(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t)h(t)\} \\
&\qquad + g(\rho)h(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)\} \\
&\qquad + f(\rho)h(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t)g(t)\} \\
&\qquad - h(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)g(t)\} \\
&\qquad - g(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)h(t)\} \\
&\qquad - f(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t)g(t)h(t)\} \\
&\qquad - f(\rho)g(\rho)h(\rho) I_{q_1}^{\alpha,\beta,\eta}\{u(t)\}.
\end{aligned} \tag{41}$$

Next, multiplying both sides of (41) by $\mathscr{F}(\rho, t, u(\rho);\ \gamma, \delta, \zeta; q_2)$ in (29) together with (30), taking q-integration of the resulting inequality with respect to ρ from 0 to t, and using (18), we get

$$\begin{aligned}
&\int_0^t \int_0^t \mathscr{F}(\tau, t, u(\tau);\alpha,\beta,\eta;q_1)\mathscr{F}(\rho, t, u(\rho);\gamma,\delta,\zeta;q_2) \\
&\qquad \times \mathscr{A}(\tau,\rho)\, d_{q_1}\tau d_{q_2}\rho \\
&\quad = I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)g(t)h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)\} \\
&\qquad + I_{q_1}^{\alpha,\beta,\eta}\{u(t)h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)f(t)g(t)\} \\
&\qquad + I_{q_1}^{\alpha,\beta,\eta}\{u(t)g(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)f(t)h(t)\} \\
&\qquad + I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{g(t)h(t)u(t)\} \\
&\qquad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)g(t)h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)f(t)\} \\
&\qquad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)g(t)\} \\
&\qquad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)g(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)h(t)\} \\
&\qquad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)f(t)g(t)h(t)\}.
\end{aligned} \tag{42}$$

Finally, by using (39) on to (42), we arrive at the desired result (37), involved in Theorem 8, after a little simplification. □

Theorem 9. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three continuous and synchronous functions on $[0,\infty)$, satisfying the following condition:*

$$\psi \leq f(x) \leq \Psi, \qquad \phi \leq g(x) \leq \Phi, \qquad \omega \leq h(x) \leq \Omega, \qquad (\phi, \psi, \omega, \Phi, \Psi, \Omega \in \mathbb{R}; x \in [0,\infty)). \tag{43}$$

Then the following inequality holds true. For $t \in (0,\infty)$,

$$\begin{aligned}
&\Big| I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t) h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t)\} \\
&\quad + I_{q_1}^{\alpha,\beta,\eta}\{u(t) h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t) f(t) g(t)\} \\
&\quad + I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t) f(t) h(t)\} \\
&\quad + I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{g(t) h(t) v(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t) h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t) f(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) h(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t) g(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t) h(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t) f(t) g(t) h(t)\} \Big| \\
&\leq I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t)\} (\Psi - \psi)(\Phi - \phi)(\Omega - \omega),
\end{aligned} \tag{44}$$

for all $\alpha > 0$, $\gamma > 0$, and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha + \beta > 0$, $\gamma + \delta > 0$, $\eta < 0$, and $\zeta < 0$.

Proof. Multiplying both sides of (41) by $\mathcal{F}(\rho, t, v(\rho); \gamma, \delta, \zeta; q_2)$ in (29) together with (30), taking the q-integration of the resulting inequality with respect to ρ from 0 to t with the aid of Definition 1, and then applying (39) on the resulting inequality, we get the desired result (44). □

Remark 10. It is easily seen that the special case $u = v$ of (44) in Theorem 9 reduces to that in Theorem 8.

Theorem 11. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three Lipschitzian functions on $[0,\infty)$ with the constants L_1, L_2, and L_3, respectively. Then the following inequality holds true. For $t \in (0,\infty)$,*

