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A new approach for solving multi variable orders differential equations with Mittag–Leffler kernel

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ABSTRACT

In this paper we consider multi variable orders differential equations (MVODEs) with non-local and no-singular kernel. The derivative is described in Atangana and Baleanu sense of variable order. We use the fifth-kind Chebyshev polynomials as basic functions to obtain operational matrices. We transfer the original equations to a system of algebraic equations using operational matrices and collocation method. The convergence analysis of the presented method is discussed. Few examples are presented to show the efficiency of the presented method.

1. Introduction

Fractional derivatives can be used to simulate various real phenomena involving long memory accurately [1]. Fractional differential equations (FDEs) have been successfully applied in modeling of physics and engineering problems such as earthquake analysis, bio-chemical, electric circuits, controller design, signal processing, viscoelasticity, diffusion equations, electromagnetic waves, and so on [2–6].

There are several definitions of fractional derivative and integral such as Riemann-Liouville, Grünwald-Letnikov and Caputo [7]. In the last decade, the concept of fractional differentiation and integration or particularly the concept of non-local operators has attracted from many fields of science [8–11].

Recently, new definitions of fractional derivative are defined for example Caputo-Fabrizio derivative by Caputo and Fabrizio [12], new Caputo-Fabrizio derivative by Losada and Nieto [10,13,14], Yang-Srivastava-Machado derivative by Yang, Srivastava and Machado [10,14], Generalized Riemann-Liouville and Generalized Caputo derivative by Atangana and Baleanu [9,15,16] which is called after that as Atangana-Baleanu derivative.

The Atangana-Baleanu and Caputo-Fabrizio fractional derivatives show crossover properties for the meansquare displacement, while the Riemann-Liouville is scale invariant. Their probability distributions are also a Gaussian to non-Gaussian crossover, with the difference that the Caputo Fabrizio kernel has a steady state between the transition. Only the Atangana-Baleanu kernel is a crossover for the waiting time distribution from stretched exponential to power law. The Caputo-Fabrizio derivative is less noisy while the Atangana-Baleanu fractional derivative provides an excellent description, due to its Mittag-Leffler memory, able to distinguish between dynamical systems taking place at different scales without steady state [17–19].

It is not easy to obtain exact solution of fractional ordinary/ partial/ integro-differential equations. There are several numerical methods for solving these equations [20–26].

Presently, the variable order fractional calculus as an development of the classical fractional calculus is getting more popular. Variable order derivative is proposed by Samko and Ross in 1993 [27]. In this case, the order of differential (or integral) operator is

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not a constant but it is a function of space, time or other variables. Since derivative operator has a kernel of the variable order, it is not simply task to obtain the solution of such equations.

Recently, few researchers have proposed several approximation and numerical methods for solving variable order of the differential equations. Li and Wu, Yang et al. used the reproducing kernel method for solving the variable order fractional functional boundary value problems [28,29]. Ganji and Jafari applied Jacobi polynomials to obtain solution the multi-variable orders differential equations [30]. Doha et al. used the spectral technique for solving variable order fractional Volterra integro-differential equations [31]. Jafari et al. obtained a the approximate solution for variable order differential equations based on Bernstein polynomials [32]. Yu and Ertürk applied a finite difference method to solve variable order fractional integro-differential equations [33]. Ganji and Jafari solved the variable order integro-differential equations via several polynomials [34].

We study the following type of MVODEs

$$\sum_{i=1}^{N} \eta_i(t)\ {}^{ABC}\!D^{\kappa_i(t)}\Theta(t) = f(t,\Theta(t)), \quad 0 \le t \le 1,$$
$$\Theta(0) = \Theta_0, \tag{1}$$

where N is a positive integer number and $\kappa_i(t)$ are bounded in interval [0,1].

$\eta_i(t)$ are known functions. $\Theta(t)$ is a continuously differentiable. ${}^{ABC}D^{\kappa_i(t)}$ denote Atangana-Baleanu-Caputo derivatives which are defined in Definition 1.

