

A FRACTIONAL MODEL FOR THE DYNAMICS OF TUBERCULOSIS INFECTION USING CAPUTO-FABRIZIO DERIVATIVE

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ABSTRACT. In the present paper, we study the dynamics of tuberculosis model using fractional order derivative in Caputo-Fabrizio sense. The number of confirmed notified cases reported by national TB program Khyber Pakhtunkhwa, Pakistan, from the year 2002 to 2017 are used for our analysis and estimation of the model biological parameters. The threshold quantity \mathcal{R}_0 and equilibria of the model are determined. We prove the existence of the solution via fixed-point theory and further examine the uniqueness of the model variables. An iterative solution of the model is computed using fractional Adams-Bashforth technique. Finally, the numerical results are presented by using the estimated values of model parameters to justify the significance of the arbitrary fractional order derivative. The graphical results show that the fractional model of TB in Caputo-Fabrizio sense gives useful information about the complexity of the model and one can get reliable information about the model at any integer or non-integer case.

1. Introduction. Tuberculosis (TB) is a serious public health problem not only for developing countries but also for developed one. This infection is the leading cause of deaths due to which it is ranked 10th and after HIV infection it is regarding the 2nd leading cause of mortality worldwide. It is a bacterial infectious disease caused by *bacillus Mycobacterium tuberculosis* (MTB). It mainly infects lungs

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(called pulmonary TB) and seldomly damage some other human body parts such as spine, central nervous system, kidneys and brain (called extra pulmonary TB). The bacteria of TB is transfer from infected person to the healthy person through air during coughing, spiting or sneezing. According to World Health Organization (WHO) reports, 10.4 million people are TB infected out of which approximately 1.7 million died each year form this infection and associated to this infection, approximately 0.36 million patients are HIV infected [1]. Around the globe about 10% of extrapulmonary cases are reported and more than 50% of TB infected people are reported with HIV co-infection every year. Mostly the TB infected people and mortality are reported in the countries like sub-continent, Philippines, Nigeria, Pakistan, China and South Africa and these countries share more than 60% of the TB burden worldwide [1].

Mathematical models play a key role and have been used extensively to analyze the dynamics and to provide useful techniques to eradicate the infectious diseases from the community. These models explore both qualitative and quantitative analysis of the disease. In the disease epidemiology, the first attempt to TB modeling goes to Waaler et al. [40]. Later on, in 1967, a mathematical model of TB infection which depends on the proportion of its prevalence is developed in [33]. A two strain and an age structure TB models with time delay were studied by Castillo et al. [17, 20]. The TB model with global stability results was discussed by Liu et al. in [28]. Egonmwan et al. [19] developed a new TB model to explore the impact of treatment and diagnoses for both latent and active TB infective population. Zhang et al. [42] proposed the TB mathematical model with hospitalized and non hospitalized infective classes and implemented the TB data of China to simulate the model. The TB model with effect of relapse and reinfection is analyzed by Robert [41]. Kim et al. studied TB dynamics with optimal control strategies and used the actual data of the Philippines population to simulate the mathematical model [27].

Fractional calculus generalized the classical integer calculus. Fractional order (FO) derivatives and fractional integrals are important concept in the study of fractional calculus because of having hereditary properties and provides a good description of the memory. Mathematical modeling with FO derivatives is the emerging field and were used as a powerful tool to explore the complex dynamics of various real phenomena in different areas of science such as [30, 31, 38, 11, 22, 23, 24]. In literature several fractional operators of order $\alpha \in [0, 1]$ are presented. The concept of most common Caputo FO derivative and related theories are developed in [36, 35]. In 2015, a new fractional order derivative based on the exponential kernel has been introduced in [16], know as Caputo-Fabrizio (CF) fractional derivative. Recently, a modification in the concept of CF fractional derivative and some of its applications have been presented in [32]. The CF derivative [16] has been applied in expressing a number of problems in various fields which can be found [34, 3, 2, 37]. In [4, 5] Abdeljawad et al. proposed a discrete version of CF fractional derivative and developed related results. In 2016, a new fractional operator with generalized Mittag-Leffler as kernel was proposed by Atangana and Baleanu [13]. The operator developed in [13] has been applied to express the complex and crossover behavior of real word phenomena [14, 26, 39]. A fractional operator with generalized discrete Mittag-Leffler function, related theories and its applications to difference equations is proposed in [6, 7, 8, 9, 12, 10, 21, 25]. All these works on fractional modeling utilized different fractional approaches to different areas of science and engineering.

Due to the recent progress on fractional calculus and its wide applications, we aim to formulate and analyzed a TB model with fractional CF fractional derivative.

Pakistan bears the highest burden of TB infection and is ranked 5th in the list of 22 countries with high TB infected cases. Increasing reported number of the TB infective cases is a serious health issue in Pakistan. In 2016, a total of 0.35 million new infected TB cases were notified which are greater than the number of TB cases reported in 2015 [1, 43]. More than seventy thousand of the population die due TB infection and about 0.5 million TB confirm new cases along with 15000 children are reported each year in Pakistan. In Khyber Pakhtunkhwa, Pakistan, more than 462920 people having TB infection are reported during 2002 to 2017 [43]. In this investigation we explore the dynamics of TB infection in t Khyber Pakhtunkhwa through a fractional compartmental model with CF derivative of order $\tau \in [0, 1]$. Further, the parameter values are fitted trough real data of TB infective people registered from 2002 till 2017 in Khyber Pakhtunkhwa obtained from the website of NTP [43]. The remaining sections of the manuscript are arranged as: In section 2, the basic results and definitions regarding CF derivative are stated. The model construction using fractional calculus, its equilibria and reproduction number are given in section 3. Existence and uniqueness results for the model variables are given in section 4. A numerical scheme based on Adams-Bashforth technique is presented in section 5. Further, in this section we present the numerical simulations of the model. At the end we give a brief conclusion in section 6.

2. Preliminaries.

Definition 2.1. The fractional derivative in CF sense for the function $\chi \in H^1(a, b)$, $b > a$, $\tau \in [0, 1]$ [16] is defined as

$$D_t^\tau(\chi(t)) = \frac{M(\tau)}{1-\tau} \int_a^t \chi'(x) \exp\left[-\tau \frac{t-x}{1-\tau}\right] dx. \quad (1)$$

$M(\tau)$ is the normalized function satisfying $M(0) = M(1) = 1$ [16]. For the case when $\chi \notin H^1(a, b)$ the above CF derivative can be expressed as

$$D_t^\tau(\chi(t)) = \frac{\tau M(\tau)}{1-\tau} \int_a^t (\chi(t) - \chi(x)) \exp\left[-\tau \frac{t-x}{1-\tau}\right] dx. \quad (2)$$

Remark 1.

If $\alpha = \frac{1-\tau}{\tau} \in [0, \infty)$, $\tau = \frac{1}{1+\alpha} \in [0, 1]$, then Eq. (2) can be written as below

$$D_t^\alpha(\chi(t)) = \frac{N(\alpha)}{\tau} \int_a^t \chi'(x) \exp\left[-\frac{t-x}{\alpha}\right] dx, \quad N(0) = N(\infty) = 1. \quad (3)$$

Moreover,

$$\lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \exp\left[-\frac{t-x}{\alpha}\right] = \delta(x-t). \quad (4)$$

The integral regarding to CF derivative is defined as below [29].