$$\begin{aligned}
&\Big| I_{q_2}^{\gamma,\delta,\zeta}\{u(t) f(t) g(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} \\
&\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t) f(t) g(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) h(t)\} \\
&\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t)\} \\
&\quad + I_{q_2}^{\gamma,\delta,\zeta}\{u(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) g(t) h(t)\} \\
&\quad - I_{q_2}^{\gamma,\delta,\zeta}\{u(t) g(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t)\} \\
&\quad - I_{q_2}^{\gamma,\delta,\zeta}\{u(t) f(t) h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t)\} \\
&\quad - I_{q_2}^{\gamma,\delta,\zeta}\{u(t) f(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) g(t) h(t)\} \\
&\quad - I_{q_2}^{\gamma,\delta,\zeta}\{u(t) g(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t) f(t) h(t)\} \Big| \\
&\leq L_1 L_2 L_3 \Big[I_{q_1}^{\alpha,\beta,\eta}\{\tau^3 u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)\} \\
&\quad + 3 I_{q_1}^{\alpha,\beta,\eta}\{\tau u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{\rho^2 u(t)\} \\
&\quad - 3 I_{q_1}^{\alpha,\beta,\eta}\{\tau^2 u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{\rho u(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{\rho^3 u(t)\} \Big],
\end{aligned} \tag{45}$$

for all $\alpha > 0$, $\gamma > 0$, and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha + \beta > 0$, $\gamma + \delta > 0$, $\eta < 0$, and $\zeta < 0$.

Proof. Let us define the following relations for all $\tau, \rho \in [0,\infty)$:

$$\begin{aligned}
|f(\tau) - f(\rho)| &\leq L_1 (\tau - \rho), \\
|g(\tau) - g(\rho)| &\leq L_2 (\tau - \rho), \\
|h(\tau) - h(\rho)| &\leq L_3 (\tau - \rho),
\end{aligned} \tag{46}$$

which implies that

$$|\mathcal{A}(\tau, \rho)| \leq L_1 L_2 L_3 (\tau - \rho)^3, \tag{47}$$

where $\mathcal{A}(\tau, \rho)$ is given by (40). Then, by taking,

$$\mathcal{B}(\tau, \rho) := L_1 L_2 L_3 (\tau - \rho)^3. \tag{48}$$

First, multiplying both sides of (48) by $\mathcal{F}(\tau, t, u(\tau); \alpha, \beta, \eta; q_1)$ and $\mathcal{F}(\rho, t, u(\rho); \gamma, \delta, \zeta; q_2)$, respectively, in (29) together with (30), taking the q-integration of the resulting inequality with respect to τ and ρ from 0 to t with the aid of Definition 1, and then applying (39) and (47) on the resulting inequality, we get the desired result (45). This completes the proof of Theorem 11. □

Theorem 12. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three Lipschitzian functions on $[0,\infty)$ with the constants L_1, L_2, and L_3,* *respectively. Then the following inequality holds true. For $t \in (0,\infty)$,*

$$\begin{aligned}
&\Big| I_{q_2}^{\gamma,\delta,\zeta}\{v(t)f(t)g(t)h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} \\
&\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t)f(t)g(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)h(t)\} \\
&\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t)h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)g(t)\} \\
&\quad + I_{q_2}^{\gamma,\delta,\zeta}\{v(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)g(t)h(t)\} \\
&\quad - I_{q_2}^{\gamma,\delta,\zeta}\{v(t)g(t)h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)\} \\
&\quad - I_{q_2}^{\gamma,\delta,\zeta}\{v(t)f(t)h(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)g(t)\} \\
&\quad - I_{q_2}^{\gamma,\delta,\zeta}\{v(t)f(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)g(t)h(t)\} \\
&\quad - I_{q_2}^{\gamma,\delta,\zeta}\{v(t)g(t)\} I_{q_1}^{\alpha,\beta,\eta}\{u(t)f(t)h(t)\}\Big| \\
&\leq L_1L_2L_3\Big[I_{q_1}^{\alpha,\beta,\eta}\{\tau^3u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{v(t)\} \\
&\quad + 3I_{q_1}^{\alpha,\beta,\eta}\{\tau u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{\rho^2 v(t)\} \\
&\quad - 3I_{q_1}^{\alpha,\beta,\eta}\{\tau^2 u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{\rho v(t)\} \\
&\quad - I_{q_1}^{\alpha,\beta,\eta}\{u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{\rho^3 v(t)\}\Big],
\end{aligned} \tag{49}$$

for all $\alpha > 0$, $\gamma > 0$, and $\beta, \eta, \delta, \zeta \in \mathbb{R}$ with $\alpha+\beta > 0$, $\gamma+\delta > 0$, $\eta < 0$, and $\zeta < 0$.