Definition 1. The Atangana-Baleanu-Caputo fractional derivative ${}^{ABC}D^{\kappa}$ of order $0<\kappa<1$ of a function $\Theta(t)\in H^1(a,\ b)$, $a\in(-\infty,t)$ is defined in the following form

$${}^{ABC}\!D^{\kappa}\,\Theta(t) = \frac{B(\kappa)}{1-\kappa}\int_a^t E_\kappa\left[-\frac{\kappa}{1-\kappa}(t-s)^\kappa\right]\Theta'(s)\ ds, \tag{2}$$

where $B(\kappa)$ is fulfilling $B(0)=B(1)=1$ and $E_\kappa(t)=\sum_{p=0}^{\infty}\frac{t^p}{\Gamma(p\kappa+1)}$ is the Mittag-Leffler function.

More details for the above derivative are given in [8,9,17,35,36]. For variable order derivative we can rewrite (2) as [37]

$${}^{ABC}\!D^{\kappa(t)}\Theta(t) = \frac{B(\kappa(t))}{1-\kappa(t)}\int_a^t E_{\kappa(t)}\left[-\frac{\kappa(t)}{1-\kappa(t)}(t-s)^{\kappa(t)}\right]\Theta'(s)ds. \tag{3}$$

It is easy to report the following result, namely

$${}^{ABC}\!D^{\kappa(t)}t^m = \begin{cases} \frac{B(\kappa(t))}{1-\kappa(t)}\sum_{p=0}^{\infty}\frac{(-\kappa(t))^p\Gamma(m+1)}{(1-\kappa(t))^p\Gamma(\kappa(t)p+m+1)}t^{\kappa(t)p+m}, & m=1,2,\ldots,\\ 0, & m=0.\end{cases} \tag{4}$$

Remark 1. According to Definition 1, $B(\kappa(t))$ is as following [13]

$$B(\kappa(t)) = \frac{2}{2-\kappa(t)}, \quad 0<\kappa(t)<1.$$

2. The fifth-kind chebyshev polynomials

The ith degree fifth-kind Chebyshev polynomials are defined on the interval $[-1,1]$ as [38]

$$\chi_i(t) = \frac{1}{\sqrt{\varepsilon_i}}\overline{\mathcal{H}}_i^{-3,2,-1,1}(t),$$

where

$$\varepsilon_i = \begin{cases}\frac{\pi}{2^{2i+1}}, & i\ even,\\ \frac{\pi\ (i+2)}{i\,2^{2i+1}}, & i\ odd,\end{cases}$$

and

$$\overline{\mathcal{H}}_i^{r,s,v,w}(t) = \left(\prod_{k=0}^{\lfloor\frac{i}{2}\rfloor-1}\frac{(2k+(-1)^{i+1}+2)w+s}{(2k+(-1)^{i+1}+2\lfloor\frac{i}{2}\rfloor)v+r}\right)\mathcal{H}_i^{r,s,v,w}(t),$$

$$\mathcal{H}_i^{r,s,v,w}(t) = \sum_{j=0}^{\lfloor\frac{i}{2}\rfloor}\left(\binom{\lfloor\frac{i}{2}\rfloor}{j}\left(\prod_{k=0}^{\lfloor\frac{i}{2}\rfloor-j-1}\frac{(2k+(-1)^{i+1}+2\lfloor\frac{i}{2}\rfloor)v+r}{(2k+(-1)^{i+1}+2)w+s}\right)t^{i-2j}\right).$$

$\chi_i(t)$ are orthonormal on $[-1,1]$

$$\int_0^1\frac{t^2}{\sqrt{1-t^2}}\ \chi_m(t)\ \chi_n(t)\ dt = \begin{cases}1, & m=n,\\ 0, & m\neq n.\end{cases}$$

The ith degree shifted fifth-kind Chebyshev polynomials are defined on the interval [0, 1] as

$$\mathcal{C}_i(t) = \chi_i(2t-1). \tag{5}$$

$\mathcal{C}_i(t)$ are orthonormal on [0, 1]

$$\int_0^1\frac{(2t-1)^2}{\sqrt{t-t^2}}\ \mathcal{C}_m(t)\ \mathcal{C}_n(t)\ dt = \begin{cases}1, & m=n,\\ 0, & m\neq n.\end{cases}$$

