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ORIGINAL ARTICLE

A numerical schemes and comparisons for fixed point results with applications to the solutions of Volterra integral equations in *dislocated extended b – metric space*

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Abstract In this article, we propose a generalization of both *b*-metric and dislocated metric, namely, dislocated extended *b*-metric space. After getting some fixed point results, we suggest a relatively simple solution for a Volterra integral equation by using the technique of fixed point in the setting of dislocated extended *b*-metric space.

1. Introduction and preliminaries

About a century ago, the first fixed point result was published. It was S. Banach who got the abstraction of the successive approximation method in solving differential equations and announced it as a contraction mapping principle. This result of Banach was not only formulated quite simply but also presented as a very convenient way for the application by showing how to achieve the desired fixed point. The fact that equations in many research areas can be easily converted to fixed point problems has made fixed point theory highly applicable to various qualitative sciences and at the same time very attractive to researchers. Since 1922, when the first fixed-point theorem was announced, there have been two claims of numerous research articles. The first is to diversify the inequality provided by a function. This trend is also known as getting new contraction types. The second is the expansion of the abstract structure in which functions are defined and inequalities arise (due to some kind of contraction). We also emphasize that fixed point has been used in fractional differential/integral equation, effectively, see e.g. [1–12].

In this article, by using these two approaches we have defined a new contraction in one of the most extended abstract spaces known. More precisely, we define " rational weak *F*-contractions " in the context of δ-metric spaces. We successively proved the existence and uniqueness of fixed point for

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such contractions in the setting of the mentioned abstract spaces.

Before giving the technical details, we, first, fix the following notations. The expression $\mathbb{R}_0^+$ stands for the set of all non-negative real numbers that is denoted by $\mathbb{R}$. Further, the letters N and $\mathbb{N}_0$ are preserved for all positive integers and all non-negative integers. Throughout this manuscript, all considered sets are non-empty. A mapping, from the cross product of a set X, that is, $X \times X$ to non-negative reals, is called distance function over X.

The contraction we deal with is inspired from the notion of *F-contraction* defined by Wardowski [13]: A self-mapping T, on a metric space (X, d), is named *F-contraction* if there exists $\kappa > 0$ such that for all $p, q \in X$,

$$d(Tp, Tq) > 0 \Rightarrow \kappa + F(d(Tp, Tq)) \leqslant F(d(p, q)),$$

where $F : \mathbb{R}_0^+ \to \mathbb{R}$ be a mapping fulfills the following conditions:

$(F1)$ For all $a, b \in \mathbb{R}_0^+$ such that if $t_1 < t_2$ then $F(t_1) < F(t_2)$(that is, F is strictly increasing.)
$(F2)$ For all sequence $\{t_n\}_{n\in\mathbb{N}} \subset (0, \infty)$

$$\lim_{n\to\infty} t_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(t_n) = -\infty;$$

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$.

We shall represent by $\mathcal{F}^*$ the set of all functions fulfills above mentioned conditions.

A self-mapping T, on a metric space (X, d), is called **Gupta-Saxena Contraction** [14] if is a continuous and satisfying the following inequality:

$$d(Tp, Tq) \leqslant a_1 \frac{[1 + d(p, Tp)]d(q, Tq)}{1 + d(p, q)} + a_2 \frac{d(p, Tp)d(q, Tq)}{1 + d(p, q)} + a_3 d(p, q)$$

for all $p, q \in X, p \neq q$, where a_1, a_2, a_3 are constants with $a_1, a_2, a_3 > 0$.

For more results pertinent to above famous contractions, the readers can refer to [15–26]. The 'theory of a metric space' has been generalized in wide directions (for more information on its generalizations and/or its developments the reader can refer to [27–40]). Recently, Kamran et al., [41] has introduced an extended b-metric space and proved the analog of Banach contraction principle in the setting of this new abstract space. After this initial results, it has taken attention of several researchers and it has been announced several results in this direction, see e.g. [42–47]. With reference to his work, we generalize an extended b-metric space as below:

Definition 1.1. Let δ be a distance function over X and $\gamma : X \times X \to [1, \infty)$. If, δ satisfies the conditions (i)-(iii) below, then it is called dislocated extended b-metric:

(i) $\delta(p, q) = \delta(q, p)$;
(ii) $\delta(p, q) = 0 \Rightarrow p = q$;
(iii) $\delta(p, q) \leqslant \gamma(p, q)[\delta(p, r) + \delta(r, q)]$,

for all $p, q, r \in X$. The pair (X, δ) is called δ-metric space, alternatively, it is called dislocated extended b-metric space. The functional δ is not continuous in general. For our purpose, we presume that the functional δ is continuous, from now on.