Definition 2.2. Let $0 < \tau < 1$, and consider the fractional derivative given below

$$D_t^\tau(\chi(t)) = g(t), \quad (5)$$

then the corresponding integral of FO τ is expressed as

$$I_t^\tau(\chi(t)) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} g(t) + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t g(s) ds, \quad t \geq 0. \quad (6)$$

Remark 2. Using the result

$$\frac{2}{2M(\tau) - \tau M(\tau)} = 1, \quad (7)$$

which gives $M(\tau) = \frac{2}{2-\tau}$, $0 < \tau < 1$, the authors in [29] give the new CF fractional derivative of order $0 < \tau < 1$ which is defined as below:

$$D_t^\tau(\chi(t)) = \frac{1}{1-\tau} \int_0^t \chi'(x) \exp\left[-\tau \frac{t-x}{1-\tau}\right] dx. \quad (8)$$

3. Formulation of fractional TB model. In the current section the mathematical model of TB infection using CF derivative of non-integer order τ is formulated. To construct the proposed model, we consider five epidemiological sub-classes that is, S, L, I, T, R , which is respectively show, the susceptible, latent, infected, under treatment and the recovered individuals and $N = S + L + I + T + R$, where N is the total population. The proposed TB model in CF sense is composed of the equations in fractional systems given below:

$$\begin{aligned} {}_0^C D_t^\tau S &= \Lambda - \frac{\beta SI}{N} - \mu S, \\ {}_0^C D_t^\tau L &= \frac{\beta SI}{N} - (\mu + \epsilon)L + (1 - \eta)\delta T, \\ {}_0^C D_t^\tau I &= \epsilon L + \eta\delta T - (\mu + \gamma + \tau_1)I, \\ {}_0^C D_t^\tau T &= \gamma I - (\mu + \delta + \tau_2 + \alpha)T, \\ {}_0^C D_t^\tau R &= \alpha T - \mu R, \end{aligned} \quad (9)$$

and the initial conditions involved in (9) are

$$S(0) = n_1, L(0) = n_2, I(0) = n_3, T(0) = n_4, \text{ and } R(0) = n_5.$$

In (9), Λ is the birth rate while β is the disease contact rate of infected people with susceptible people. The latent individuals become infected at the rate ϵ and move to I . The TB infected individuals are treated and join the class T at the rate γ and then goes to R class at the rate α after successful treatment. The rate at which the individual leave the class T is δ and enter to either class L or I due to treatment failure. The parameter η is the treatment failure rate and μ denotes the natural death rate. Whereas τ_1 and τ_2 are disease related rates in I and T classes respectively. The details of the models parameters estimations, we refer the reader to the see [43, 44], are obtained and is tabulated in Table 1. The total dynamics of the fractional TB model are obtained by (9) and is given by the following equation:

$$\begin{aligned} N'(t) &= \Lambda - \mu N - \tau_1 I + \tau_2 T, \\ &\leq \Lambda - \mu N, \end{aligned}$$

when t goes ∞ , then, we can get, $N \rightarrow \frac{\Lambda}{\mu}$. So, the biological feasible region for the fractional Caputo-Fabrizio model where the dynamics of the model and their existence and other epidemiological properties holds:

$$\Xi = \left[(S, L, I, T, R) \in R_+^5 : 0 \leq S + L + I + T + R \leq \frac{\Lambda}{\mu} \right].$$

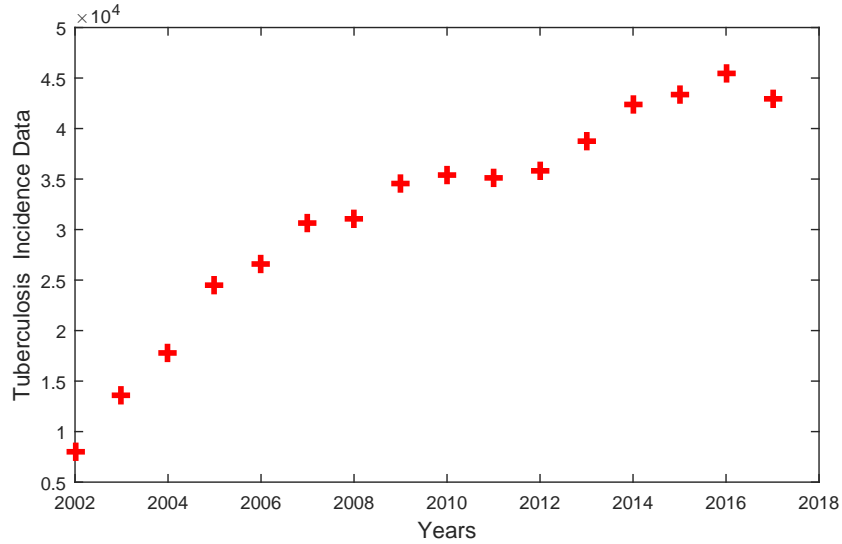


Figure 1: The incidence data of TB from Khyber Pakhtunkhwa, Pakistan

Parameter	Definition	value	Ref.
Λ	Birth rate	450,862.20088626	Estimated
β	Disease contact rate	0.5433	Fitted
α	Progression from T class to R	0.3968	Fitted
γ	Transmission from I class to T	0.2873	Fitted
μ	Natural mortality rate	1/67.7	[44]
τ_1	Disease related motility rate of infected individuals	0.2202	Fitted
τ_2	Disease related death rate in T	0.0550	Fitted
δ	Leaving rate of the individuals from class T	1.1996	Fitted
η	Treatment failure rate	0.1500	Fitted
ϵ	Moving rate from L class to I	0.2007	Fitted

Table 1: Fitting of the model parameters and its estimations for The TB infected cases of Khyber Pakhtunkhwa, Pakistan.

The parameters and their definitions are presented in Table 1, which are parameterized by using the curve fitting technique to the differential equations, namely, the least square curve fitting. In these setting we mentioned the Table 1 some parameters that are obtained from the literature and others are fitted. The TB cases are shown in Figure 1 represent the annual cases of TB occurred in Khyber Pakhtunkhwa, province Pakistan. Figure 2 is the curve fitting to the realistic data of TB which shows that the model behavior to the realistic data is reasonable. Further, we give a long time behavior of the TB cases vs time (years) in Figure 3. Here, from Figure 3, we can see that the data is accurately fit to the model curve and further, one can observe that the cases with time on long term behavior grows exponentially. This case could be alarming that the incidence may increases further in the coming years if the health department not applied the proper treatment strategies for the TB elimination.