Proof. Multiplying both sides of (48) by $\mathcal{F}(\tau, t, u(\tau); \alpha, \beta, \eta; q_1)$ and $\mathcal{F}(\rho, t, v(\rho); \gamma, \delta, \zeta; q_2)$, respectively, in (29) together with (30), taking the q-integration of the resulting inequality with respect to τ and ρ from 0 to t with the aid of Definition 1, and then applying (39) and (47) on the resulting inequality, we get the desired result (49). This completes the proof of Theorem 12. □

Remark 13. It is easily seen that the special case $u = v$ of (49) in Theorem 12 reduces to that in Theorem 11.

3. Special Cases and Concluding Remarks

In this section, we consider some consequences of the main results derived in the preceding sections. Following Garg and Chanchlani [16], the operator (18) would reduce immediately to the extensively investigated q-analogue of Erdélyi-Kober and Riemann-Liouville type fractional integral operators in (19) and (20), respectively (see also [1]).

For example, if we consider $\beta = -\alpha$ and $\delta = -\gamma$ (and $v = u$ additionally for Theorem 6) and make use of the relation (20), Theorems 5 and 6 provide, respectively, the known fractional integral inequalities due to Sroysang [17].

Here we derive certain (presumably) new integral inequalities by setting $\beta = -\alpha$ and $\delta = -\gamma$ in Theorems 8 to 12, respectively, and, applying the integral operator (19) to the resulting inequalities, we obtain four integral inequalities involving q-Riemann-Liouville fractional integral operators stated in Corollaries 14 to 17, respectively, below.

Corollary 14. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three continuous and synchronous functions on $[0,\infty)$, satisfying the following condition:*

$$\psi \leq f(x) \leq \Psi, \qquad \phi \leq g(x) \leq \Phi, \qquad \omega \leq h(x) \leq \Omega, \qquad (\phi, \psi, \omega, \Phi, \Psi, \Omega \in \mathbb{R}; x \in [0,\infty)). \tag{50}$$

Then the following inequality holds true. For $t \in (0,\infty)$,

$$\begin{aligned}
&\Big| I_{q_1}^{\alpha}\{u(t)f(t)g(t)h(t)\} I_{q_2}^{\gamma}\{u(t)\} \\
&\quad + I_{q_1}^{\alpha}\{u(t)h(t)\} I_{q_2}^{\gamma}\{u(t)f(t)g(t)\} \\
&\quad + I_{q_1}^{\alpha}\{u(t)g(t)\} I_{q_2}^{\gamma}\{u(t)f(t)h(t)\} \\
&\quad + I_{q_1}^{\alpha}\{u(t)f(t)\} I_{q_2}^{\gamma}\{g(t)h(t)u(t)\} \\
&\quad - I_{q_1}^{\alpha}\{u(t)g(t)h(t)\} I_{q_2}^{\gamma}\{u(t)f(t)\} \\
&\quad - I_{q_1}^{\alpha}\{u(t)f(t)h(t)\} I_{q_2}^{\gamma}\{u(t)g(t)\} \\
&\quad - I_{q_1}^{\alpha}\{u(t)f(t)g(t)\} I_{q_2}^{\gamma}\{u(t)h(t)\} \\
&\quad - I_{q_1}^{\alpha}\{u(t)\} I_{q_2}^{\gamma}\{u(t)f(t)g(t)h(t)\}\Big| \\
&\leq I_{q_1}^{\alpha}\{u(t)\} I_{q_2}^{\gamma}\{u(t)\}(\Psi-\psi)(\Phi-\phi)(\Omega-\omega),
\end{aligned} \tag{51}$$

for all $\alpha > 0$ and $\gamma > 0$.

Corollary 15. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three continuous and synchronous functions on $[0,\infty)$, satisfying the following condition:*

$$\psi \leq f(x) \leq \Psi, \qquad \phi \leq g(x) \leq \Phi, \qquad \omega \leq h(x) \leq \Omega, \qquad (\phi, \psi, \omega, \Phi, \Psi, \Omega \in \mathbb{R}; x \in [0,\infty)). \tag{52}$$