We can rewrite $\mathcal{C}_i(t)$ as

$$\mathcal{C}_i(t) = \sum_{\iota=0}^{i}\sigma_{\iota,i}\ t^\iota,$$

where

$$\sigma_{\iota,i} = \frac{2^{2\iota+\frac{3}{2}}}{\sqrt{\pi}(2\iota)!}\begin{cases}2\sum_{k=\lfloor\frac{\iota+1}{2}\rfloor}^{\frac{i}{2}}\frac{(-1)^{\frac{i}{2}+k-\iota}k\ \delta_k\ (2k+\iota-1)!}{(2k-\iota)!}, & i\ even,\\ \frac{1}{\sqrt{i(i+2)}}\sum_{k=\lfloor\frac{\iota}{2}\rfloor}^{\frac{i-1}{2}}\frac{(-1)^{\frac{i+1}{2}+k-\iota}(2k+1)^2\ (2k+\iota)!}{(2k-\iota+1)!}, & i\ odd,\end{cases}$$

and

$$\delta_k = \begin{cases}\frac{1}{2}, & k=0,\\ 1, & k>0.\end{cases} \tag{6}$$

The fifth-kind Chebyshev polynomials are bounded on [0, 1] for all $i\geq0$ as [38]

$$|\mathcal{C}_i(t)| < \sqrt{\frac{2}{\pi}}(i+2), \quad \forall\ t\in[0,1]. \tag{7}$$

We can express the shifted fifth-kind Chebyshev basis polynomials in the matrix form

$$\varphi(t) = [\mathcal{C}_0(t),\mathcal{C}_1(t),\ldots,\mathcal{C}_n(t)]^T = AT_n(t), \tag{8}$$

where

$$T_n(t) = [1,t,\ldots,t^n]^T,$$

and

$$A = \begin{bmatrix}\sigma_{0,0} & 0 & 0 & \ldots & 0\\ \sigma_{0,1} & \sigma_{1,1} & 0 & \ldots & 0\\ \sigma_{0,2} & \sigma_{1,2} & \sigma_{2,2} & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \sigma_{0,n} & \sigma_{1,n} & \sigma_{2,n} & \ldots & \sigma_{n,n}\end{bmatrix},$$

where $\sigma_{0,2i} = \sqrt{\frac{2}{\pi}}$ and the matrix A is invertible.

3. Operational matrix of variable order derivative operators

We can expand a function $\Theta(t)\in L^2[0,\ 1]$ in terms of the shifted fifth-kind Chebyshev polynomials by the following infinite series

$$\Theta(t)=\sum_{i=0}^{\infty}a_i\ \mathcal{C}_i(t). \tag{9}$$

Also, we can approximate $\Theta(t)$ by the first $n+1$ terms of the shifted fifth-kind Chebyshev polynomials as

$$\Theta(t)\approx\Theta_n(t)=\sum_{i=0}^{n}a_i\,\mathcal{C}_i(t)=\Lambda^T\varphi(t), \tag{10}$$

where $\Lambda=[a_0,a_1,\ldots,a_n]^T$ is the shifted fifth-kind Chebyshev coefficients vector.

We can obtain the shifted fifth-kind Chebyshev coefficients by

$$a_i=\int_0^1\frac{(2t-1)^2}{\sqrt{t-t^2}}\ \Theta(t)\ \mathcal{C}_i(t)\,dt.$$

a_i are bounded as [38]

$$|a_i|<\frac{\sqrt{2\pi}\ L}{2i^3},\quad \forall\ i>3,$$

where $|\Theta^{(3)}(t)|\leq L$.

