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Research Article

On Local Fractional Continuous Wavelet Transform

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We introduce a new wavelet transform within the framework of the local fractional calculus. An illustrative example of local fractional wavelet transform is also presented.

1. Introduction

Wavelet transforms have been applied successfully in the areas of signals analysis, data compression, and sound processing (see, for details, [1–6] and the references cited therein). Although there is scaled and shifted versions of a mother wavelet, the daughter wavelets are structured as follows (see [3–5]):

$$\varphi_{a,b}(t) = \frac{1}{a^{1/2}} \varphi\left(\frac{t-b}{a}\right), \tag{1}$$

where a is the dyadic dilation, b is the dyadic position, and $a^{-1/2}$ is the normalization factor. The expression of a one-dimensional wavelet transform for a given continuous signal $f(t)$ is given by

$$W_\varphi f(a,b) = \int_{-\infty}^{\infty} f(t)\, \overline{\varphi_{a,b}(t)} dt \tag{2}$$

and the reconstruction formula becomes

$$f(x) = C_\varphi \iint_{-\infty}^{\infty} \frac{1}{a^2} W_\varphi f(a,b)\, \varphi_{a,b}(t)\, da\, db, \tag{3}$$

where

$$C_\varphi = \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|} dx. \tag{4}$$

Recently, fractional wavelet transform, as a generalization of the classical wavelet transform, was proposed in [7]. The one-dimensional fractal wavelet transform of a continuous signal $f(t)$ has the following form:

$$W_\varphi f(a,b) = \iint_{-\infty}^{\infty} B(x,t)\, f(t)\, \overline{\varphi_{a,b}(x)} dt\, dx, \tag{5}$$

where $B(x,t)$ denotes a bulk optics kernel.

The reconstructing formula of the input is defined as given by the following expression:

$$f(x) = \frac{1}{C_\varphi} \iint_{-\infty}^{\infty} \frac{1}{a^3} W_\varphi f(a,b)\, B(x,t)\, \varphi_{a,b}(t)\, da\, db\, dt\, dt. \tag{6}$$

We notice that the fractional wavelet transforms was applied to image encryption [8], to the simultaneous spectral analysis in [9], and to the composite signals in [10, 11]. For other definition of fractional wavelet transform, see [12] and the references cited therein.

Keeping in mind the study of the fractal signals (local fractional continuous signals), a new local fractional wavelet transform was developed in [13] based upon the local fractional Fourier transform [14] via local fractional calculus [15–18]. In this paper, we investigate the local fractional Fourier transform to deal with the local fractional wavelet transforms by implementing the local fractional calculus.

The organization of the paper is as follows. Section 2 presents the concept of local fractional Fourier transform and wavelet. Section 3 discusses the derivation of the local fractional continuous wavelet transform. Section 4 studies the wave space and Section 5 present an illustrative example. Finally, Section 6 outlines the main conclusions of our present investigation.

2. Local Fractional Fourier Transform and Wavelet

Let $f(x)$ be local fractional continuous function, which is denoted as follows (see [18]):

$$f(x) \in C_\alpha(-\infty, \infty). \tag{7}$$

The space of local fractional continuous functions $C_{p,\alpha}[a,b]$, under p-norm, is given by (see [13])

$$\|f\|_{p,\alpha} = \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p (dx)^\alpha\right)^{1/p}, \quad \text{for } 1 \le p < \infty, \tag{8}$$

where the operator is local fractional operator.

The space $L_{p,\alpha}[\mathbb{R}]$ norm on $C_{p,\alpha}[\mathbb{R}]$ is defined by

$$\|f\|_{p,\alpha} =: \left(\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} |f(x)|^p (dx)^\alpha\right)^{1/p} < \infty \tag{9}$$

for $1 \le p < \infty$. This is infinite for a and b.

