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Research Article

Lower and Upper Solutions Method for Positive Solutions of Fractional Boundary Value Problems

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We apply the lower and upper solutions method and fixed-point theorems to prove the existence of positive solution to fractional boundary value problem $D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0$, $0 < t < 1$, $2 < \alpha \leq 3$, $u(0) = u'(0) = 0$, $D_{0^+}^{\alpha-1}u(1) = \beta u(\xi)$, $0 < \xi < 1$, where $D_{0^+}^{\alpha}$ denotes Riemann-Liouville fractional derivative, β is positive real number, $\beta\xi^{\alpha-1} \geq 2\Gamma(\alpha)$, and f is continuous on $[0,1] \times [0,\infty)$. As an application, one example is given to illustrate the main result.

1. Introduction

In the recent years, fractional calculus has been one of the most interesting issues that have attracted many scientists, especially in the fields of mathematics and engineering sciences. Many natural phenomena can be presented by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, and viscoelasticity try to present a model of these phenomena by boundary value problems of fractional differential equations [1–4]. In order to achieve extra information in fractional calculus, interested readers can refer to more valuable books that are written by other authors [5–20].

The existence and multiplicity of solutions or positive solutions of nonlinear fractional differential equation (FDE) by the use of fixed point theorems, Leray-Shauder theory, and so forth are mentioned in some papers [6, 8, 12, 20, 21]. Few papers have considered the boundary value problems of fractional differential equations [12, 14]. By the use of some fixed point theorems on cones, Zhang [15] obtained the existence of positive solution for the equation

$$D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \ 1 < \alpha \leq 2, \tag{1}$$

with the boundary conditions

$$u(0) = u(1) = 0. \tag{2}$$

In [22], Liang and Zhang applied lower and upper solutions method and fixed point theorems to obtain some results on the existence of positive solutions for the following BVPs:

$$\begin{gathered} D_{0^+}^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1, \ 3 < \alpha \leq 4, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered} \tag{3}$$

where D^{α} denotes Riemann-Liouville fractional derivative.

In this paper, we investigate the existence of positive solution for a nonlocal BVP of FDE,

$$D_{0^+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\ 2<\alpha\leq 3, \tag{4}$$

$$u(0)=u'(0)=0,\quad D_{0^+}^{\alpha-1}u(1)=\beta u(\xi), \tag{5}$$

using lower and upper solutions method and fixed point theorem, where D^{α} denotes standard Riemann-Liouville fractional derivative, $\beta\xi^{\alpha-1}\geq 2\Gamma(\alpha)$, and $f\in C([0,1]\times[0,\infty),[0,\infty))$.

The main result of this paper can be seen in Theorem 10. In Theorem 10, we use the following conditions:

(H_1) $f(t,u(t))\in C([0,1]\times[0,\infty),\mathbb{R}^+)$ is nondecreasing with respect to u,

(H_2) $f(t,\varrho(t))\neq 0$ for $t\in(0,1)$,

(H_3) there exist a positive constant $\lambda<1$ such that $k^{\lambda}f(t,u)\leq f(t,ku)$, for all $0\leq k\leq 1$, and the Schauder fixed-point theorem to show that problem (4)-(5) has a positive solution.

2. Basic Definitions and Preliminaries

In this section, we present the necessary definitions and lemmas that will be used to prove our new results.

Definition 1 (see [5, 6]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$I_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds,\quad n-1<\alpha\leq n, \tag{6}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2 (see [5, 6]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$D_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^t(t-s)^{\alpha-1}f(s)\,ds,\qquad n-1<\alpha\leq n, \tag{7}$$

where $n=[\alpha]+1$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 3 (see [5, 6]). A function $\mu(t)\in C^2[0,1]$ is called a lower solution of problem (4)-(5) if $\mu(t)$ satisfies

$$\begin{gathered}-D_{0^+}^{\alpha}\mu(t)\leq f(t,\mu(t)),\quad 0<t<1,\ 2<\alpha\leq 3,\\ \mu(0)\leq 0,\quad \mu'(0)\leq 0,\quad D_{0^+}^{\alpha}\mu(t)\leq\beta\mu(\xi).\end{gathered} \tag{8}$$