We find the operational matrices of derivative operators of orders $\kappa_i(t)$ for vector $\varphi(t)$ as

$${}^{ABC}\!\mathfrak{D}^{\kappa_i(t)}\varphi(t)={}^{ABC}\!\mathfrak{D}^{\kappa_i(t)}[AT_n(t)]=A\ {}^{ABC}\!\mathfrak{D}^{\kappa_i(t)}[1\ t\ \ldots t^n]^T.$$

According to (4), we can write

$$\begin{aligned}{}^{ABC}\!\mathfrak{D}^{\kappa_i(t)}\varphi(t)&=A\Bigg[0\quad\frac{B(\kappa_i(t))}{1-\kappa_i(t)}\sum_{p=0}^{\infty}\frac{(-\kappa_i(t))^p\,t^{\kappa_i(t)p+1}}{(1-\kappa_i(t))^p\Gamma(\kappa_i(t)p+2)}\\&\qquad\ldots\ \frac{B(\kappa_i(t))}{1-\kappa_i(t)}\sum_{p=0}^{\infty}\frac{(-\kappa_i(t))^p\,\Gamma(n+1)\,t^{\kappa_i(t)p+n}}{(1-\kappa_i(t))^p\Gamma(\kappa_i(t)p+n+1)}\Bigg]^T\\&=A\,\Psi_i\,T_n(t),\end{aligned}$$

where Ψ_i are $(n+1)\times(n+1)$ matrices as

$$\Psi_i=[\psi_{il,j}]_{(n+1)\times(n+1)}$$

where

$$\psi_{il,j}=\begin{cases}\frac{B(\kappa_i(t))}{1-\kappa_i(t)}\sum_{p=0}^{\infty}\frac{(-\kappa_i(t))^p\,\Gamma(j+1)\,t^{\kappa_i(t)p}}{(1-\kappa_i(t))^p\Gamma(\kappa_i(t)p+j+1)}, & l=j>0,\\ 0, & otherwise.\end{cases}$$

From (8), we have $T_n(t)=A^{-1}\varphi(t)$, then

$${}^{ABC}\!\mathfrak{D}^{\kappa_i(t)}\varphi(t)=A\Psi_iA^{-1}\varphi(t),\qquad i=1,2,\ldots,N.$$

We rewrite

$$A\Psi_iA^{-1}\varphi(t)=\Omega_i\varphi(t), \tag{11}$$

where Ω_i are the operational matrices for variable orders derivatives based on the fifth-type Chebyshev polynomials.

4. Proposed method for solving MVODEs

To find solution $\Theta(t)$, we present an algorithm for it that includes the following steps

Step 1. Consider Eq. (1).

Step 2. Approximate the unknown function $(\Theta(t))$ as (10) and substitute in Eq. (1).

Step 3. Calculate the operational matrices (11) and substitute in Eq. (1).

• The results steps 2 and 3 are as below

$$\begin{aligned}&\sum_{i=1}^{N}\eta_i(t)\ \Lambda^T\Omega_i\varphi(t)=f(t,\Lambda^T\varphi(t)),\\&\Lambda^T\varphi(0)=\Theta_0.\end{aligned} \tag{12}$$

Step 4. Calculate the residual function.

• The residual function can be calculated as

$$R(t,a_0,a_1,\ldots,a_n)=\sum_{i=1}^{N}\eta_i(t)\ \Lambda^T\Omega_i\varphi(t)-f(t,\Lambda^T\varphi(t)).$$

Step 5. Let $t_j=\frac{j}{n+1}$ for $j=1,2,\ldots,n$ be the collocation points. Then, solve obtained system and gain the unique coefficients. Finally, substitute the results into step 2.

• By solving below system, coefficients a_i can be calculated.

$$\begin{aligned}&R(t_j,a_0,a_1,\ldots,a_n)=0,\quad t_j=\frac{j}{n+1},\quad j=1,2,\ldots,n,\\&\Lambda^T\varphi(0)=\Theta_0.\end{aligned}$$

5. Error analysis

In this section, we investigate the convergence analysis of our proposed method.

Theorem 1 [38]. *Assume that a function $\Theta(t)\in L^2[0,\ 1]$ with $|\Theta^{(3)}(t)|\leq L$, and assume that it has the expansion as (9). If $E_n(t)=\sum_{i=n+1}^{\infty}a_i\ \mathcal{C}_i(t)$ be the global error, then $E_n(t)$ can be estimated as*

$$|E_n(t)|<\frac{3\ L}{n}.$$

Theorem 2. *Suppose that $\Theta(t)$ satisfies in Theorem 1 and $\Theta_n(t)=\sum_{i=0}^{n}a_i\ \mathcal{C}_i(t)$ be the approximate solution obtained via the presented method. Then, we have*

$$\sup_{t\in[0,1]}|\Theta(t)-\Theta_n(t)|\leq\frac{3L}{n}+\varepsilon_n\,\|\tilde{\Lambda}-\Lambda\|_2.$$