Then the following inequality holds true. For $t \in (0,\infty)$,

$$\begin{aligned}
&\Big| I_{q_1}^{\alpha}\{u(t)f(t)(t)h(t)\} I_{q_2}^{\gamma}\{v(t)\} \\
&\quad + I_{q_1}^{\alpha}\{u(t)h(t)\} I_{q_2}^{\gamma}\{v(t)f(t)g(t)\} \\
&\quad + I_{q_1}^{\alpha}\{u(t)g(t)\} I_{q_2}^{\gamma}\{v(t)f(t)h(t)\} \\
&\quad + I_{q_1}^{\alpha}\{u(t)f(t)\} I_{q_2}^{\gamma}\{g(t)h(t)v(t)\} \\
&\quad - I_{q_1}^{\alpha}\{u(t)g(t)h(t)\} I_{q_2}^{\gamma}\{v(t)f(t)\} \\
&\quad - I_{q_1}^{\alpha}\{u(t)f(t)h(t)\} I_{q_2}^{\gamma}\{v(t)g(t)\} \\
&\quad - I_{q_1}^{\alpha}\{u(t)f(t)g(t)\} I_{q_2}^{\gamma}\{v(t)h(t)\} \\
&\quad - I_{q_1}^{\alpha}\{u(t)\} I_{q_2}^{\gamma}\{v(t)f(t)g(t)h(t)\}\Big| \\
&\leq I_{q_1}^{\alpha}\{u(t)\} I_{q_2}^{\gamma}\{v(t)\}(\Psi-\psi)(\Phi-\phi)(\Omega-\omega),
\end{aligned} \tag{53}$$

for all $\alpha > 0$ and $\gamma > 0$.

Corollary 16. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three Lipschitzian functions on $[0,\infty)$ with the constants L_1, L_2, and L_3, respectively. Then the following inequality holds true. For $t \in (0,\infty)$,*

$$\begin{aligned}
&\Big| I_{q_2}^{\gamma}\{u(t)f(t)g(t)h(t)\} I_{q_1}^{\alpha}\{u(t)\} \\
&\quad + I_{q_2}^{\gamma}\{u(t)f(t)g(t)\} I_{q_1}^{\alpha}\{u(t)h(t)\} \\
&\quad + I_{q_2}^{\gamma}\{u(t)h(t)\} I_{q_1}^{\alpha}\{u(t)f(t)g(t)\} \\
&\quad + I_{q_2}^{\gamma}\{u(t)\} I_{q_1}^{\alpha}\{u(t)f(t)g(t)h(t)\} \\
&\quad - I_{q_2}^{\gamma}\{u(t)g(t)h(t)\} I_{q_1}^{\alpha}\{u(t)f(t)\} \\
&\quad - I_{q_2}^{\gamma}\{u(t)f(t)h(t)\} I_{q_1}^{\alpha}\{u(t)g(t)\} \\
&\quad - I_{q_2}^{\gamma}\{u(t)f(t)\} I_{q_1}^{\alpha}\{u(t)g(t)h(t)\} \\
&\quad - I_{q_2}^{\gamma}\{u(t)g(t)\} I_{q_1}^{\alpha}\{u(t)f(t)h(t)\}\Big| \\
&\le L_1L_2L_3\Big[I_{q_1}^{\alpha}\{\tau^3u(t)\} I_{q_2}^{\gamma}\{u(t)\} \\
&\qquad + 3I_{q_1}^{\alpha}\{\tau u(t)\} I_{q_2}^{\gamma}\{\rho^2u(t)\} \\
&\qquad - 3I_{q_1}^{\alpha}\{\tau^2u(t)\} I_{q_2}^{\gamma}\{\rho u(t)\} \\
&\qquad - I_{q_1}^{\alpha}\{u(t)\} I_{q_2}^{\gamma}\{\rho^3u(t)\}\Big],
\end{aligned} \tag{54}$$

for all $\alpha > 0$ and $\gamma > 0$.

Corollary 17. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three Lipschitzian functions on $[0,\infty)$ with the constants L_1, L_2, and L_3, respectively. Then the following inequality holds true. For $t \in (0,\infty)$,*