The local fractional Fourier transforms in fractal space is defined as follows (see [13, 14]):

$$\begin{aligned} F_\alpha\{f(x)\} &= f_\omega^{F,\alpha}(\omega) \\ &:= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} E_\alpha(-i^\alpha\omega^\alpha x^\alpha) f(x)(dx)^\alpha. \end{aligned} \tag{10}$$

Its inverse is formulated as follows (see [13, 14]):

$$\begin{aligned} f(x) &= F_\alpha^{-1}\left(f_\omega^{F,\alpha}(\omega)\right) \\ &:= \frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty} E_\alpha(i^\alpha\omega^\alpha x^\alpha) f_\omega^{F,\alpha}(\omega)(d\omega)^\alpha, \quad x > 0. \end{aligned} \tag{11}$$

Let $\varphi(x) \in L_{2,\alpha}[\mathbb{R}]$ and let

$$\varphi_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} E_\alpha(-i^\alpha\omega^\alpha x^\alpha)\varphi(x)(dx)^\alpha, \quad 0 < \alpha \le 1. \tag{12}$$

When

$$\varphi_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} \varphi(x)(dx)^\alpha = 0, \quad 0 < \alpha \le 1, \tag{13}$$

the function $\varphi(x)$ is called a local fractional wavelet [13].

Let $\varphi(x) \in L_{2,\alpha}[\mathbb{R}]$. Then, we have

$$\begin{aligned} \|\varphi_{a,b,\alpha}(t)\|_{1,\alpha}^2 &= \frac{1}{a^\alpha\Gamma(1+\alpha)}\int_{-\infty}^{\infty} |\varphi_{a,b,\alpha}(t)|^2 (dx)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} |\varphi(t)|^2 (dx)^\alpha = \|\varphi\|_{1,\alpha}^2, \end{aligned} \tag{14}$$

so that

$$\varphi_{a,b,\alpha}(t) = \frac{1}{a^{\alpha/2}}\varphi\left(\frac{t-b}{a}\right), \tag{15}$$

where $a, b \in \mathbb{R}$ and $a \neq 0$.

3. Local Fractional Continuous Wavelet Transform

Let $\varphi \in L_{2,\alpha}[\mathbb{R}]$. Then, we arrive at the following relation:

$$\begin{aligned} \|\varphi_{a,b,\alpha}(t)\|_{2,\alpha}^1 &= \frac{1}{a^\alpha\Gamma(1+\alpha)}\int_{-\infty}^{\infty} |\varphi_{a,b,\alpha}(t)|^2 (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} |\varphi(t)|^2 (dt)^\alpha = \|\varphi\|_{2,\alpha}^1, \end{aligned} \tag{16}$$

where $\varphi_{a,b,\alpha}(t) = (1/a^{\alpha/2})\varphi((t-b)/a)$, $a, b \in \mathbb{R}$, and $a \neq 0$.

Similarly, we get

$$\|\varphi_{b,\alpha}(t)\|_{2,\alpha}^1 = \|\varphi\|_{2,\alpha}^1. \tag{17}$$

Taking $\varphi_{b,\alpha}(t)$ in place of $\varphi_{a,b,\alpha}(t)E_\alpha(-i^\alpha\omega^\alpha t^\alpha)$, we obtain

$$\begin{aligned} \Theta_{\varphi_{a,b,\alpha}} f(a,b) &= \langle f(t), \varphi_{a,b,\alpha}(t)\rangle \\ &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} f(t)\overline{\varphi_{a,b,\alpha}(t)}(dt)^\alpha. \end{aligned} \tag{18}$$

In the special case when $f(t) = 1$, we have the following relation:

$$\Theta_{\varphi_{a,b,\alpha}}(a,b) = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} \overline{\varphi_{a,b,\alpha}(t)}(dt)^\alpha \tag{19}$$

such that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} \Theta_{\varphi_{a,b,\alpha}}(a,b)\varphi_{a,b,\alpha}(t)(db)^\alpha \\ &\quad = |a|^\alpha\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\varphi(t)(dt)^\alpha\right]^2. \end{aligned} \tag{20}$$

Hence, there exists the following relation:

$$\begin{aligned} &\frac{1}{\Gamma^2(1+\alpha)}\iint_{-\infty}^{\infty} a^{-2\alpha}\Theta_{\varphi_{a,b,\alpha}}(a,b)\varphi_{a,b,\alpha}(t)(da)^\alpha(db)^\alpha \\ &\quad = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} |x|^{-\alpha}(dx)^\alpha. \end{aligned} \tag{21}$$