Proof. Assume $\tilde{\Theta}_n(t)$ be an approximate solution of Eq. (1). Then, we can write

$$|\Theta(t)-\Theta_n(t)|\leq|\Theta(t)-\tilde{\Theta}_n(t)|+|\tilde{\Theta}_n(t)-\Theta_n(t)|. \tag{13}$$

According to Theorem 1, we have

$$\left|\Theta(t)-\tilde{\Theta}_n(t)\right|\leq\frac{3L}{n}. \tag{14}$$

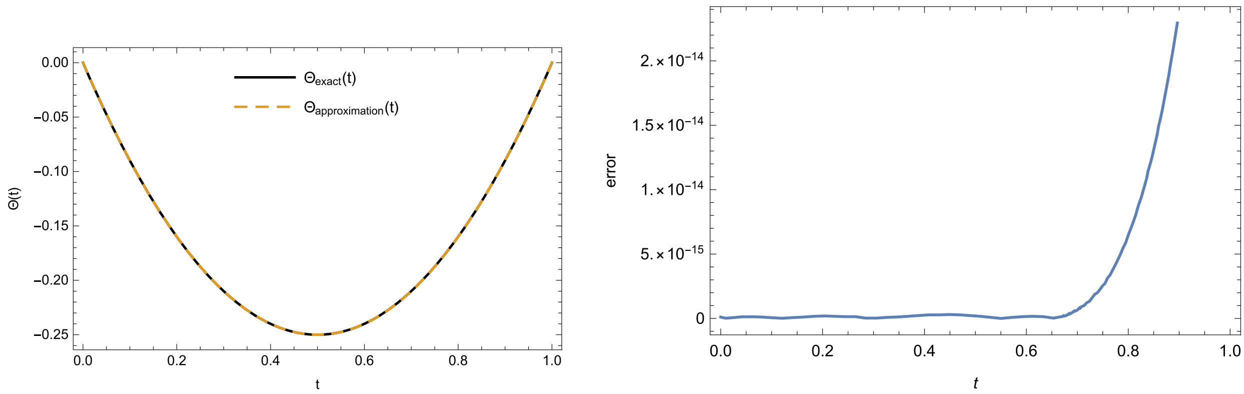
Also, using Cauchy–Schwarz inequality we can write

$$\begin{aligned}|\tilde{\Theta}_n(t)-\Theta_n(t)|&=\left|\sum_{i=0}^{n}\tilde{a}_i\ \mathcal{C}_i(t)-\sum_{i=0}^{n}a_i\ \mathcal{C}_i(t)\right|\\&=\left|\sum_{i=0}^{n}(\tilde{a}_i-a_i)\ \mathcal{C}_i(t)\right|\\&\leq\left(\sum_{i=0}^{n}|\tilde{a}_i-a_i|^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{n}|\mathcal{C}_i(t)|^2\right)^{\frac{1}{2}}.\end{aligned}$$

In view of (7), we have

$$\left(\sum_{i=0}^{n}|\mathcal{C}_i(t)|^2\right)^{\frac{1}{2}}\leq\varepsilon_n,$$

where $\varepsilon_n=\left(\sum_{i=0}^{n}\frac{2}{\pi}(i+2)^2\right)^{\frac{1}{2}}$.



(a) (b)

Fig. 1. (a) The exact and the approximate solutions (b) The absolute errors ($n=5$).

Let $\tilde{\Lambda}=[\tilde{a}_0,\tilde{a}_1,\dots,\tilde{a}_n]^T$, then we have

$$|\tilde{\Theta}_n(t)-\Theta_n(t)|\le\varepsilon\|\tilde{\Lambda}-\Lambda\|_2. \tag{15}$$

In view of (13), (14) and (15) we have

$$\sup_{t\in[0,1]}|\Theta(t)-\Theta_n(t)|\le\frac{3L}{n}+\varepsilon_n\|\tilde{\Lambda}-\Lambda\|_2.$$

6. Test examples

In this section, we present four examples to compare the approximate solution with the exact solution. The absolute errors are defined as

$$e(t)=|\chi(t)-\Lambda^T\varphi(t)|,\quad t\in[0,1].$$

For solving examples this section, we approximate in all relations the upper index of Σ to n and suppose $n\to\infty$.