$$\begin{aligned}
&\Big| I_{q_2}^{\gamma}\{v(t)f(t)g(t)h(t)\} I_{q_1}^{\alpha}\{u(t)\} \\
&\quad + I_{q_2}^{\gamma}\{v(t)f(t)g(t)\} I_{q_1}^{\alpha}\{u(t)h(t)\} \\
&\quad + I_{q_2}^{\gamma}\{v(t)h(t)\} I_{q_1}^{\alpha}\{u(t)f(t)g(t)\} \\
&\quad + I_{q_2}^{\gamma}\{v(t)\} I_{q_1}^{\alpha}\{u(t)f(t)g(t)h(t)\} \\
&\quad - I_{q_2}^{\gamma}\{v(t)g(t)h(t)\} I_{q_1}^{\alpha}\{u(t)f(t)\} \\
&\quad - I_{q_2}^{\gamma}\{v(t)f(t)h(t)\} I_{q_1}^{\alpha}\{u(t)g(t)\} \\
&\quad - I_{q_2}^{\gamma}\{v(t)f(t)\} I_{q_1}^{\alpha}\{u(t)g(t)h(t)\} \\
&\quad - I_{q_2}^{\gamma}\{v(t)g(t)\} I_{q_1}^{\alpha}\{u(t)f(t)h(t)\}\Big| \\
&\le L_1L_2L_3\Big[I_{q_1}^{\alpha}\{\tau^3u(t)\} I_{q_2}^{\gamma}\{v(t)\} \\
&\qquad + 3I_{q_1}^{\alpha}\{\tau u(t)\} I_{q_2}^{\gamma}\{\rho^2v(t)\} \\
&\qquad - 3I_{q_1}^{\alpha}\{\tau^2u(t)\} I_{q_2}^{\gamma}\{\rho v(t)\} \\
&\qquad - I_{q_1}^{\alpha}\{u(t)\} I_{q_2}^{\gamma}\{\rho^3v(t)\}\Big],
\end{aligned} \tag{55}$$

for all $\alpha > 0$ and $\gamma > 0$.

Furthermore, if one sets $\beta = 0 = \delta$ in Theorems 5 to 12, and applying the integral operator (20) to the resulting inequalities, one obtains the following (presumably) new inequalities involving q-analogue of the Erdélyi-Kober type fractional integral operators asserted by Corollaries 18 to 23 below.

Corollary 18. *Let $q_1, q_2 \in (0,t)$, f, g, and h be three continuous and synchronous functions on $[0,\infty)$, and $u : [0,\infty) \to [0,\infty)$ be a continuous function. Then the following inequality holds true. For $t \in (0,\infty)$,*

$$\begin{aligned}
&I_{q_2}^{\zeta,\gamma}\{u(t)f(t)g(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)f(t)g(t)\} I_{q_1}^{\eta,\alpha}\{u(t)h(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)h(t)\} \\
&\ge I_{q_2}^{\zeta,\gamma}\{u(t)g(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)f(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)g(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)f(t)\} I_{q_1}^{\eta,\alpha}\{u(t)g(t)h(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)g(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)h(t)\},
\end{aligned} \tag{56}$$

for all $\alpha > 0$, $\gamma > 0$, and $\eta, \zeta \in \mathbb{R}$.

Corollary 19. *Let $q_1, q_2 \in (0,t)$, f, g, and h be three continuous and synchronous functions on $[0,\infty)$, and $u, v : [0,\infty) \to [0,\infty)$ be a continuous function. Then the following inequality holds true. For $t \in (0,\infty)$,*

$$\begin{aligned}
&I_{q_2}^{\zeta,\gamma}\{v(t)f(t)g(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)f(t)g(t)\} I_{q_1}^{\eta,\alpha}\{u(t)h(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)h(t)\} \\
&\ge I_{q_2}^{\zeta,\gamma}\{v(t)g(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)f(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)g(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)f(t)\} I_{q_1}^{\eta,\alpha}\{u(t)g(t)h(t)\} \\
&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)g(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)h(t)\},
\end{aligned} \tag{57}$$

for all $\alpha > 0$, $\gamma > 0$, and $\eta, \zeta \in \mathbb{R}$.

Corollary 20. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three continuous and synchronous functions on $[0,\infty)$, satisfying the following condition:*

$$\begin{gathered}
\psi \le f(x) \le \Psi, \qquad \phi \le g(x) \le \Phi, \qquad \omega \le h(x) \le \Omega, \\
(\phi, \psi, \omega, \Phi, \Psi, \Omega \in \mathbb{R};\ x \in [0,\infty)).
\end{gathered} \tag{58}$$