The notion of δ-metric space is very predominant-indeed, δ-metric space becomes dislocated metric space if $\gamma(p, q) = 1$.

Hereinafter referred to as, unless otherwise specified, (X, δ) represents metric space.

Example 1.4. If $X = \mathbb{R}_0^+$. Define a distance function δ over X as $\delta(p, q) = (p + q)^2$ and $\gamma : X \times X \to [1, \infty)$ as $\gamma(p, q) = 2p + 3q + 5$. Then (X, δ) forms a δ-metric space.

Example 1.5. Define a distance function δ over $X = \{1, 2, 3\}$ as follows:

$$\delta(1, 1) = \delta(2, 2) = 1 \text{ and } \delta(3, 3) = 2;$$

$$\delta(1, 2) = \delta(2, 1) = 2;$$

$$\delta(2, 3) = \delta(3, 2) = 7;$$

$$\delta(3, 1) = \delta(1, 3) = 5;$$

Consider $\gamma : X \times X \to [1, \infty)$ as $\gamma\ (p, q) = 1 + pq$. Now let us consider the modified triangle inequality for all possibilities. For $\delta(1, 2) = 2$, we have only one possibility: $\gamma(1, 2)[\delta(1, 3) + \delta(3, 2)] = 36$. Thus, we have

$$\delta(1, 2) \leqslant \gamma(1, 2)[\delta(1, 3) + \delta(3, 2)].$$

Similarly, for $\delta(2, 3) = 7$, the only possible case is $\gamma(2, 3)[\delta(2, 1) + \delta(1, 3)] = 49$. Hence, the modified triangle inequality holds for this case. Finally, for $\delta(3, 1) = 5$, we have again only one possible case: $\gamma(3, 1)[\delta(3, 2) + \delta(2, 1)] = 36$. Accordingly, we have

$$\delta(3, 1) \leqslant \gamma(3, 1)[\delta(3, 2) + \delta(2, 1)].$$

Thus, all the conditions has been satisfied. Hence $\delta(p, q) \leqslant \gamma(p, q)[\delta(p, r) + \delta(r, q)]$ for all $p, q, r \in X$. Therefore, the pair (X, δ) forms a δ-metric space.

Definition 1.6. A sequence $\{p_n\}$ in (X, δ) is said to be

(1) Cauchy sequence if and only if for given $\epsilon > 0$, there corresponds $n_0 \in \mathbb{N}$ such that for all $n, m \geqslant n_0$ then $\delta(p_m, p_n) < \epsilon$; this can be also written as $\lim_{n,m\to\infty} \delta(p_n, p_m) = 0$.
(2) converges to p if and only if $\lim_{n,m\to\infty} \delta(p_n, p) = 0$. In this scenario, p is called a limit of $\{p_n\}$.

Definition 1.7. Theδ-metric space (X, δ) is complete if and only if every Cauchy sequence in X converges to a point $p \in X$.

For convenience, hereafter, the pair (X^*, δ) denotes complete metric space unless otherwise stated.

2. Analog of Banach contraction principle

In this section, we provide a constructive method to find fixed points in new type of generalized b-dislocated metric space, which we call as dislocated extended b-metric space.

Theorem 2.4. *Let T be a self-mapping on (X^*, δ) that satisfies*

$$\delta(Tp, Tq) \leqslant k\delta(p, q) \tag{1}$$

for all $p, q \in X$, *where* $k \in [0, 1)$ *be such that for each* $p_0 \in X$, $\lim_{n,m\to\infty}\gamma(p_n, p_m) < \frac{1}{k}$, *here* $p_n = T^n p_0$, $n = 1, 2, \dots$ *Then* T *possesses a unique fixed point* η.