Example 1. Let $\kappa_1(t)=t,\ \eta_1(t)=1,\ \Theta(0)=0$ and

$$f(t,\Theta(t))=\sum_{p=0}^{n}\frac{(-t)^p\,B(t)\,t^{t\,p}}{(1-t)^{p+1}\,\Gamma(t\,p+2)}\left(\frac{2t^2}{t\,p+2}-t\right),\qquad t\in[0,1],$$

where the exact solution is $\Theta(t)=t^2-t$. By applying the presented method, the numerical results are shown in Fig. 1. As seen from Fig. 1 (a), it is evident that the numerical solution obtained converges to the analytical solution. Also, Fig. 1 (b) shows that the absolute errors are small. Therefore, it indicates the accuracy of the presented method.

Example 2. Let $\kappa_1(t)=1-0.5e^{-t},\ \eta_1(t)=1,\ \Theta(0)=1$ and

$$f(t,\Theta(t))=e^t\left(1+\sum_{p=0}^{n}\frac{(-\kappa_1(t))^p B(\kappa_1(t))\ t^{\kappa_1(t)\,p+1}\alpha}{(1-\kappa_1(t))^{p+1}\Gamma(\kappa_1(t)\,p+2)}\right)-\Theta(t),$$
$$t\in[0,1],$$

where $\alpha=Hypergeometric1F1[\kappa_1(t)\,p+1,\kappa_1(t)\,p+2,-t]$. The exact ($\Theta(t)=e^t$), the approximate solutions and the errors are shown in Fig. 2 when we applied the presented method. The absolute errors for various n are shown in Table 1. According to the results shown in Table 1 and Fig. 2, the presented method provides an acceptable approximate solution even using a few number of the fifth-kind orthonormal Chebyshev polynomials and also increasing the number of these basis functions improves the accuracy exponentially.

Example 3. Let $\kappa_1(t)=t,\ \kappa_2(t)=1-0.5e^{-t},\ \eta_1(t)=1,\ \eta_2(t)=\sin(t),\ \Theta(0)=0$ and

$$f(t,\Theta(t))=\sum_{p=0}^{n}\frac{6(-\kappa_1(t))^p\ B(\kappa_1(t))\ t^{\kappa_1(t)p+3}}{(1-\kappa_1(t))^{p+1}\Gamma(\kappa_1(t)p+4)}$$

Table 1
The absolute errors for various n.

t	$n=3$	$n=5$	$n=7$	$n=9$
0.1	$3.12568e-4$	$7.38305e-7$	$6.74379e-10$	$2.11386e-13$
0.2	$1.87544e-4$	$6.61186e-8$	$1.49564e-10$	$9.96980e-14$
0.3	$2.07343e-5$	$6.69348e-8$	$2.34189e-10$	$1.13021e-13$
0.4	$8.02715e-5$	$2.52627e-7$	$5.72256e-11$	$9.63674e-14$
0.5	$1.05715e-4$	$1.95098e-7$	$1.14717e-10$	$1.08802e-13$
0.6	$4.84578e-4$	$1.10856e-7$	$3.57683e-10$	$1.23679e-13$
0.7	$8.38521e-4$	$1.16466e-6$	$4.23163e-10$	$3.29070e-13$
0.8	$7.67235e-4$	$3.44932e-6$	$3.51789e-9$	$1.20703e-12$
0.9	$3.31304e-4$	$2.27271e-6$	$6.52878e-9$	$6.59117e-12$

$$+6\sin t\sum_{p=0}^{n}\frac{(-\kappa_2(t))^p\ B(\kappa_2(t))\ t^{\kappa_2(t)p+3}}{(1-\kappa_2(t))^{p+1}\Gamma(\kappa_2(t)p+4)}$$
$$+t^3\cos t-\cos(t)\Theta(t),$$

where $t\in[0,\ 1]$ and the exact solution is $\Theta(t)=t^3$. By using the presented method, the numerical results are shown in Fig. 3.