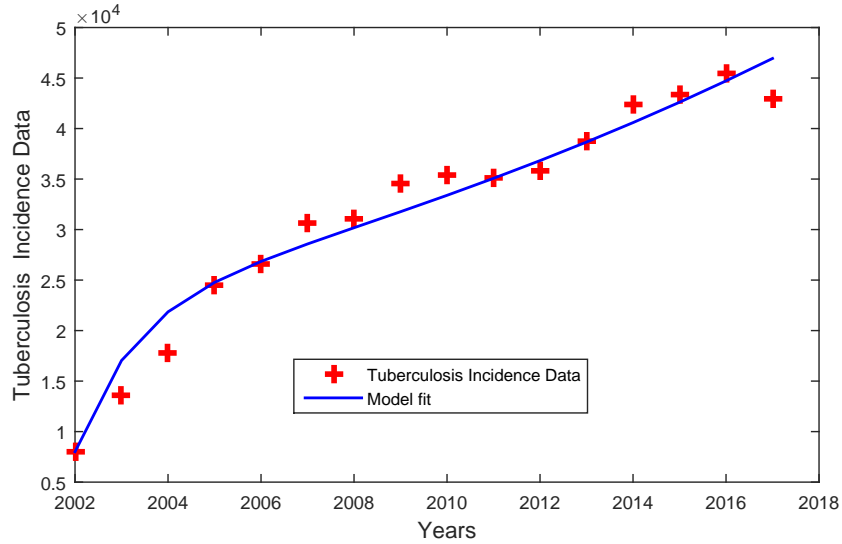


Figure 2: The incidence data of TB from Khyber Pakhtunkhwa, Pakistan and the model fit for $\tau = 1$.

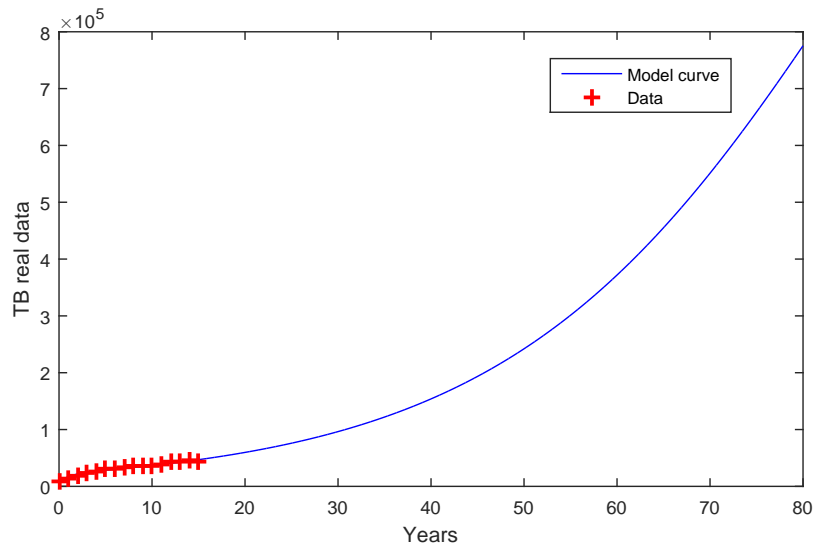


Figure 3: Long term behavior of the CF model with realistic data when $\tau = 1$.

Next, we present the basic properties associated to the fractional CF model (9).

3.1. Equilibria and threshold number. The disease free equilibrium (DFE) denoted by \mathcal{K}_0 of fractional order TB model (1) is evaluated by solving the following equations:

$${}_0^{CF}D_t^\tau S = {}_0^{CF}D_t^\tau L = {}_0^{CF}D_t^\tau I = {}_0^{CF}D_t^\tau T = {}_0^{CF}D_t^\tau R = 0,$$

and is given as below.

$$\mathcal{K}_0 = (S^0, 0, 0, 0, 0),$$

where $S^0 = \frac{\Lambda}{\mu}$. By applying the next generation technique presented in [18], the basic reproduction number \mathcal{R}_0 is obtained as follows.

$$\mathcal{R}_0 = \frac{\beta\epsilon(\mu + \delta + \tau_2 + \alpha) + (1 - \eta)\gamma\delta\epsilon + (\mu + \epsilon)\gamma\delta\eta}{(\mu + \gamma + \tau_1)(\mu + \epsilon)(\mu + \delta + \tau_2 + \alpha)}.$$

A unique endemic equilibria $\mathcal{K}_1(S^*, L^*, I^*, T^*)$ of the model (9) exists if $\mathcal{R}_0 > 1$ and is given below.

$$\begin{cases} S^* = \frac{N^*}{\mathcal{R}_0}, \\ L^* = \frac{((\mu + \tau_1) + \gamma(\mu + \tau_2 + \alpha) + \gamma\delta(1 - \eta))N^*(\mathcal{R}_0 - 1)}{\mathcal{R}_0\{(\mu + \tau_1) + \gamma(\mu + \tau_2 + \alpha) + \gamma\delta(1 - \eta) + \epsilon(\gamma + \mu + \delta + \tau_2 + \alpha)\}}, \\ I^* = \frac{\epsilon k_3 N^*(\mathcal{R}_0 - 1)}{\mathcal{R}_0\{(\mu + \tau_1) + \gamma(\mu + \tau_2 + \alpha) + \gamma\delta(1 - \eta) + \epsilon(\gamma + \mu + \delta + \tau_2 + \alpha)\}}, \\ T^* = \frac{\epsilon \gamma N^*(\mathcal{R}_0 - 1)}{\mathcal{R}_0\{(\mu + \tau_1) + \gamma(\mu + \tau_2 + \alpha) + \gamma\delta(1 - \eta) + \epsilon(\gamma + \mu + \delta + \tau_2 + \alpha)\}}. \end{cases}$$

4. Existence and uniqueness. The present section aims to show the existence and then uniqueness of the model (9) solution by applying fixed-point results. For convenience the proposed system (9) can be re-write in the equivalent form given below.

$$\begin{aligned} {}_0^C D_t^\tau [S(t)] &= \mathcal{F}_1(t, S), \\ {}_0^C D_t^\tau [L(t)] &= \mathcal{F}_2(t, L), \\ {}_0^C D_t^\tau [I(t)] &= \mathcal{F}_3(t, I), \\ {}_0^C D_t^\tau [T(t)] &= \mathcal{F}_4(t, T), \\ {}_0^C D_t^\tau [R(t)] &= \mathcal{F}_5(t, R). \end{aligned} \tag{10}$$

By applying the definition of CF fractional integral operator given in [29], the above system (10), reduces to the following integral equation of Volterra type with the CF fractional integral of order $0 < \tau < 1$.

$$\begin{aligned} S(t) - S(0) &= 2 \frac{(1 - \tau)}{(2 - \tau)M(\tau)} \mathcal{F}_1(t, S) + 2 \frac{\tau}{(2 - \tau)M(\tau)} \int_0^t \mathcal{F}_1(\zeta, S) d\zeta, \\ L(t) - L(0) &= 2 \frac{(1 - \tau)}{(2 - \tau)M(\tau)} \mathcal{F}_2(t, L) + 2 \frac{\tau}{(2 - \tau)M(\tau)} \int_0^t \mathcal{F}_2(\zeta, L) d\zeta, \\ I(t) - I(0) &= 2 \frac{(1 - \tau)}{(2 - \tau)M(\tau)} \mathcal{F}_3(t, I) + 2 \frac{\tau}{(2 - \tau)M(\tau)} \int_0^t \mathcal{F}_3(\zeta, I) d\zeta, \\ T(t) - T(0) &= 2 \frac{(1 - \tau)}{(2 - \tau)M(\tau)} \mathcal{F}_4(t, T) + 2 \frac{\tau}{(2 - \tau)M(\tau)} \int_0^t \mathcal{F}_4(\zeta, T) d\zeta, \\ R(t) - R(0) &= 2 \frac{(1 - \tau)}{(2 - \tau)M(\tau)} \mathcal{F}_5(t, R) + 2 \frac{\tau}{(2 - \tau)M(\tau)} \int_0^t \mathcal{F}_5(\zeta, R) d\zeta. \end{aligned} \tag{11}$$

Now we prove that the kernels $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4,$ and \mathcal{F}_5 fulfill the Lipchitz condition and contraction under some assumptions. In the following theorem we prove for \mathcal{F}_1 and can proceeds for the rest in a similar patron.