Proof. Let us define the iterative sequence $\{p_n\}$ by, for $p_0 \in X$

$$p_0, Tp_0 = p_1,\ p_2 = Tp_1 = T(Tp_0) = T^2p_0, \dots p_n = T^np_0 \dots$$

By successively applying inequality (1), we obtain,

$$\delta(p_n, p_{n+1}) \leqslant k^n\delta(p_0, p_1) \quad \text{forall} \quad n = 1, 2, \dots$$

Consequently, if $n < m$, by triangle inequality, we get

$$\begin{aligned}
&\delta\ (p_n, p_m) \leqslant \gamma\ (p_n, p_m)\ k^n\delta\ (p_0, p_1) \\
&\quad +\gamma\ (p_n, p_m)\ \gamma\ (p_{n+1}, p_m)\ k^{n+1}\delta\ (p_0, p_1) \\
&\quad +\gamma\ (p_n, p_m)\ \gamma\ (p_{n+1}, p_m)\ \gamma\ (p_{n+2}, p_m)\ \dots\dots \\
&\quad \gamma\ (p_{m-2}, p_m)\ \gamma\ (p_{m-1}, p_m)\ k^{m-1}\delta\ (p_0, p_1) \\
&\quad \leqslant \delta\ (p_0, p_1)\ [\gamma\ (p_1, p_m)\ \gamma\ (p_2, p_m)\ \dots\gamma\ (p_{n-1}, p_m)\ \gamma\ (p_n, p_m)\ k^n \\
&\quad +\gamma\ (p_1, p_m)\ \gamma\ (p_2, p_m)\ \dots\gamma\ (p_n, p_m)\ \gamma\ (p_{n+1}, p_m)\ k^{n+1} + \dots \\
&\qquad\qquad\qquad\vdots \\
&\quad +\gamma\ (p_1, p_m)\ \gamma\ (p_2, p_m) \dots\ \gamma\ (p_{m-2}, p_m)\ \gamma\ (p_{m-1}, p_m)\ k^{m-1}]\ .
\end{aligned}$$

Since $\lim_{n,m\to\infty}\gamma(p_{n+1}, p_m)k < 1$, the series

$$\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n}\gamma(p_i, p_m)$$

converges due to the ratio test for each $m \in \mathbb{N}$.

Let

$$S = \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n}\gamma(p_i, p_m) \text{ and } S_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j}\gamma(p_i, p_m).$$

Thus for $m > n$, the inequality above yields

$$\delta(T^np_0, T^mp_0) = \delta(p_n, p_m) \leqslant \delta(p_0, p_1)[S_{m-1} - S_{n-1}].$$

Letting $n, m \to \infty$, we conclude that the sequence $\{T^np\}$ is Cauchy. Keeping the completeness of the δ-space (X, δ), one can find $\eta \in X$ such that $\lim_{n\to\infty}p_n \to \eta$.

Since T is continuous,

$$\lim_{n\to\infty} T(p_n) \to T(\eta)$$

$$i.e., \lim_{n\to\infty} p_{n+1} \to T(\eta)$$

Now consider, $\delta(\eta, T(\eta)) = \lim_{n\to\infty}\delta(p_n, p_{n+1})$.

Since $\delta(p_n, p_{n+1}) \leqslant k^n\delta(p_0, p_1)$ and $0 < k < 1$, $\lim_{n\to\infty}k^n\delta(p_0, p_1) = 0$. Hence $\delta(\eta, T(\eta)) = 0$. Thus η is a fixed point.

It is very easy to prove T has unique fixed point. Hence omitted. □

3. Common fixed point problems for rational weak *F*-contraction

Definition 3.1. A pair (S, T) of self-mappings on (X, δ) is called a rational weak F-contractions if for all $p, q \in \{TS(p_n)\}$ then we have

$$\kappa + F(\delta(Sp, Tq)) \leqslant F(\mathcal{A}(p, q)) \tag{2}$$

where $F \in \mathcal{F}^*, \kappa > 0$ and

$$\mathcal{A}(p, q) = \max\{\delta(p, q), \frac{\delta(p, Sp)}{1 + \delta(p, Sp)}, \frac{\delta(q, Tq)}{1 + \delta(q, Tq)}, \delta(p, Sp), \delta(q, Tq)\}. \tag{3}$$

Theorem 3.2. *Let* (S, T) *be a pair of self-mappings on* (X^*, δ) *that forms a pair of rational weak* F*-contractions such that for each* $p_0 \in X$, $\lim\sup_{n,m\to\infty}\gamma(p_n, p_m) < \frac{1}{k}$. *Then* S *and* T *have a common fixed point* p^* *in* X. *Moreover if* (2) *also holds for* p^*, *then* $\delta(p^*, p^*) = 0$.