Example 4. Let $\kappa_1(t)=\sin t,\ \ \kappa_2(t)=1-\frac{t}{2},\ \ \kappa_3(t)=1-\cos t,$ $\eta_1(t)=1,\ \eta_2(t)=e^t,\ \eta_3(t)=\frac{2}{2t+1},\ \Theta(0)=0$ and

$$f(t,\Theta(t))=\sum_{p=0}^{n}\frac{2(-\kappa_1(t))^p\ B(\kappa_1(t))\ t^{\kappa_1(t)p+2}}{(1-\kappa_1(t))^{p+1}\Gamma(\kappa_1(t)p+3)}$$
$$+2e^t\sum_{p=0}^{n}\frac{(-\kappa_2(t))^p\ B(\kappa_2(t))\ t^{\kappa_2(t)p+2}}{(1-\kappa_2(t))^{p+1}\Gamma(\kappa_2(t)p+3)}$$
$$+\frac{4}{2t+1}\sum_{p=0}^{n}\frac{(-\kappa_3(t))^p\ B(\kappa_3(t))\ t^{\kappa_3(t)p+2}}{(1-\kappa_3(t))^{p+1}\Gamma(\kappa_3(t)p+3)}$$
$$+t^{\frac{5}{2}}-\sqrt{t}\Theta(t),$$

where $t\in[0,\ 1]$. The exact solution ($\Theta(t)=t^2$), the approximate solution and the errors are shown in Fig. 4 when we used the presented method.

7. Conclusion

In this paper, multi variable orders differential equations with non-local and no-singular kernel have been solved using operational matrices based on the fifth-kind orthonormal Chebyshev polynomials. The derivative is described in Atangana and Baleanu sense of variable order. We used the fifth-kind orthonormal Chebyshev polynomials as basic functions. First, we approximated the unknown function and its derivatives in terms of the fifth-kind

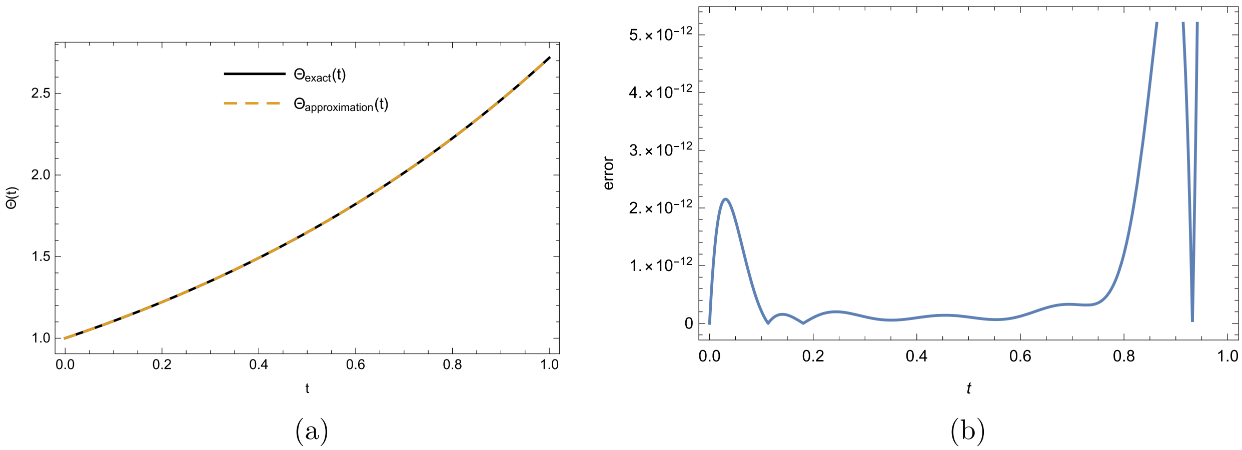


Fig. 2. (a) The exact and the approximate solutions (b) The absolute errors ($n = 9$).