Then the following inequality holds true. For $t \in (0,\infty)$,

$$\begin{aligned}&\Big| I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)h(t)\} I_{q_2}^{\zeta,\gamma}\{u(t)\}\\&\quad + I_{q_1}^{\eta,\alpha}\{u(t)h(t)\} I_{q_2}^{\zeta,\gamma}\{u(t)f(t)g(t)\}\\&\quad + I_{q_1}^{\eta,\alpha}\{u(t)g(t)\} I_{q_2}^{\zeta,\gamma}\{u(t)f(t)h(t)\}\\&\quad + I_{q_1}^{\eta,\alpha}\{u(t)f(t)\} I_{q_2}^{\zeta,\gamma}\{g(t)h(t)u(t)\}\\&\quad - I_{q_1}^{\eta,\alpha}\{u(t)g(t)h(t)\} I_{q_2}^{\zeta,\gamma}\{u(t)f(t)\}\\&\quad - I_{q_1}^{\eta,\alpha}\{u(t)f(t)h(t)\} I_{q_2}^{\zeta,\gamma}\{u(t)g(t)\}\\&\quad - I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)\} I_{q_2}^{\zeta,\gamma}\{u(t)h(t)\}\\&\quad - I_{q_1}^{\eta,\alpha}\{u(t)\} I_{q_2}^{\zeta,\gamma}\{u(t)f(t)g(t)h(t)\}\Big|\\&\leq I_{q_1}^{\eta,\alpha}\{u(t)\} I_{q_2}^{\zeta,\gamma}\{u(t)\}(\Psi-\psi)(\Phi-\phi)(\Omega-\omega),\end{aligned} \tag{59}$$

for all $\alpha > 0$, $\gamma > 0$, and $\eta, \zeta \in \mathbb{R}$.

Corollary 21. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three continuous and synchronous functions on $[0,\infty)$, satisfying the following condition:*

$$\begin{aligned}&\psi \leq f(x) \leq \Psi, \qquad \phi \leq g(x) \leq \Phi, \qquad \omega \leq h(x) \leq \Omega,\\&\qquad (\phi,\psi,\omega,\Phi,\Psi,\Omega \in \mathbb{R}; x \in [0,\infty)).\end{aligned} \tag{60}$$

Then the following inequality holds true. For $t \in (0,\infty)$,

$$\begin{aligned}&\Big| I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)h(t)\} I_{q_2}^{\zeta,\gamma}\{v(t)\}\\&\quad + I_{q_1}^{\eta,\alpha}\{u(t)h(t)\} I_{q_2}^{\zeta,\gamma}\{v(t)f(t)g(t)\}\\&\quad + I_{q_1}^{\eta,\alpha}\{u(t)g(t)\} I_{q_2}^{\zeta,\gamma}\{v(t)f(t)h(t)\}\\&\quad + I_{q_1}^{\eta,\alpha}\{u(t)f(t)\} I_{q_2}^{\zeta,\gamma}\{g(t)h(t)v(t)\}\\&\quad - I_{q_1}^{\eta,\alpha}\{u(t)g(t)h(t)\} I_{q_2}^{\zeta,\gamma}\{v(t)f(t)\}\\&\quad - I_{q_1}^{\eta,\alpha}\{u(t)f(t)h(t)\} I_{q_2}^{\zeta,\gamma}\{v(t)g(t)\}\\&\quad - I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)\} I_{q_2}^{\zeta,\gamma}\{v(t)h(t)\}\\&\quad - I_{q_1}^{\eta,\alpha}\{u(t)\} I_{q_2}^{\zeta,\gamma}\{v(t)f(t)g(t)h(t)\}\Big|\\&\leq I_{q_1}^{\eta,\alpha}\{u(t)\} I_{q_2}^{\zeta,\gamma}\{v(t)\}(\Psi-\psi)(\Phi-\phi)(\Omega-\omega),\end{aligned} \tag{61}$$

for all $\alpha > 0$, $\gamma > 0$, and $\eta, \zeta \in \mathbb{R}$.