Proof. Proof: We construct an iterative sequence p_n such that $p_{2n+1} = Sp_{2n}$ and $p_{2n+2} = Tp_{2n+1}$. If $\mathcal{A}(p, q) = 0$, this claims $p = q$ is a common fixed point of S and T. Hence there is nothing left to prove and our proof is complete. Let $\mathcal{A}(p, q) > 0$ for all $p, q \in \{TS(p_n)\}$ with $p \neq q$. The from a pair of rational weak F-contractions and Lemma 1.9, we get,

$$\begin{aligned} F(\delta(p_{2k+1}, p_{2k+2}) &= F(\delta(Sp_{2k}, Tp_{2k+1})) \\ &\leqslant F(\mathcal{A}(p_{2k}, p_{2k+1})) - \kappa \end{aligned} \tag{4}$$

for all $k \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned} \mathcal{A}(p_{2k}, p_{2k+1}) &= \max\left\{\delta(p_{2k}, p_{2k+1}), \tfrac{\delta(p_{2k}, Sp_{2k})}{1+\delta(p_{2k}, Sp_{2k})}, \tfrac{\delta(p_{2k+1}, Tp_{2k+1})}{1+\delta(p_{2k+1}, Tp_{2k+1})},\right. \\ &\quad \left.\delta(p_{2k}, Sp_{2k}), \delta(p_{2k+1}, Tp_{2k+1})\right\} \\ &= \max\left\{\delta(p_{2k}, p_{2k+1}), \tfrac{\delta(p_{2k}, p_{2k+1})}{1+\delta(p_{2k}, p_{2k+1})}, \tfrac{\delta(p_{2k+1}, p_{2k+2})}{1+\delta(p_{2k+1}, p_{2k+2})},\right. \\ &\quad \left.\delta(p_{2k}, p_{2k+1}), \delta(p_{2k+1}, p_{2k+2})\right\} \\ &= \max\{\delta(p_{2k}, p_{2k+1}), \delta(p_{2k+1}, p_{2k+2})\} \end{aligned} \tag{5}$$

If $\mathcal{A}(p_{2k}, p_{2k+1}) = \delta(p_{2k+1}, p_{2k+2})$ then $F(\delta(p_{2k+1}, p_{2k+2})) \leqslant F(\delta(p_{2k+1}, p_{2k+2})) - \kappa$, this is a contradiction according to F_1.

Hence $\mathcal{A}(p_{2k}, p_{2k+1}) = \delta(p_{2k}, p_{2k+1})$. Then from (1),

$$F(\delta(p_{2k+1}, p_{2k+2})) \leqslant F(\delta(p_{2k}, p_{2k+1})) - \kappa; \quad \forall k \in \mathbb{N} \cup \{0\} \tag{6}$$

Similarly, we have

$$F(\delta(p_{2k}, p_{2k+1})) \leqslant F(\delta(p_{2k-1}, p_{2k})) - \kappa \tag{7}$$

By using (6) and (7), we get

$$F(\delta(p_{2k+1}, p_{2k+2})) \leqslant F(\delta(p_{2k}, p_{2k+1})) - 2\kappa \tag{8}$$

Repeating consequently the iterative sequence, we get,

$$F(\delta(p_{2k+1}, p_{2k+2})) \leqslant F(\delta(p_0, p_1)) - (2e + 1)\kappa \tag{9}$$

Similarly,

$$F(\delta(p_{2k}, p_{2k+1})) \leqslant F(\delta(p_0, p_1)) - 2e\kappa \tag{10}$$

From (9) and (10), we get,

$$F(\delta(p_n, p_{n+1})) \leqslant F(\delta(p_0, p_1)) - n\kappa \tag{11}$$

On taking $\lim_{n\to\infty}$ on both sides of (11), we have

$$\lim_{n\to\infty} F(\delta(p_n, p_{n+1})) = -\infty \tag{12}$$

From F_2,

$$\lim_{n\to\infty} \delta(p_n, p_{n+1}) = 0 \tag{13}$$

From (11), for all $n \in \mathbb{N}$, we get

$$(\delta(p_n, p_{n+1}))^k (F(\delta(p_n, p_{n+1})) - F(\delta(p_0, p_1))) \leqslant -(\delta(p_n, p_{n+1}))^k n\kappa \leq 0 \tag{14}$$