Definition 4 (see [7, 8]). A function $\mu(t)\in C^2[0,1]$ is called an upper solution of problem (4)-(5) if $\mu(t)$ satisfies

$$\begin{gathered}-D_{0^+}^{\alpha}\mu(t)\geq f(t,\mu(t)),\quad 0<t<1,\ 2<\alpha\leq 3,\\ \mu(0)\geq 0,\quad \mu'(0)\geq 0,\quad D_{0^+}^{\alpha}\mu(t)\geq\beta\mu(\xi).\end{gathered} \tag{9}$$

Lemma 5 (see [7, 8]). *Let $u\in C(0,1)\cap L^1(0,1)$. Then the fractional differential equation*

$$D^{\alpha}u(t)=0 \tag{10}$$

has

$$\begin{gathered}u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},\\ \text{for some } c_i\in\mathbb{R},\ i=1,\dots,n,\end{gathered} \tag{11}$$

as a unique solution.

Lemma 6 (see [7, 8]). *Let $u\in C(0,1)\cap L^1(0,1)$ with a fractional derivative of order $\alpha>0$, $n-1<\alpha\leq n$ $(n\in\mathbb{N})$, that belongs to $C(0,1)\cap L(0,1)$. Then*

$$\begin{gathered}I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},\\ \text{for some } c_i\in\mathbb{R},\ i=1,\dots,n.\end{gathered} \tag{12}$$

Lemma 7. *If $\beta\xi^{\alpha-1}\geq 2\Gamma(\alpha)$, then for $0\leq y(t)\in C[0,1]$, the problem,*

$$D_{0^+}^{\alpha}u(t)+y(t)=0,\quad 0<t<1,\ 2<\alpha\leq 3, \tag{13}$$

$$u(0)=u'(0)=0,\quad D_{0^+}^{\alpha-1}u(1)=\beta u(\xi), \tag{14}$$

has a unique positive solution

$$\begin{aligned}u(t)=&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds\\ &+\frac{1}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}\int_0^1t^{\alpha-1}y(s)\,ds\\ &+\frac{\beta}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_0^{\xi}t^{\alpha-1}(\xi-s)^{\alpha-1}y(s)\,ds.\end{aligned} \tag{15}$$

Proof. We can apply Lemma 6 to reduce (13) to an equivalent integral equation

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}, \tag{16}$$

for some $c_1,c_2,c_3\in\mathbb{R}$. From $u(0)=0$ and $u'(0)=0$ in (14), we have $c_2=c_3=0$. On the other hand, $D^{\alpha-1}u(1)=\beta u(\xi)$ yields

$$\begin{aligned}c_1=&\frac{1}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}\int_0^t y(s)\,ds\\ &+\frac{\beta}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}y(s)\,ds.\end{aligned} \tag{17}$$

Then, the unique solution of problem is given by $u(t)$. Obviously, $u(t)\geq 0$ if $y(t)\geq 0$ on $t\in[0,1]$. The proof is complete. □

3. Main Result

In this section, we present and prove our main result.

Lemma 8. *Suppose that $\beta\xi^{\alpha-1} \geq 2\Gamma(\alpha)$. Given that $y \in C[0,1]$, the Green function for the problem (13)-(14) is given by*

$$G(t,s) = \begin{cases} \left(-(t-s)^{\alpha-1}\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right] + \Gamma(\alpha)t^{\alpha-1} + \beta t^{\alpha-1}(\xi-s)^{\alpha-1}\right) \times \left(\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)\right)^{-1}, & 0 \leq s \leq t \leq 1,\ s \leq \xi, \\ \left(-(t-s)^{\alpha-1}\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right] + \Gamma(\alpha)t^{\alpha-1}\right) \times \left(\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)\right)^{-1}, & 0 \leq \xi \leq s \leq t \leq 1, \\ \dfrac{\Gamma(\alpha)t^{\alpha-1} + \beta t^{\alpha-1}(\xi-s)^{\alpha-1}}{\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)}, & 0 \leq t \leq s \leq \xi \leq 1, \\ \dfrac{t^{\alpha-1}}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}, & t \leq s,\ s \geq \xi. \end{cases} \tag{18}$$