Theorem 4.1. *The kernel \mathcal{F}_1 satisfies the Lipchitz condition and contraction if the inequality given below holds*

$$0 \leq (\beta_{\varsigma_1} + \mu) < 1.$$

Proof. For S and S_1 we proceed as below:

$$\begin{aligned} \|\mathcal{F}_1(t, S) - \mathcal{F}_1(t, S_1)\| &= \left\| -\frac{\beta I}{N}(S(t) - S(t_1)) - \mu(S(t) - S(t_1)) \right\| \\ &\leq \beta \|I(t)\| \|S(t) - S(t_1)\| + \mu \|S(t) - S(t_1)\| \\ &\leq \{\beta_{\varsigma_1} + \mu\} \|S(t) - S(t_1)\| \\ &\leq l_1 \|S(t) - S(t_1)\|. \end{aligned} \tag{12}$$

Taking $l_1 = \{\beta_{\varsigma_1} + \mu\}$, where $\|I(t)\| \leq \varsigma_1$ is bounded function, which implies that

$$\|\mathcal{F}_1(t, S) - \mathcal{F}_1(t, S_1)\| \leq l_1 \|S(t) - S(t_1)\|. \tag{13}$$

Hence, for \mathcal{F}_1 the Lipschitz condition is obtained and if an additionally $0 \leq (\beta_{\varsigma_1} + \mu) < 1$ which gives a contraction. The Lipschitz condition can be easily verified for the rest of the cases and given as follows:

$$\begin{aligned} \|\mathcal{F}_2(t, L) - \mathcal{F}_2(t, L_1)\| &\leq l_2 \|L(t) - L(t_1)\|, \\ \|\mathcal{F}_3(t, I) - \mathcal{F}_3(t, I_1)\| &\leq l_3 \|I(t) - I(t_1)\|, \\ \|\mathcal{F}_4(t, T) - \mathcal{F}_4(t, T_1)\| &\leq l_4 \|T(t) - T(t_1)\|, \\ \|\mathcal{F}_5(t, R) - \mathcal{F}_5(t, R_1)\| &\leq l_5 \|R(t) - R(t_1)\|. \end{aligned} \tag{14}$$

□

The difference between successive terms of system (9) in recursive form is given below:

$$\begin{aligned} \phi_{1n}(t) &= S_n(t) - S_{n-1}(t) = \frac{2(1-\tau)}{(2-\tau)M(\tau)} (\mathcal{F}_1(t, S_{n-1}) - \mathcal{F}_1(t, S_{n-2})) \\ &\quad + 2\frac{\tau}{(2-\tau)M(\tau)} \int_0^t (\mathcal{F}_1(\zeta, S_{n-1}) - \mathcal{F}_1(\zeta, S_{n-2}))d\zeta, \\ \phi_{2n}(t) &= L_n(t) - L_{n-1}(t) = 2\frac{(1-\tau)}{(2-\tau)M(\tau)} (\mathcal{F}_2(t, L_{n-1}) - \mathcal{F}_2(t, L_{n-2})) + \\ &\quad 2\frac{\tau}{(2-\tau)M(\tau)} \int_0^t (\mathcal{F}_2(\zeta, L_{n-1}) - \mathcal{F}_2(\zeta, L_{n-2}))d\zeta, \\ \phi_{3n}(t) &= I_n(t) - I_{n-1}(t) = 2\frac{(1-\tau)}{(2-\tau)M(\tau)} (\mathcal{F}_3(t, I_{n-1}) - \mathcal{F}_3(t, I_{n-2})) + \\ &\quad 2\frac{\tau}{(2-\tau)M(\tau)} \int_0^t (\mathcal{F}_3(\zeta, I_{n-1}) - \mathcal{F}_3(\zeta, I_{n-2}))d\zeta, \\ \phi_{4n}(t) &= T_n(t) - T_{n-1}(t) = 2\frac{(1-\tau)}{(2-\tau)M(\tau)} (\mathcal{F}_4(t, T_{n-1}) - \mathcal{F}_4(t, T_{n-2})) + \end{aligned}$$

$$\begin{aligned} & \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t (\mathcal{F}_4(\zeta, T_{n-1}) - \mathcal{F}_4(\zeta, T_{n-2}))d\zeta, \\ \phi_{5n}(t) = & R_n(t) - R_{n-1}(t) = 2\frac{(1-\tau)}{(2-\tau)M(\tau)} (\mathcal{F}_5(t, R_{n-1}) - \mathcal{F}_5(t, R_{n-2})) + \\ & 2\frac{\tau}{(2-\tau)M(\tau)} \int_0^t (\mathcal{F}_5(\zeta, R_{n-1}) - \mathcal{F}_5(\zeta, R_{n-2}))d\zeta. \end{aligned} \tag{15}$$

With below initial conditions

$$S_0(t) = S(0), L_0(t) = L(0), I_0(t) = I(0), T_0(t) = T(0), R_0(t) = R(0).$$

Taking norm of the first equation in system (15)

$$\begin{aligned} \|\phi_{1n}(t)\| = \|S_n(t) - S_{n-1}(t)\| = & \left\| \frac{2(1-\tau)}{(2-\tau)M(\tau)} (\mathcal{F}_1(t, S_{n-1}) - \mathcal{F}_1(t, S_{n-2})) \right. \\ & \left. + \frac{2\tau}{M(\tau)(2-\tau)} \int_0^t (\mathcal{F}_1(\zeta, S_{n-1}) - \mathcal{F}_1(\zeta, S_{n-2}))d\zeta \right\|. \end{aligned} \tag{16}$$

Applying the triangular inequality, Eq. (16) gives

$$\begin{aligned} \|S_n(t) - S_{n-1}(t)\| \leq & \frac{2(1-\tau)}{(2-\tau)M(\tau)} \|(\mathcal{F}_1(t, S_{n-1}) - \mathcal{F}_1(t, S_{n-2}))\| \\ & + \frac{2\tau}{M(\tau)(2-\tau)} \left\| \int_0^t (\mathcal{F}_1(\zeta, S_{n-1}) - \mathcal{F}_1(\zeta, S_{n-2}))d\zeta \right\|. \end{aligned} \tag{17}$$

Using Lipschitz condition (13) we obtained

$$\begin{aligned} \|S_n(t) - S_{n-1}(t)\| \leq & \frac{2(1-\tau)}{(2-\tau)M(\tau)} l_1 \|S_{n-1} - S_{n-2}\| + \frac{2\tau}{(2-\tau)M(\tau)} l_1 \\ & \times \int_0^t \|S_{n-1} - S_{n-2}\|d\zeta. \end{aligned} \tag{18}$$