By using (12), (13) and letting $n \to \infty$ in (14), we get,

$$\lim_{n\to\infty} (n(\delta(p_n, p_{n+1}))^k) = 0 \tag{15}$$

From (15), there exists $n_1 \in \mathbb{N}$ such that $n(\delta(p_n, p_{n+1}))^k \leqslant 1$ for all $n \geqslant n_1$ or

$$\delta(p_n, p_{n+1}) \leqslant \frac{1}{n^{\frac{1}{k}}} \quad \forall n \geqslant n_1 \tag{16}$$

From (16), we get from $m > n > n_1$

$$\begin{aligned}\delta(p_n,p_m) &\leqslant \gamma(p_n,p_m)[\delta(p_n,p_{n+1}) + \delta(p_{n+1},p_m)] \\ &\leqslant \gamma(p_n,p_m)\delta(p_n,p_{n+1}) + \gamma(p_n,p_m)\gamma(p_{n+1},p_m)[\delta(p_{n+1},p_{n+2}) \\ &+\delta(p_{n+2},p_m)] \\ &\leqslant \gamma(p_n,p_m)\delta(p_n,p_{n+1}) + \gamma(p_n,p_m)\gamma(p_{n+1},p_m)\delta(p_{n+1},p_{n+2}) + \dots \\ &+\gamma(p_n,p_m)\gamma(p_{n+1},p_m)\gamma(p_{n+2},p_m)\dots\gamma(p_{m-2},p_m)\gamma(p_{m-1},p_m) \\ &\delta(p_{m-1},p_m) \\ &\leqslant \gamma(p_1,p_m)\gamma(p_2,p_m)\dots\gamma(p_n,p_m)\delta(p_n,p_{n+1}) \\ &+\gamma(p_1,p_m)\gamma(p_2,p_m)\dots\gamma(p_{n+1},p_m)\delta(p_{n+1},p_{n+2}) + \dots \\ &+\gamma(p_1,p_m)\gamma(p_2,p_m)\dots\gamma(p_{m-1},p_m)\delta(p_{m-1},p_m).\end{aligned}$$

Note that this series

$$\sum_{n=1}^{\infty} \delta(p_n, p_{n+1}) \prod_{i=1}^{n} \gamma(p_i, p_m) \quad \text{converges.}$$

Since,

$$\begin{aligned}\sum_{n=1}^{\infty} \delta(p_n, p_{n+1}) \prod_{i=1}^{n} \gamma(p_i, p_m) &\leqslant \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \prod_{i=1}^{n} \gamma(p_i, p_m) \\ &\leqslant \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}; \text{ which is convergent.}\end{aligned}$$

Let

$$S = \sum_{n=1}^{\infty} \delta(p_n, p_{n+1}) \prod_{i=1}^{n} \gamma(p_i, p_m).$$

$$S_n = \sum_{j=1}^{n} \delta(p_j, p_{j+1}) \prod_{i=1}^{j} \gamma(p_i, p_m).$$

Thus for $m > n$ above inequality implies

$$\delta(p_n, p_m) \leqslant S_{m-1} - S_{n-1}.$$

Letting $n \to \infty$, we conclude that $\{TS(p_n)\}$ is a Cauchy sequence in (X, δ). Since (X, δ) is a complete δ-metric space, there exists $p^* \in X$ such that $\{TS(p_n)\} \to p^*$, in other notation,

$$\lim_{n\to\infty} \delta(p_n, p^*) = 0 \tag{17}$$

Now by Lemma 1.9, we have

$$\kappa + F\,(\delta\,(p_{2n+1}, Tp^*)) \leqslant \kappa + F\,(\delta\,(Sp_{2n}, Tp^*)) \tag{18}$$

The inequality (2) holds good for p^*, then we have,

$$\kappa + F(\delta(p_{2n+1}, Tp^*)) \leqslant F(\boldsymbol{\mathcal{A}}(p_{2n}, p^*)) \tag{19}$$

where

$$\begin{aligned}\boldsymbol{\mathcal{A}}(p_{2n}, p^*) &= \max\Big\{\delta(p_{2n}, p^*), \tfrac{\delta(p_{2n}, Sp_{2n})}{1+\delta(p_{2n}, Sp_{2n})}, \tfrac{\delta(p^*, Tp^*)}{1+\delta(p^*, Tp^*)}, \\ &\quad \delta(p_{2n}, Sp_{2n}), \delta(p^*, Tp^*)\Big\} \\ &= \max\Big\{\delta(p_{2n}, p^*), \tfrac{\delta(p_{2n}, p_{2n+1})}{1+\delta(p_{2n}, p_{2n+1})}, \tfrac{\delta(p^*, Tp^*)}{1+\delta(p^*, Tp^*)}, \\ &\quad \delta(p_{2n}, p_{2n+1}), \delta(p^*, Tp^*)\Big\}.\end{aligned} \tag{20}$$