Proof. By Lemma 7, for $t \leq \xi$, we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds + \frac{1}{\beta\xi^{\alpha-1}-\Gamma(\alpha)} \times \left[\left(\int_0^t + \int_t^\xi + \int_\xi^1\right) t^{\alpha-1} y(s)\,ds\right] \\ &\quad + \frac{\beta}{\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)} \times \left[\left(\int_0^t + \int_t^\xi\right) t^{\alpha-1}(\xi-s)^{\alpha-1} y(s)\,ds\right] \\ &= \int_0^t \Big(\Big(-(t-s)^{\alpha-1}\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right] + \Gamma(\alpha)t^{\alpha-1} + \beta t^{\alpha-1}(\xi-s)^{\alpha-1}\Big) \times \left(\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)\right)^{-1}\Big) y(s)\,ds \\ &\quad + \int_t^\xi \frac{\Gamma(\alpha)t^{\alpha-1} + \beta t^{\alpha-1}(\xi-s)^{\alpha-1}}{\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)} y(s)\,ds + \int_\xi^1 \frac{\Gamma(\alpha)t^{\alpha-1}}{\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)} y(s)\,ds \\ &= \int_0^1 G(t,s)\,y(s)\,ds. \end{aligned} \tag{19}$$

For $t \geq \xi$, we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\left[\left(\int_0^\xi \int_\xi^t\right)\right](t-s)^{\alpha-1} y(s)\,ds + \frac{1}{\beta\xi^{\alpha-1}-\Gamma(\alpha)} \times \left[\left(\int_0^\xi + \int_\xi^t + \int_t^1\right) t^{\alpha-1} y(s)\,ds\right] \\ &\quad + \frac{\beta}{\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)}\int_0^\xi t^{\alpha-1}(\xi-s)^{\alpha-1} y(s)\,ds \\ &= \int_0^\xi \Big(\Big(-(t-s)^{\alpha-1}\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right] + \Gamma(\alpha)t^{\alpha-1} + \beta t^{\alpha-1}(\xi-s)^{\alpha-1}\Big) \times \left(\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)\right)^{-1}\Big) y(s)\,ds \\ &\quad + \int_\xi^t \Big(\Big(-(t-s)^{\alpha-1}\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right] + \Gamma(\alpha)t^{\alpha-1}\Big) \times \left(\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)\right)^{-1}\Big) y(s)\,ds \\ &\quad + \int_t^1 \frac{\Gamma(\alpha)t^{\alpha-1}}{\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]\Gamma(\alpha)} y(s)\,ds \\ &= \int_0^1 G(t,s)\,y(s)\,ds. \end{aligned} \tag{20}$$

The proof is complete. □

Lemma 9. *Suppose that $u(t) \in C^2[0,1]$ and is a positive solution of (4)-(5). Then*

$$m\varrho(t) \leq u(t) \leq M\varrho(t), \tag{21}$$

where

$$\varrho(t) = \frac{1}{c\Gamma(\alpha+1)}\left[\beta\xi^{\alpha} - \Gamma(\alpha+1)\right]t^{\alpha-1} - ct^{\alpha}, \tag{22}$$

where $c = \beta\xi^{\alpha-1} - \Gamma(\alpha)$, $\beta\xi^{\alpha-1} \neq \Gamma(\alpha)$, and m and M are two constants.

Proof. Since $u(t) \in C^2[0,1]$, there exists $T > 0$ so that $|u(t)| \leq T$ for $t \in [0,1]$. We define

$$\begin{aligned} m &:= \min_{(t,u)\in[0,1]\times[0,T]} f(t,u(t)), \\ M &:= \max_{(t,u)\in[0,1]\times[0,T]} f(t,u(t)). \end{aligned} \tag{23}$$

Therefore, we have

$$m\int_0^1 G(t,s)\,ds \leq \int_0^1 G(t,s)\,f(s,u(s))\,ds \leq M\int_0^1 G(t,s)\,ds. \tag{24}$$

On the other hand, by direct computation, we get

$$\int_0^1 G(t,s)\,ds = \frac{1}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\,\Gamma(\alpha+1)} \times \left(\left[\beta\xi^{\alpha-1}-\Gamma(\alpha+1)\right]t^{\alpha-1}-\left[\beta\xi^{\alpha-1}-\Gamma(\alpha)\right]t^{\alpha}\right). \tag{25}$$

This completes the proof of the lemma. □

Theorem 10. *The fractional boundary value problem* (4)-(5) *has a positive solution $u(t)$ if the conditions* (H_1)–(H_3) *are satisfied.*