Thus, we have

$$\|\phi_{1n}(t)\| \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} l_1 \|\phi_{n-1}(t)\| + \frac{2\tau}{(2-\tau)M(\tau)} l_1 \int_0^t \|\phi_{1(n-1)}(\zeta)\|d\zeta. \tag{19}$$

Similarly, for the rest of equations in system (15) we obtained

$$\begin{aligned} \|\phi_{2n}(t)\| & \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} l_2 \|\phi_{2(n-1)}(t)\| + \frac{2\tau}{M(\tau)(2-\tau)} l_2 \int_0^t \|\phi_{2(n-1)}(\zeta)\|d\zeta, \\ \|\phi_{3n}(t)\| & \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} l_3 \|\phi_{3(n-1)}(t)\| + \frac{2\tau}{M(\tau)(2-\tau)} l_3 \int_0^t \|\phi_{3(n-1)}(\zeta)\|d\zeta, \\ \|\phi_{4n}(t)\| & \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} l_4 \|\phi_{4(n-1)}(t)\| + \frac{2\tau}{M(\tau)(2-\tau)} l_4 \int_0^t \|\phi_{4(n-1)}(\zeta)\|d\zeta, \\ \|\phi_{5n}(t)\| & \leq \frac{2(1-\tau)}{(2-\tau)M(\tau)} l_5 \|\phi_{5(n-1)}(t)\| + \frac{2\tau}{M(\tau)(2-\tau)} l_5 \int_0^t \|\phi_{5(n-1)}(\zeta)\|d\zeta. \end{aligned} \tag{20}$$

From above we can write that

$$\begin{cases} S_n(t) = \sum_{i=1}^n \phi_{1i}(t), \\ L_n(t) = \sum_{i=1}^n \phi_{2i}(t), \\ I_n(t) = \sum_{i=1}^n \phi_{3i}(t), \\ T_n(t) = \sum_{i=1}^n \phi_{4i}(t), \\ R_n(t) = \sum_{i=1}^n \phi_{5i}(t). \end{cases} \tag{21}$$

Hence, in order to confirm solution existence we state the below theorem.

Theorem 4.2. *A system of solutions given by the fractional TB model (9) exist if one can find t_1 for which the following inequality holds*

$$\frac{2(1-\tau)}{(2-\tau)M(\tau)}l_i + \frac{2\tau t_1}{(2-\tau)M(\tau)}l_i < 1,$$

for $i=1,2,\dots,5$.

Proof. As we have shown that the kernels condition given in (13) holds. So, by considering the Eqs. (19) and (20), and by applying the recursive technique we obtained the succeeding results as below:

$$\begin{aligned} \|\phi_{1n}(t)\| &\leq \|S_n(0)\| \left[\left(\frac{2(1-\tau)}{M(\tau)(2-\tau)}l_1 \right) + \left(\frac{2\tau}{M(\tau)(2-\tau)}l_1t \right)^n \right], \\ \|\phi_{2n}(t)\| &\leq \|L_n(0)\| \left[\left(\frac{2(1-\tau)}{M(\tau)(2-\tau)}l_2 \right) + \left(\frac{2\tau}{M(\tau)(2-\tau)}l_2t \right)^n \right], \\ \|\phi_{3n}(t)\| &\leq \|I_n(0)\| \left[\left(\frac{2(1-\tau)}{M(\tau)(2-\tau)}l_3 \right) + \left(\frac{2\tau}{M(\tau)(2-\tau)}l_3t \right)^n \right], \\ \|\phi_{4n}(t)\| &\leq \|T_n(0)\| \left[\left(\frac{2(1-\tau)}{M(\tau)(2-\tau)}l_4 \right) + \left(\frac{2\tau}{M(\tau)(2-\tau)}l_4t \right)^n \right], \\ \|\phi_{5n}(t)\| &\leq \|R_n(0)\| \left[\left(\frac{2(1-\tau)}{M(\tau)(2-\tau)}l_5 \right) + \left(\frac{2\tau}{M(\tau)(2-\tau)}l_5t \right)^n \right]. \end{aligned} \tag{22}$$

Hence, the system solution exists and also it is continuous. In order to confirm that above functions construct solution for the model (9), we consider as

$$\begin{aligned} S(t) - S(0) &= S_n(t) - \mathcal{A}_{1n}(t), \\ L(t) - L(0) &= L_n(t) - \mathcal{A}_{2n}(t), \\ I(t) - I(0) &= I_n(t) - \mathcal{A}_{3n}(t), \\ T(t) - T(0) &= T_n(t) - \mathcal{A}_{4n}(t), \\ R(t) - R(0) &= R_n(t) - \mathcal{A}_{5n}(t). \end{aligned} \tag{23}$$

Therefore, we get

$$\begin{aligned} &\|\mathcal{A}_{1n}(t)\| \\ &= \left\| \frac{2(1-\tau)}{M(\tau)(2-\tau)}(\mathcal{F}_1(t, S) - \mathcal{F}_1(t, S_{n-1})) + \frac{2\tau}{M(\tau)(2-\tau)} \int_0^t (\mathcal{F}_1(\zeta, S) - \mathcal{F}_1(\zeta, S_{n-1}))d\zeta \right\|, \end{aligned}$$

$$\begin{aligned} &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\|(\mathcal{F}_1(t, S) - \mathcal{F}_1(t, S_{n-1}))\| + \frac{2\tau}{(2-\tau)M(\tau)} \int_0^t \|(\mathcal{F}_1(\zeta, S) - \mathcal{F}_1(\zeta, S_{n-1}))\|d\zeta, \\ &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}l_1\|S - S_{n-1}\| + \frac{2\tau}{(2-\tau)M(\tau)}l_1\|S - S_{n-1}\|t. \end{aligned} \tag{24}$$

On repeating the same procedure we obtained

$$\|\mathcal{A}_{1n}(t)\| \leq \left(\frac{2(1-\tau)}{2M(\tau) - \tau M(\tau)} + \frac{2\tau}{2M(\tau) - \tau M(\tau)}t \right)^{n+1} l_1^{n+1}b. \tag{25}$$

At t_1 , we have

$$\|\mathcal{A}_{1n}(t)\| \leq \left(\frac{2(1-\tau)}{M(\tau)(2-\tau)} + \frac{2\tau}{M(\tau)(2-\tau)}t_1 \right)^{n+1} l_1^{n+1}b. \tag{26}$$

Applying limit on Eq. (25) as n approaches to ∞ , we get $\|\mathcal{A}_{1n}(t)\| \rightarrow 0$. In similar way we can proceed to show that

$$\|\mathcal{A}_{2n}(t)\| \rightarrow 0, \|\mathcal{A}_{3n}(t)\| \rightarrow 0, \|\mathcal{A}_{4n}(t)\| \rightarrow 0, \|\mathcal{A}_{5n}(t)\| \rightarrow 0.$$

□

Further for uniqueness of the solution let suppose that there exists another solution of the proposed model say $S_1(t)$, $L_1(t)$, $I_1(t)$, $T_1(t)$, and $R_1(t)$ then proceeds as follows:

$$\begin{aligned} S(t) - S_1(t) &= \frac{2(1-\tau)}{(2-\tau)M(\tau)}(\mathcal{F}_1(t, S) - \mathcal{F}_1(t, S_1)) + \frac{2\tau}{(2-\tau)M(\tau)} \times \\ &\int_0^t (\mathcal{F}_1(\zeta, S) - \mathcal{F}_1(\zeta, S_1))d\zeta. \end{aligned} \tag{27}$$