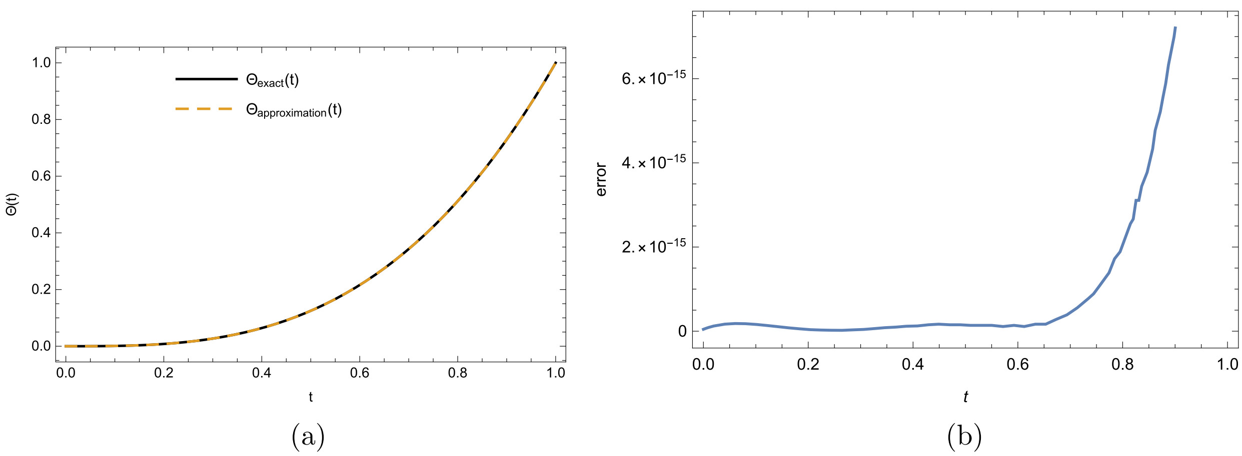


Fig. 3. (a) The exact and the approximate solutions (b) The absolute errors ($n = 5$).

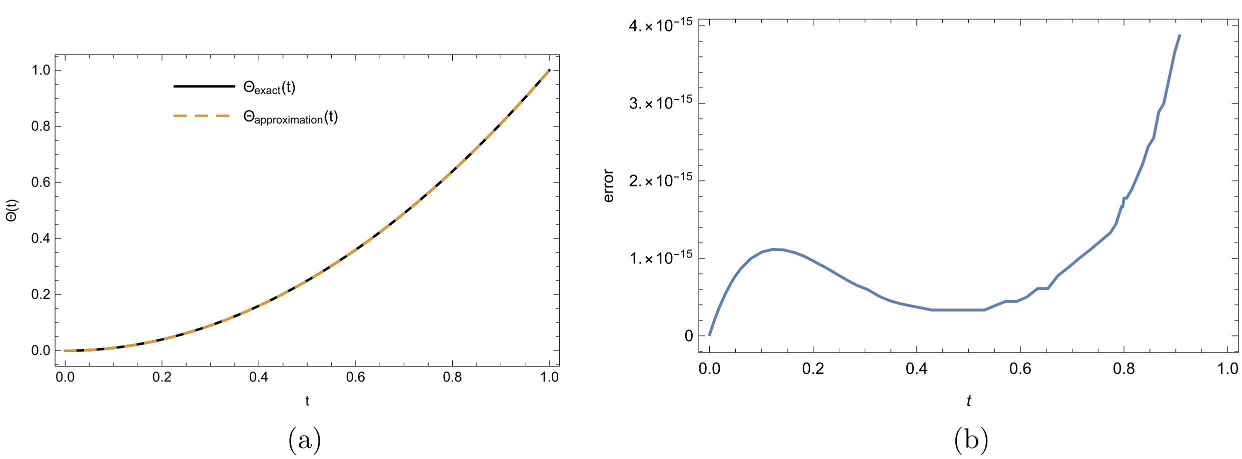


Fig. 4. (a) The exact and the approximate solutions (b) The absolute errors ($n = 5$).

orthonormal Chebyshev polynomials. Then, we substituted these approximations in multi variable orders differential equations and obtained an algebraic system. By applying collocation method, we obtained the approximate solution. The convergence of the presented method is discussed.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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