Taking $\lim_{n\to\infty}$ and using (17) we get,

$$\lim_{n\to\infty} \boldsymbol{\mathcal{A}}(p_{2n}, p^*) = \delta(p^*, Tp^*) \tag{21}$$

Since $\lim_{n\to\infty}\delta(p_{2n}, p^*) = 0$ and

$$\lim_{n\to\infty} \delta(p_{2n}, p_{2n+1}) \leqslant \lim_{n\to\infty} \gamma(p_{2n}, p_{2n+1})[\delta(p_{2n}, p^*) + \delta(p^*, p_{2n+1})] \to 0.$$

Accordingly, from (19), we find that

$$\begin{aligned}F(\delta(p_{2n+1}, Tp^*)) &\leqslant F(\boldsymbol{\mathcal{A}}(p_{2n}, p^*)) - \kappa \\ &< F(\boldsymbol{\mathcal{A}}(p_{2n}, p^*))\end{aligned} \tag{22}$$

By using F_1, we get

$$\delta(p_{2n+1}, Tp^*) < \boldsymbol{\mathcal{A}}(p_{2n}, p^*)$$

Applying limits as $n \to \infty$ and using (21), we get $\delta(p^*, Tp^*) < \delta(p^*, Tp^*)$, which is a contradiction. Hence $\delta(p^*, Tp^*) = 0$.

Similarly using (17) and Lemma 1.9,

$$\kappa + F(\delta(p_{2n+2}, Sp^*)) \leqslant \kappa + F(\delta(Tp_{2n+1}, Sp^*))$$

We can prove that $\delta(p^*, Sp^*) = 0$ or $p^* = Sp^*$. Hence S and T have a common fixed point p^* in X.

Now if $\delta(p^*, p^*) \neq 0$ then

$$\begin{aligned}F(\delta(p^*, p^*)) &\leqslant F(\delta(Sp^*, Tp^*)) \\ &\leqslant F(\boldsymbol{\mathcal{A}}(p^*, p^*)) - \kappa\end{aligned} \tag{23}$$

where

$$\begin{aligned}\boldsymbol{\mathcal{A}}(p^*, p^*) &= \max\Big\{\delta(p^*, p^*), \tfrac{\delta(p^*, Sp^*)}{1+\delta(p^*, Sp^*)}, \tfrac{\delta(p^*, Tp^*)}{1+\delta(p^*, Tp^*)}, \\ &\quad \delta(p^*, Sp^*), \delta(p^*, Tp^*)\Big\} \\ &= \max\{\delta(p^*, p^*)\} \\ &= \delta(p^*, p^*)\end{aligned} \tag{24}$$

Thus from (23), $F(\delta(p^*, p^*)) \leqslant F(\delta(p^*, p^*)) - \kappa$, which is a contradiction.

Hence $\delta(p^*, p^*) = 0$. This completes the proof of the theorem. □

Special Cases of the Theorem 3.2: If we take,

(1) $\gamma(p, q) = \gamma(\geqslant 1)$, then above theorem reduces to rational weak F-contractions in b-dislocated metric space.
(2) $\gamma(p, q) = 1$, then above theorem reduces to rational weak F-contractions in dislocated metric space.
(3) $S = T$, then above theorem reduces to single mapping which also holds good for rational weak F-contractions in δ-metric space.
(4) $S = T$ and $\gamma(p, q) = \gamma(\geqslant 1)$, then above theorem reduces to rational weak F-contractions in b-dislocated metric space.
(5) $S = T$ and $\gamma(p, q) = 1$, then above theorem reduces to rational weak F-contractions in dislocated metric space.

Apart from the above special cases, we can establish variety of results as consequences on rational weak F-contractions by arranging the below different consecutive values of $\mathcal{A}(p,q)$ in Eq. (3).