By taking norm of Eq. (27), we obtained

$$\begin{aligned} \|S(t) - S_1(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}\|\mathcal{F}_1(t, S) - \mathcal{F}_1(t, S_1)\| + \frac{2\tau}{(2-\tau)M(\tau)} \times \\ &\int_0^t \|\mathcal{F}_1(\zeta, S) - \mathcal{F}_1(\zeta, S_1)\|d\zeta. \end{aligned} \tag{28}$$

Using Lipschitz condition (13) it simplifies to

$$\begin{aligned} \|S(t) - S_1(t)\| &\leq \frac{2(1-\tau)}{(2-\tau)M(\tau)}l_1\|S(t) - S_1(t)\| + \frac{2\tau}{(2-\tau)M(\tau)} \times \\ &l_1t\|S(t) - S_1(t)\|. \end{aligned} \tag{29}$$

It simplifies to

$$\|S(t) - S_1(t)\| \left(1 - \frac{2(1-\tau)}{(2-\tau)M(\tau)}l_1 - \frac{2\tau}{(2-\tau)M(\tau)}l_1t \right) \leq 0. \tag{30}$$

Theorem 4.3. *If the condition given below holds*

$$\left(1 - \frac{2(1-\tau)}{(2-\tau)M(\tau)}l_1 - \frac{2\tau}{(2-\tau)M(\tau)}l_1t \right) > 0,$$

then the model solution will be unique.

Proof. Let the condition given by (30) holds, so

$$\|S(t) - S_1(t)\| \left(1 - \frac{2(1-\tau)}{(2-\tau)M(\tau)}l_1 - \frac{2\tau}{(2-\tau)M(\tau)}l_1t \right) \leq 0. \tag{31}$$

Implies that

$$\|S(t) - S_1(t)\| = 0. \tag{32}$$

Then, we get

$$S(t) = S_1(t). \tag{33}$$

A similar equality can be shown for the rest. Hence, the model solution is unique. \square

5. Numerical scheme and simulations. The present part of the paper provides an approximate solution for the fractional order TB model (9) using two-step fractional Adams-Bashforth technique for the CF fractional derivative [15]. The system is written in fractional Volterra type using fundamental theorem of integration. To obtain the desired iterative scheme first we consider only the first equation of system (9) and proceeds as below. With the help of fundamental theorem of integration we get the following from the first equation of system (10)

$$S(t) - S(0) = \frac{(1 - \tau)}{M(\tau)} \mathcal{F}_1(t, S) + \frac{\tau}{M(\tau)} \int_0^t \mathcal{F}_1(\zeta, S) d\zeta. \tag{34}$$

For $t = t_{n+1}$, $n = 0, 1, 2, \dots$, we obtain

$$S(t_{n+1}) - S_0 = \frac{1 - \tau}{M(\tau)} \mathcal{F}_1(t_n, S_n) + \frac{\tau}{M(\tau)} \int_0^{t_{n+1}} \mathcal{F}_1(t, S) dt. \tag{35}$$

The successive terms difference is given as follows:

$$S_{n+1} - S_n = \frac{1 - \tau}{M(\tau)} \{ \mathcal{F}_1(t_n, S_n) - \mathcal{F}_1(t_{n-1}, S_{n-1}) \} + \frac{\tau}{M(\tau)} \int_{t_n}^{t_{n+1}} \mathcal{F}_1(t, S) dt. \tag{36}$$

Over the close interval $[t_k, t_{k+1}]$, the function $\mathcal{F}_1(t, S)$ can be approximated by the interpolation polynomial

$$P_k(t) \cong \frac{f(t_k, y_k)}{h} (t - t_{k-1}) - \frac{f(t_{k-1}, y_{k-1})}{h} (t - t_k), \tag{37}$$

where $h = t_n - t_{n-1}$. Calculating the integral in (36) using above polynomial approximation we get

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \mathcal{F}_1(t, S) dt &= \int_{t_n}^{t_{n+1}} \frac{\mathcal{F}_1(t_n, S_n)}{h} (t - t_{n-1}) - \frac{\mathcal{F}_1(t_{n-1}, S_{n-1})}{h} (t - t_n) dt \\ &= \frac{3h}{2} \mathcal{F}_1(t_n, S_n) - \frac{h}{2} \mathcal{F}_1(t_{n-1}, S_{n-1}). \end{aligned} \tag{38}$$

Putting (38) in (36) and after simplification we obtained

$$S_{n+1} = S_n + \left(\frac{1 - \tau}{M(\tau)} + \frac{3h}{2M(\tau)} \right) \mathcal{F}_1(t_n, S_n) - \left(\frac{1 - \tau}{M(\tau)} + \frac{\tau h}{2M(\tau)} \right) \mathcal{F}_1(t_{n-1}, S_{n-1}). \tag{39}$$

In similar way for the rest of equations of system (10) we obtained the recursive formulae as below

$$L_{n+1} = L_0 + \left(\frac{1 - \tau}{M(\tau)} + \frac{3h}{2M(\tau)} \right) \mathcal{F}_2(t_n, L_n) - \left(\frac{1 - \tau}{M(\tau)} + \frac{\tau h}{2M(\tau)} \right) \mathcal{F}_2(t_{n-1}, L_{n-1}),$$

$$\begin{aligned}
I_{n+1} &= I_0 + \left(\frac{1-\tau}{M(\tau)} + \frac{3h}{2M(\tau)} \right) \mathcal{F}_3(t_n, I_n) - \left(\frac{1-\tau}{M(\tau)} + \frac{\tau h}{2M(\tau)} \right) \mathcal{F}_3(t_{n-1}, I_{n-1}), \\
T_{n+1} &= T_0 + \left(\frac{1-\tau}{M(\tau)} + \frac{3h}{2M(\tau)} \right) \mathcal{F}_4(t_n, T_n) - \left(\frac{1-\tau}{M(\tau)} + \frac{\tau h}{2M(\tau)} \right) \mathcal{F}_4(t_{n-1}, T_{n-1}), \\
R_{n+1} &= R_0 + \left(\frac{1-\tau}{M(\tau)} + \frac{3h}{2M(\tau)} \right) \mathcal{F}_5(t_n, R_n) - \left(\frac{1-\tau}{M(\tau)} + \frac{\tau h}{2M(\tau)} \right) \mathcal{F}_5(t_{n-1}, R_{n-1}).
\end{aligned} \tag{40}$$

Moreover, we describe the numerical simulations to observe the dynamics of propose model stated in (9) for various values of arbitrary order of CF derivative $\tau \in [0, 1]$ and model relevant parameters. The incidence of TB cases and its comparison with the proposed model (9) when the fractional order parameter is $\tau = 1$, is depicted in Figure 1 and 2 and Figure 3 is the behavior of infected TB individuals with the realistic data for the long term and shows that the that at a long term level the model curve is accurately match to the realistic data. The numerical values used in the solution of the fractional TB Caputo-Fabrizio model (9) are given in Table 1 where some of the parameter values are considered from the literature. We implement the above scheme presented in Eqs. (39) and (40). The time interval taken in the simulations are considered up to 100 unit (years). In Fig 4, we shown the dynamics of the susceptible susceptible individuals by considering various values of the fractional order parameter τ . Clearly, the susceptible population increases with the decreasing values of the fractional parameter τ . Figures 5-8 show the influence of the variation in the fractional order τ on the biological behavior of the rest of classes of model 9. It is noted from these Figures that population in all the infected classes of the model have a decreasing effect when we decrease τ . The behavior of cumulative TB infective for various values of τ is addressed in Fig 9. It is clear that the total TB infective decrease significantly when the fractional order decreases. The impact of treatment rate (denoted by parameter γ) and fractional order τ on the behavior of total TB infected individuals is despite in Fig 10. Form here we analyzed that if we increase the value of γ and decrease τ , the TB infective people decreases at significant rate. Finally, Fig 11 shows the impact of treatment failure rate (denoted by parameter η) and fractional parameter τ on the behavior of cumulative TB active population of Khyber Pakhtunkhwa. Decrease in both parameters η and τ also decreases the total TB infected individuals.