In general, we also deduce the following identities:

$$f(x) = \frac{\int_{-\infty}^{\infty} \left(\left| f(x) \right|^2 / |x|^\alpha \right) (dx)^\alpha}{\Gamma^3 (1+\alpha)} \times \iint_{-\infty}^{\infty} a^{-2\alpha} \Theta_{\varphi_{a,b,\alpha}} f(a,b) \varphi_{a,b,\alpha}(t) (db)^\alpha (da)^\alpha,$$
$$\Theta_{\varphi_{a,b,\alpha}} f(a,b) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha. \tag{22}$$

Now, we establish the following relations:

$$\Theta_{\varphi_{a,b,\alpha}} \widetilde{f(a,b)} = \frac{a^{-\alpha/2}}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha, \quad 0 < \alpha \le 1,$$
$$f(x) = \frac{\int_{-\infty}^{\infty} \left(\left| f(x) \right|^2 / |x|^\alpha \right) (dx)^\alpha}{\Gamma^3 (1+\alpha)} \times \iint_{-\infty}^{\infty} a^{-2\alpha} \Theta_{\varphi_{a,b,\alpha}} \widetilde{f(a,b)} \varphi_{a,b,\alpha} \times (t) (da)^\alpha (db)^\alpha. \tag{23}$$

Hence, the local fractional continuous wavelet transform takes the following form (see [13]):

$$W_{\varphi,\alpha} f(a,b) = \frac{a^{-\alpha/2}}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha, \quad 0 < \alpha \le 1. \tag{24}$$

And the inversion formula of local fractional continuous wavelet transform is derived as follows (see [14]):

$$f(x) = \frac{C_{\varphi,\alpha}}{\Gamma^2(1+\alpha)} \times \iint_{-\infty}^{\infty} a^{-2\alpha} W_{\varphi,\alpha} f(a,b) \varphi_{a,b,\alpha}(t) (da)^\alpha (db)^\alpha, \quad 0 < \alpha \le 1, \tag{25}$$

where

$$C_{\varphi,\alpha} = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \frac{\left| f(x) \right|^2}{|x|^\alpha} (dx)^\alpha, \quad 0 < \alpha \le 1. \tag{26}$$

4. The Wavelet Space

In order to differ the classical wavelets from fractional wavelets, here we formulate a wavelet space as follows. In fact, a wavelet space is defined by

$$W_{\varphi,\alpha}[\mathbb{R}] = \left\{ (\varphi,\alpha) : W_{\varphi,\alpha} f(a,b) = \frac{a^{-\alpha/2}}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha, \ 0 < \alpha \le 1 \right\}. \tag{27}$$

When the fractal dimension α is equal to 1, from (27), we deduce (see [3–5])

$$W_{\varphi,1}[\mathbb{R}] = \left\{ (\varphi,1) : W_{\varphi,1} f(a,b) = a^{-1/2} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,1}(t)} dt, \ \alpha = 1 \right\}, \tag{28}$$

where $f(t)$ is continuous and $W_{\varphi,1} f(a,b) \in W_{\varphi,1}[\mathbb{R}]$.

Taking the fractal dimension $0 < \alpha < 1$, we derive a formula given by

$$W_{\varphi,\alpha} f(a,b) = \frac{a^{-\alpha/2}}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha \tag{29}$$

with $W_{\varphi,\alpha} f(a,b) \in W_{\varphi,\alpha}[\mathbb{R}]$, where $f(t)$ is a local fractional continuous function.

5. An Illustrative Example

In order to construct the local fractional continuous wavelet, we suppose that $\phi(t)$ is $m\alpha$ times the local fractional differentiable function.

We define the local fractional wavelet $\varphi(t)$ by means of the following expression:

$$\varphi(t) = \frac{d^{m\alpha} y}{dx^{m\alpha}}, \tag{30}$$

where the differential operator is the local fractional operator proposed by Yang [18] (for other definition, see [19] and the references cited therein).