Proof. Suppose that

$$\begin{aligned}
\alpha_1 &= \min\left\{1, \inf_{t\in[0,1]} f(t,\varrho(t))\right\},\\
\alpha_2 &= \max\left\{1, \sup_{t\in[0,1]} f(t,\varrho(t))\right\},\\
0 < k_1 &\le \min\left\{\frac{1}{\alpha_2}, (\alpha_1)^{\lambda/(1-\lambda)}\right\},\\
k_2 &\ge \max\left\{\frac{1}{\alpha_2}, (\alpha_1)^{\lambda/(1-\lambda)}\right\},
\end{aligned} \tag{26}$$

and $h(t)=\int_0^1 G(t,s)f(s,\varrho(s))ds$. We show that $\mu(t)=k_1h(t)$ and $\nu(t)=k_2h(t)$ are lower and upper solutions of (4)-(5), respectively. From Lemma 7, $h(t)$ is a positive solution of the following problem:

$$\begin{aligned}
&-D_{0^+}^{\alpha}u(t) \le f(t,\varrho(t)), \quad 0<t<1,\ 2<\alpha\le 3,\\
&u(0)=u'(0)=0, \qquad D_{0^+}^{\alpha-1}u(1)=\beta u(\xi).
\end{aligned} \tag{27}$$

We know that $\alpha_1\varrho(t) \le h(t) \le \alpha_2\varrho(t)$. Now, using the assumption of the theorem, we get

$$\begin{aligned}
&k_1\alpha_1 \le \frac{h(t)}{\varrho(t)} \le k_2\alpha_2, \qquad \frac{1}{k_1\alpha_2} \le \frac{\varrho(t)}{h(t)} \le \frac{1}{k_2\alpha_1} \le 1,\\
&(k_1\alpha_1)^{\lambda} \ge k_1, \quad (k_2\alpha_2)^{\lambda} \ge k_2.
\end{aligned} \tag{28}$$

Therefore, from (H_3) and since $(k_1\alpha_1)^{\lambda} \ge k_1$, the following relations satisfy

$$\begin{aligned}
f(t,\mu(t)) &= f\left(t,\frac{\mu(t)}{\varrho(t)}\varrho(t)\right) \ge \left(\frac{\mu(t)}{\varrho(t)}\right)^{\lambda} f(t,\varrho(t))\\
&\ge (k_1\alpha_1)^{\lambda} f(t,\varrho(t)) > k_1 f(t,\varrho(t)),\\
k_2 f(t,\varrho(t)) &= f\left(t,\frac{\varrho(t)}{\nu(t)}\nu(t)\right) \ge k_2\left(\frac{\varrho(t)}{\nu(t)}\right)^{\lambda} f(t,\nu(t))\\
&\ge k_2(k_1\alpha_1)^{-\lambda} f(t,\nu(t)) > f(t,\nu(t)).
\end{aligned} \tag{29}$$

Consequently

$$\begin{aligned}
&-D_{0^+}^{\alpha}\mu(t) = k_1 f(t,\varrho(t)) \le f(t,\mu(t)),\\
&\qquad 0<t<1,\ 2<\alpha\le 3,\\
&-D_{0^+}^{\alpha}\nu(t) = k_2 f(t,\varrho(t)) \ge f(t,\nu(t)),\\
&\qquad 0<t<1,\ 2<\alpha\le 3.
\end{aligned} \tag{30}$$

Since $\mu(t)=k_1h(t)$ and $\nu(t)=k_2h(t)$ satisfy the boundary conditions, $\mu(t)$ and $\nu(t)$ are lower and upper solutions of (4)-(5), respectively. Now, we suppose that

$$g(t,u(t)) = \begin{cases} f(t,\mu(t)), & u(t)\le\mu(t),\\ f(t,u(t)), & \mu(t)\le u(t)\le\nu(t),\\ f(t,\nu(t)), & u(t)\le\nu(t), \end{cases} \tag{31}$$