6. Conclusion. A fractional order transmission model using CF derivative for the TB dynamics is analyzed. The equilibria and threshold number \mathcal{R}_0 of the model are determined. We examine the existence of the variables of the model and proved its uniqueness. Using two-step Adams-Bashforth technique [15] for the CF fractional derivative is implemented to derived an iterative solution of the propose TB model of fractional order $\tau \in [0, 1]$. Finally, the numerical simulations of the model are plotted and discussed briefly for various values of the fractional order τ . From the graphical results we conclude that the fractional order derivative of CF type provide more realistic analysis than the classical integer-order TB model. From the data reported by NTP Pakistan we parameterized the parameters and give useful information about the nature of the disease spread and control. The present situation of the TB infected plot is threatening and a serious health issue of population of

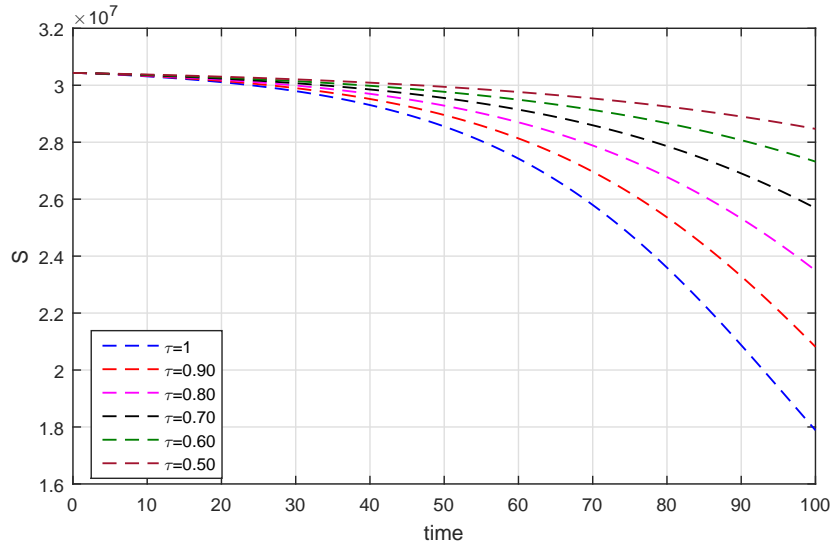


Figure 4: Simulation of S with τ .

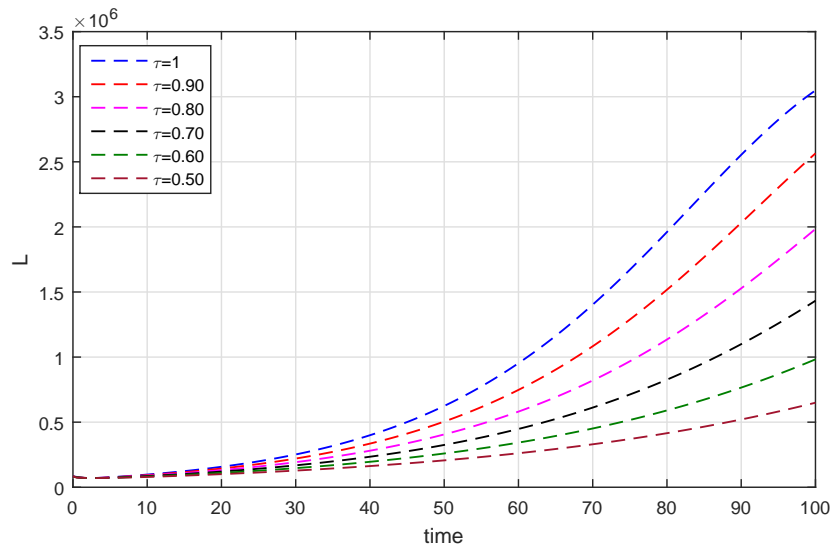


Figure 5: Simulation of L with τ .

Khyber Pakhtunkhwa. The main reasons of the current situation of this infection includes lack of awareness in the community, treatment failure and lack of proper follow up of under treatment patients. The main goal of the present study is to help the government of Pakistan to initiate various programs to eliminate TB infection in the community.

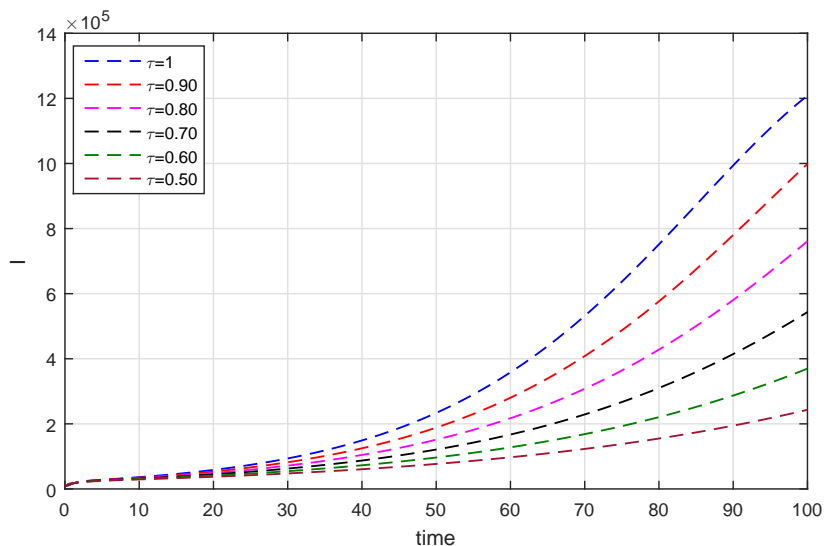


Figure 6: Simulation of I with τ .