Corollary 22. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three Lipschitzian functions on $[0,\infty)$ with the constants L_1, L_2, and L_3, respectively. Then the following inequality holds true. For $t \in (0,\infty)$,*

$$\begin{aligned}&\Big| I_{q_2}^{\zeta,\gamma}\{u(t)f(t)g(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)\}\\&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)f(t)g(t)\} I_{q_1}^{\eta,\alpha}\{u(t)h(t)\}\\&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)\}\\&\quad + I_{q_2}^{\zeta,\gamma}\{u(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)h(t)\}\\&\quad - I_{q_2}^{\zeta,\gamma}\{u(t)g(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)\}\\&\quad - I_{q_2}^{\zeta,\gamma}\{u(t)f(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)g(t)\}\\&\quad - I_{q_2}^{\zeta,\gamma}\{u(t)f(t)\} I_{q_1}^{\eta,\alpha}\{u(t)g(t)h(t)\}\\&\quad - I_{q_2}^{\zeta,\gamma}\{u(t)g(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)h(t)\}\Big|\\&\leq L_1L_2L_3\Big[I_{q_1}^{\eta,\alpha}\{\tau^3u(t)\} I_{q_2}^{\gamma,\delta,\zeta}\{u(t)\}\\&\qquad + 3I_{q_1}^{\eta,\alpha}\{\tau u(t)\} I_{q_2}^{\zeta,\gamma}\{\rho^2u(t)\}\\&\qquad - 3I_{q_1}^{\eta,\alpha}\{\tau^2u(t)\} I_{q_2}^{\zeta,\gamma}\{\rho u(t)\}\\&\qquad - I_{q_1}^{\eta,\alpha}\{u(t)\} I_{q_2}^{\zeta,\gamma}\{\rho^3u(t)\}\Big],\end{aligned} \tag{62}$$

for all $\alpha > 0$, $\gamma > 0$, and $\eta, \zeta \in \mathbb{R}$.

Corollary 23. *Let $q_1, q_2 \in (0,t)$, $u : [0,\infty) \to [0,\infty)$ be a continuous function, and f, g, and h be three Lipschitzian functions on $[0,\infty)$ with the constants L_1, L_2, and L_3, respectively. Then the following inequality holds true. For $t \in (0,\infty)$,*

$$\begin{aligned}&\Big| I_{q_2}^{\zeta,\gamma}\{v(t)f(t)g(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)\}\\&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)f(t)g(t)\} I_{q_1}^{\eta,\alpha}\{u(t)h(t)\}\\&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)\}\\&\quad + I_{q_2}^{\zeta,\gamma}\{v(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)g(t)h(t)\}\\&\quad - I_{q_2}^{\zeta,\gamma}\{v(t)g(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)\}\\&\quad - I_{q_2}^{\zeta,\gamma}\{v(t)f(t)h(t)\} I_{q_1}^{\eta,\alpha}\{u(t)g(t)\}\\&\quad - I_{q_2}^{\zeta,\gamma}\{v(t)f(t)\} I_{q_1}^{\eta,\alpha}\{u(t)g(t)h(t)\}\\&\quad - I_{q_2}^{\zeta,\gamma}\{v(t)g(t)\} I_{q_1}^{\eta,\alpha}\{u(t)f(t)h(t)\}\Big|\end{aligned}$$

$$\leq L_1 L_2 L_3 \left[I_{q_1}^{\eta,\alpha} \left\{ \tau^3 u(t) \right\} I_{q_2}^{\zeta,\gamma} \left\{ \nu(t) \right\} + 3 I_{q_1}^{\eta,\alpha} \left\{ \tau u(t) \right\} I_{q_2}^{\zeta,\gamma} \left\{ \rho^2 \nu(t) \right\} - 3 I_{q_1}^{\eta,\alpha} \left\{ \tau^2 u(t) \right\} I_{q_2}^{\zeta,\gamma} \left\{ \rho \nu(t) \right\} - I_{q_1}^{\eta,\alpha} \left\{ u(t) \right\} I_{q_2}^{\zeta,\gamma} \left\{ \rho^3 \nu(t) \right\} \right], \tag{63}$$

for all $\alpha > 0$, $\gamma > 0$, *and* $\eta, \zeta \in \mathbb{R}$.

It is also noted that, if we consider $g(t)$ as a unity and $\beta = -\alpha$ and $\delta = -\gamma$ in Theorems 5 and 6, respectively, and make use of the relation (19), then we get the known fractional integral inequalities due to Brahim and Taf [15].

We conclude our present investigation by remarking further that the results obtained here are general in character and useful in deriving various inequalities involving fractional and q-fractional integral operators. Moreover, they are expected to find some applications for establishing uniqueness of solutions in fractional boundary value problems and in fractional partial differential equations in the theory of quantum calculus.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors should express their deep thanks for the editorial and reviewer board members for accepting this paper in their reputable journal.

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