Consequences: If we take,

(1) $\mathcal{A}(p,q)=\delta(p,q)$
(2) $\mathcal{A}(p,q)=\frac{\delta(p,Sp)}{1+\delta(p,Sp)}$
(3) $\mathcal{A}(p,q)=\frac{\delta(q,Tq)}{1+\delta(q,Tq)}$
(4) $\mathcal{A}(p,q)=\delta(p,Sp)$
(5) $\mathcal{A}(p,q)=\delta(q,Tq)$
(6) $\mathcal{A}(p,q)=\max\{\delta(p,q),\frac{\delta(p,Sp)}{1+\delta(p,Sp)}\}$
(7) $\mathcal{A}(p,q)=\max\{\delta(p,q),\frac{\delta(q,Tq)}{1+\delta(q,Tq)}\}$
(8) $\mathcal{A}(p,q)=\max\{\delta(p,q),\delta(p,Sp),\delta(q,Tq)\}$
(9) $\mathcal{A}(p,q)=\max\{\delta(p,q),\frac{\delta(p,Sp)}{1+\delta(p,Sp)},\frac{\delta(q,Tq)}{1+\delta(q,Tq)}\}$

4. Common fixed point problems for rational Gupta-Saxena type *F*-contraction

Definition 4.1. Let $S,T:X\to X$ be two self-mappings on (X^*,δ) . The pair (S,T) is denoted as a pair of rational Gupta-Saxena type F-contractions if for all $p,q\in X$

$$\kappa+F(\delta(Sp,Tq))\leqslant F(\mathcal{A}(p,q)) \tag{25}$$

where $F\in\mathcal{F}^*,\kappa>0$ and

$$\mathcal{A}(p,q)=\max\left\{\frac{[1+\delta(p,Sp)]\delta(q,Tq)}{1+\delta(p,q)},\frac{\delta(p,Sp)\delta(q,Tq)}{1+\delta(p,q)},\delta(p,q)\right\}. \tag{26}$$

Theorem 4.2. *Let (S,T) be a pair o self-mappings on (X^*,δ) that forms a pair of rational Gupta-Saxena type F-contractions such that for each $p_0\in X$ we have $\limsup_{n,m\to\infty}\gamma(p_n,p_m)<\frac{1}{k}$. Then S and T have a common fixed point v in X. Moreover if (2) also holds for v, then $\delta(v,v)=0$.*

Proof. We construct an iterative sequence $\{p_n\}$ such that $p_{2n+1}=Sp_{2n}$ and $p_{2n+1}=Tp_{2n+1}$. If $\mathcal{A}(p,q)=0$, this claims $p=q$ is a common fixed point of S and T. Then there is nothing left to prove and our proof is complete. Let $\mathcal{A}(p,q)>0$ for all $p,q\in X$. From a pair of rational Gupta-Saxena type F-contractions, we get,

$$\begin{aligned}F(\delta(p_{2k+1},p_{2k+2}))&\leqslant F(\delta(Sp_{2k},Tp_{2k+1}))\\&\leqslant F(\mathcal{A}(p_{2k},p_{2k+1}))-\kappa,\end{aligned} \tag{27}$$

for all $i\in\mathbb{N}\cup\{0\}$, where

$$\begin{aligned}\mathcal{A}(p_{2k},p_{2k+1})&=\max\Big\{\tfrac{[1+\delta(p_{2k},Sp_{2k})]\delta(p_{2k+1},Tp_{2k+1})}{1+\delta(p_{2k},p_{2k+1})},\\&\quad\tfrac{\delta(p_{2k},Sp_{2k})\delta(p_{2k+1},Tp_{2k+1})}{1+\delta(p_{2k},p_{2k+1})},\delta(p_{2k},p_{2k+1})\Big\}\\&=\max\Big\{\tfrac{[1+\delta(p_{2k},p_{2k+1})]\delta(p_{2k+1},p_{2k+2})}{1+\delta(p_{2k},p_{2k+1})},\\&\quad\tfrac{\delta(p_{2k},p_{2k+1})\delta(p_{2k+1},p_{2k+2})}{1+\delta(p_{2k},p_{2k+1})},\delta(p_{2k},p_{2k+1})\Big\}\\&=\max\{\delta(p_{2k+1},p_{2k+2}),\delta(p_{2k+1},p_{2k+2}),\\&\qquad\delta(p_{2k},p_{2k+1})\}\\&=\max\{\delta(p_{2k},p_{2k+1}),\delta(p_{2k+1},p_{2k+2})\}\end{aligned} \tag{28}$$

If $\mathcal{A}(p_{2k},p_{2k+1})=\delta(p_{2k+1},p_{2k+2})$ then $F(\delta(p_{2k+1},p_{2k+2}))\leqslant F(\delta(p_{2k+1},p_{2k+2}))-\kappa$, this is a contradiction according to $(F1)$.