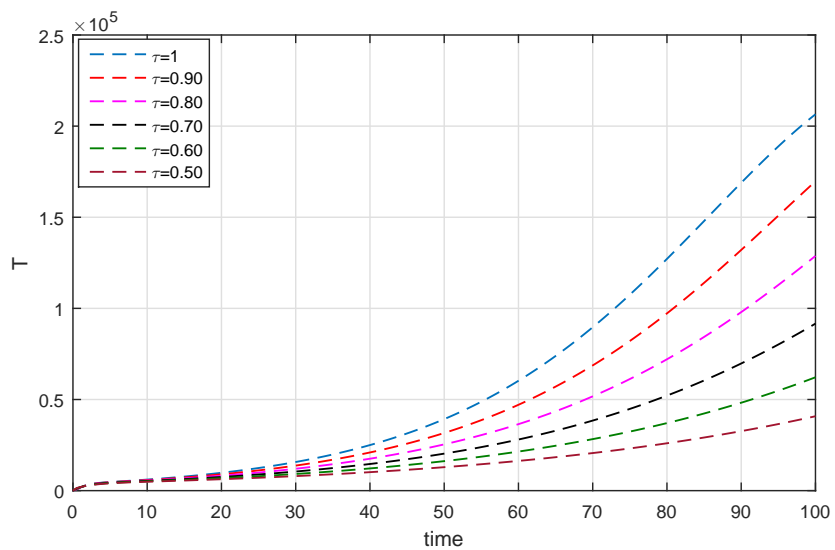


Figure 7: Simulation of T with τ .

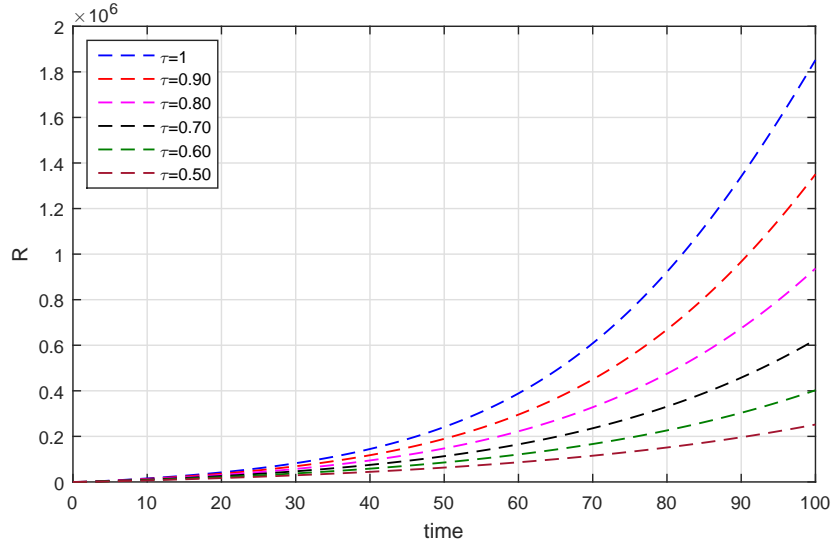


Figure 8: Simulation of R with τ .

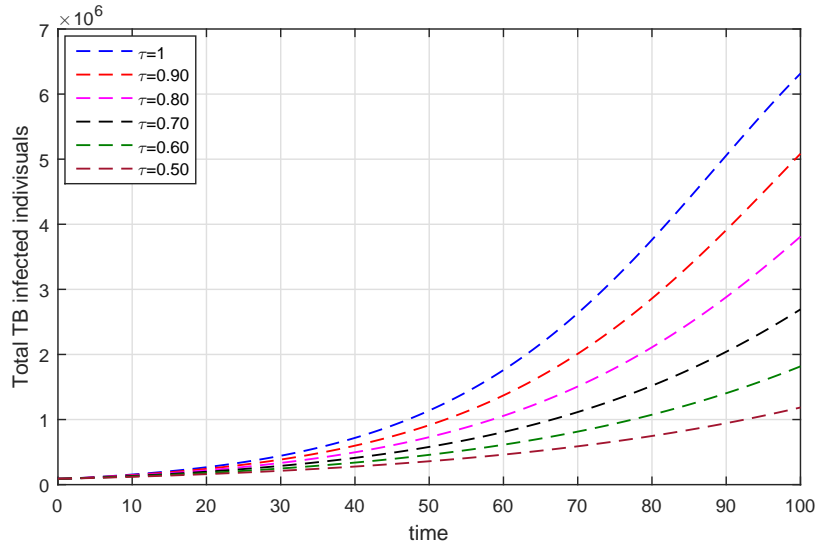


Figure 9: Simulation of cumulative TB infected people with τ .

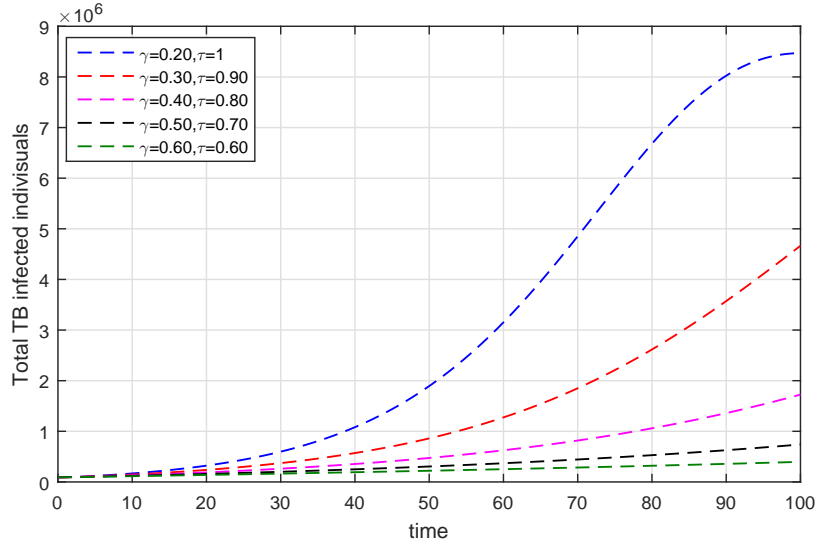


Figure 10: The graphical result of the total infected people for several values of the parameter γ (treatment rate) and τ (fractional parameter).

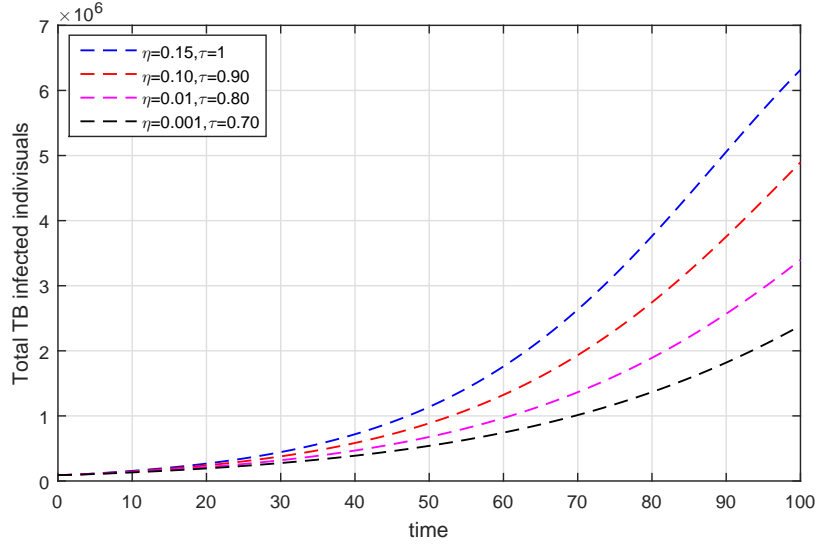


Figure 11: The graphical result of the total infective with TB individuals for various values of the parameter η (treatment failure rate) and τ (fractional parameter).

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