and prove that FBVP,

$$-D_{0^+}^{\alpha}u(t) = g(t,u(t)), \quad 0<t<1,\ 2<\alpha\le 3, \tag{32}$$

$$u(0)=u'(0)=0, \qquad D_{0^+}^{\alpha-1}u(1)=\beta u(\xi), \tag{33}$$

has a solution. Consider operator $T: C^2[0,1] \to C^2[0,1]$, with $Tu(t)=\int_0^1 G(t,s)g(s,u(s))ds$, where $G(t,s)$ is defined as in Lemma 8. It is easy to see that T is continuous in $C^2[0,1]$. Since f is nondecreasing in u (from (H_1)), for $u\in C^2[0,1]$, we have

$$f(t,\mu(t)) \le g(t,u(t)) \le f(t,\nu(t)), \quad t\in[0,1]. \tag{34}$$

So, there exists a positive constant M, such that $|g(t,u(t))| \le M$. We will show that the operator T is equicontinuous.

Case 1. If $s\le\xi$,

$$\begin{aligned}
&|Tu(t_1)-Tu(t_2)|\\
&\le \left|\int_0^1 [G(t_2,s)-G(t_1,s)]\,g(s,u(s))\,ds\right|\\
&= \left|\int_0^{t_2} \frac{-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s))\,ds\right.\\
&\quad + \frac{\Gamma(\alpha)\,t_2^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\,\Gamma(\alpha)} \int_0^{t_2} g(s,u(s))\,ds\\
&\quad + \frac{\beta t_2^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\,\Gamma(\alpha)} \int_0^{t_2} (\xi-s)^{\alpha-1} g(s,u(s))\,ds\\
&\quad + \frac{\Gamma(\alpha)\,t_2^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\,\Gamma(\alpha)} \times \int_{t_2}^1 g(s,u(s))\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta t_2^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_{t_2}^{1}(\xi-s)^{\alpha-1}g(s,u(s))\,ds\\
&-\int_0^{t_1}\frac{-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,u(s))\,ds\\
&-\frac{\Gamma(\alpha)t_1^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_0^{t_1}g(s,u(s))\,ds\\
&-\frac{\beta t_1^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_0^{t_1}(\xi-s)^{\alpha-1}g(s,u(s))\,ds\\
&-\frac{\Gamma(\alpha)t_2^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\times\int_{t_1}^{1}g(s,u(s))\,ds\\
&-\frac{\beta t_1^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_{t_1}^{1}(\xi-s)^{\alpha-1}g(s,u(s))\,ds\Bigg|\\
&=\Bigg|\int_0^{t_1}\left[\frac{-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right]g(s,u(s))\,ds\\
&+\int_{t_1}^{t_2}\frac{-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,u(s))\,ds\\
&+\frac{\Gamma(\alpha)\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_0^{t_1}g(s,u(s))\,ds\\
&+\frac{\Gamma(\alpha)t_2^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_{t_1}^{t_2}g(s,u(s))\,ds\\
&+\frac{\beta\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_0^{t_1}(\xi-s)^{\alpha-1}g(s,u(s))\,ds\\
&+\frac{\beta t_2^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\times\int_{t_1}^{t_2}(\xi-s)^{\alpha-1}g(s,u(s))\,ds\\
&+\frac{\Gamma(\alpha)\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_{t_2}^{1}g(s,u(s))\,ds\\
&-\frac{\Gamma(\alpha)t_1^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_{t_1}^{t_2}g(s,u(s))\,ds\\
&+\frac{\beta\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\int_{t_2}^{1}(\xi-s)^{\alpha-1}g(s,u(s))\,ds\\
&-\frac{\beta t_1^{\alpha-1}}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\times\int_{t_1}^{t_2}(\xi-s)^{\alpha-1}g(s,u(s))\,ds\Bigg|\\
&\leq\frac{2M}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}+\frac{M\Gamma(\alpha)\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)t_1}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\\
&+\frac{M\Gamma(\alpha)t_2^{\alpha-1}(t_2-t_1)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\\
&+\frac{M\beta\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)t_1}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}+\frac{M\beta t_2^{\alpha-1}(t_2-t_1)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\\
&+\frac{M\Gamma(\alpha)\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)(1-t_2)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}+\frac{M\Gamma(\alpha)t_1^{\alpha-1}(t_2-t_1)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}\\
&+\frac{M\beta\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)(1-t_2)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}+\frac{M\beta t_1^{\alpha-1}(t_2-t_1)}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]\Gamma(\alpha)}.
\end{aligned}\tag{35}
$$