Then, we get

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \varphi(t) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)} (dt)^\alpha = 0 \quad (m = 0, 1, 2, \ldots, m). \tag{31}$$

Let us consider the following nondifferentiable signal, namely,

$$\phi_{H(\alpha)}(t) = \begin{cases} \dfrac{t^\alpha}{\Gamma(1+\alpha)}, & 0 \le t < \dfrac{1}{2}, \\ \dfrac{(1-t)^\alpha}{\Gamma(1+\alpha)}, & \dfrac{1}{2} \le t < 1, \\ 0, & \text{else}. \end{cases} \tag{32}$$

For $0 \le t < 1/2$, we obtain

$$\frac{d^\alpha \phi_{H(\alpha)}(t)}{dt^\alpha} = \frac{d^\alpha}{dt^\alpha} \frac{t^\alpha}{\Gamma(1+\alpha)} = 1. \tag{33}$$

For $1/2 \le t < 1$, we obtain

$$\frac{d^\alpha \phi_{H(\alpha)}(t)}{dt^\alpha} = \frac{d^\alpha}{dt^\alpha} \frac{(1-t)^\alpha}{\Gamma(1+\alpha)} = -1. \tag{34}$$

In view of (33)-(34), we get a local fractional wavelet given by

$$\varphi_{H(\alpha)}(t)=\begin{cases}1, & 0\le t<\dfrac{1}{2},\\ -1, & \dfrac{1}{2}\le t<1,\\ 0, & \text{else}.\end{cases} \quad (35)$$

Following (35), we obtain

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\varphi_{H(\alpha)}(t)(dt)^{\alpha}&=0,\\ \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\varphi_{H(\alpha)}^{2}(t)(dt)^{\alpha}&=1.\end{aligned} \quad (36)$$

In view of (15), taking $a=2^{-j}$ and $b=k2^{-j}$, we have

$$\begin{aligned}\varphi_{a,b,\alpha}(t)&=\frac{1}{a^{\alpha/2}}\varphi\left(\frac{t-b}{a}\right)=\varphi_{j,k,\alpha}(t)\\ &=\varphi_{2^{-j},k2^{-j},\alpha}(t)=2^{j\alpha/2}\varphi\left(2^{j}t-k\right)\end{aligned} \quad (37)$$

for integers $j,k\in\mathbb{Z}$.

Hence, we get the following equation:

$$\varphi_{H(\alpha)}^{j,k}(t)=2^{j\alpha/2}\varphi_{H(\alpha)}\left(2^{j}t-k\right). \quad (38)$$

We thus conclude that

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[\varphi_{H(\alpha)}^{j,k}(t)\right]^{2}(dt)^{\alpha}\\ &=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[2^{j\alpha/2}\varphi_{H(\alpha)}\left(2^{j}t-k\right)\right]^{2}(dt)^{\alpha}\\ &=2^{j\alpha}\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[\varphi_{H(\alpha)}\left(2^{j}t-k\right)\right]^{2}(dt)^{\alpha}\\ &=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[\varphi_{H(\alpha)}\left(2^{j}t-k\right)\right]^{2}\left(d\left(2^{j}t-k\right)\right)^{\alpha}\\ &=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[\varphi_{H(\alpha)}(t)\right]^{2}(dt)^{\alpha}\\ &=1,\\ &\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\varphi_{H(\alpha)}^{j,k}(t)(dt)^{\alpha}\\ &=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1/2}2^{j\alpha/2}\varphi_{H(\alpha)}\left(2^{j}t-k\right)(dt)^{\alpha}\\ &\quad-\frac{1}{\Gamma(1+\alpha)}\int_{1/2}^{1}2^{j\alpha/2}\varphi_{H(\alpha)}\left(2^{j}t-k\right)(dt)^{\alpha}\\ &=0.\end{aligned} \quad (39)$$

6. Concluding Remarks and Observations

A novel local fractional wavelet transformation was investigated by using Fourier transform based upon local fractional calculus. This transform has been found to be advantageous in dealing with the functions in fractal space. The wave space is considered and an illustrative example is shown.

Conflict of Interests

The authors declare that they have no conflict of interests regarding this paper.

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