Case 2. If $s\geq\xi$,

$$
\begin{aligned}
&\left|Tu(t_1)-Tu(t_2)\right|\\
&\leq\left|\int_0^1[G(t_2,s)-G(t_1,s)]\,g(s,u(s))\,ds\right|\\
&=\Bigg|\int_0^{t_2}\frac{-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,u(s))\,ds\\
&\quad+\frac{1}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]}\int_0^{t_2}t_2^{\alpha-1}g(s,u(s))\,ds\\
&\quad+\frac{1}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}\int_{t_2}^{1}t_2^{\alpha-1}g(s,u(s))\,ds\\
&\quad+\int_0^{t_1}\frac{-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,u(s))\,ds\\
&\quad-\frac{1}{[\beta\xi^{\alpha-1}-\Gamma(\alpha)]}\int_0^{t_1}t_1^{\alpha-1}g(s,u(s))\,ds\\
&\quad-\frac{1}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}\int_{t_1}^{1}t_1^{\alpha-1}g(s,u(s))\,ds\Bigg|\\
&=\Bigg|\int_0^{t_1}\left[\frac{-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right]g(s,u(s))\,ds\\
&\quad+\int_{t_1}^{t_2}\frac{-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,u(s))\,ds\\
&\quad+\frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}\int_0^{t_1}g(s,u(s))\,ds\\
&\quad+\frac{t_2^{\alpha-1}}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}\int_{t_1}^{t_2}g(s,u(s))\,ds\\
&\quad+\frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}\int_{t_2}^{1}g(s,u(s))\,ds\\
&\quad+\frac{t_1^{\alpha-1}}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}\int_{t_1}^{t_2}g(s,u(s))\,ds\Bigg|\\
&\leq\frac{2M}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}+\frac{M\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)t_1}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}
\end{aligned}
$$

$$+\frac{Mt_2^{\alpha-1}(t_2-t_1)}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}+\frac{M\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)(1-t_2)}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}$$
$$+\frac{Mt_1^{\alpha-1}(t_2-t_1)}{\beta\xi^{\alpha-1}-\Gamma(\alpha)}. \quad (36)$$

Therefore, the operator T is equicontinuous, and by Arzela-Ascoli theorem, T is a compact operator. Now, the Schauder fixed-point theorem [23] shows that the operator T has a fixed-point theorem and so FBVP (32)-(33) has a solution. Finally, we will prove that FBVP (4)-(5) has a positive solution. Suppose that $u_1(t)$ is a solution of FBVP (32)-(33). Since the function f is nondecreasing in u, we have

$$f(t,\mu(t))\le g(t,u_1(t))\le f(t,\nu(t)),\quad t\in[0,1]. \quad (37)$$

Assuming $X(t)=\nu(t)-u_1(t)$,

$$\begin{gathered}D^{\alpha}X(t)\ge f(t,\nu(t))-g(t,u_1(t))\ge 0,\\ X(0)=X'(0)=0,\qquad D^{\alpha-1}X(1)=\beta X(\xi).\end{gathered} \quad (38)$$

By Lemma 7, $X(t)\ge 0$; that is, $u_1(t)\le\nu(t)$ for $t\in[0,1]$. Similarly, $\mu(t)\le u_1(t)$ for $t\in[0,1]$. Therefore $u_1(t)$ is a positive solution of FBVP (4)-(5). The proof is complete. □

Example 11. Consider the following fractional boundary value problem:

$$\begin{gathered}-D_{0^+}^{5/2}u(t)=f(t,u(t)),\quad 0<t<1,\\ u(0)=u'(0)=0,\qquad D_{0^+}^{\alpha-1}u(1)=\beta u(\xi),\end{gathered} \quad (39)$$

where

$$f(t,u(t))=\sin\left(\frac{\pi t}{2}\right)+\sqrt{u}. \quad (40)$$

For $0\le k\le 1$, we have $\sqrt{k}\le 1$. Therefore

$$\begin{aligned}k^{1/2}f(t,u(t))&=k^{1/2}\left(\sin\frac{\pi}{2}t+\sqrt{u}\right)\\ &\le\sin\frac{\pi}{2}t+\sqrt{ku}=f(t,ku(t)).\end{aligned} \quad (41)$$

Now, by Theorem 10, we obtain that the FBVP (39) has a positive solution.

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