Therefore $\mathcal{A}(p_{2k},p_{2k+1})=\delta(p_{2k},p_{2k+1})$. Then from (4),

$$F(\delta(p_{2k+1},p_{2k+2}))\leqslant F(\delta(p_{2k},p_{2k+1}))-\kappa;\quad\forall k\in\mathbb{N}\cup\{0\} \tag{29}$$

Similarly, we have

$$F(\delta(p_{2k},p_{2k+1}))\leqslant F(\delta(p_{2k-1},p_{2k}))-\kappa. \tag{30}$$

By using (6) and (7), we get

$$F(\delta(p_{2k+1},p_{2k+2}))\leqslant F(\delta(p_{2k-1},p_{2k}))-2\kappa. \tag{31}$$

Repeating consequently, the iterative sequence, we get

$$F(\delta(p_{2k+1},p_{2k+2}))\leqslant F(\delta(p_0,p_1))-(2k+1)\kappa \tag{32}$$

Similarly,

$$F(\delta(p_{2k},p_{2k+1}))\leqslant F(\delta(p_0,p_1))-2k\kappa \tag{33}$$

From (9) and (10), we get,

$$F(\delta(p_n,p_{n+1}))\leqslant F(\delta(p_0,p_1))-n\kappa \tag{34}$$

On taking $\lim_{n\to\infty}$ on both sides of (11), we have

$$\lim_{n\to\infty}F(\delta(p_n,p_{n+1}))=-\infty \tag{35}$$

From F_2,

$$\lim_{n\to\infty}\delta(p_n,p_{n+1})=0 \tag{36}$$

From (11), for all $n\in\mathbb{N}$, we get

$$(\delta(p_n,p_{n+1}))^k(F(\delta(p_n,p_{n+1}))-F(\delta(p_0,p_1)))\leqslant-(\delta(p_n,p_{n+1}))^k n\kappa\leq 0 \tag{37}$$

By using (12), (13) and letting $n\to\infty$ in (14), we get,

$$\lim_{n\to\infty}(n(\delta(p_n,p_{n+1}))^k)=0 \tag{38}$$

From (15), there exists $n_1\in\mathbb{N}$ such that $n(\delta(p_n,p_{n+1}))^k\leqslant 1$ for all $n\geqslant n_1$ or

$$\delta(p_n,p_{n+1})\leqslant\frac{1}{n^{\frac{1}{k}}}\quad\forall n\geqslant n_1 \tag{39}$$

From (16), we get from $m>n>n_1$

$$\begin{aligned}\delta(p_n,p_m)&\leqslant\gamma(p_n,p_m)[\delta(p_n,p_{n+1})+\delta(p_{n+1},p_m)]\\&\leqslant\gamma(p_n,p_m)\delta(p_n,p_{n+1})+\gamma(p_n,p_m)\gamma(p_{n+1},p_m)\\&\quad[\delta(p_{n+1},p_{n+2})+\delta(p_{n+2},p_m)]\\&\leqslant\gamma(p_n,p_m)\delta(p_n,p_{n+1})+\gamma(p_n,p_m)\gamma(p_{n+1},p_m)\\&\quad\delta(p_{n+1},p_{n+2})+\ldots+\gamma(p_n,p_m)\gamma(p_{n+1},p_m)\\&\quad\gamma(p_{n+2},p_m)\ldots\gamma(p_{m-2},p_m)\gamma(p_{m-1},p_m)\delta(p_{m-1},p_m)\\&\leqslant\gamma(p_1,p_m)\gamma(p_2,p_m)\ldots\gamma(p_n,p_m)\delta(p_n,p_{n+1})\\&\quad+\gamma(p_1,p_m)\gamma(p_2,p_m)\ldots\gamma(p_{n+1},p_m)\delta(p_{n+1},p_{n+2})+\ldots\\&\quad+\gamma(p_1,p_m)\gamma(p_2,p_m)\ldots\gamma(p_{m-1},p_m)\delta(p_{m-1},p_m).\end{aligned}$$

Note that this series

$$\sum_{n=1}^{\infty}\delta(p_n,p_{n+1})\prod_{i=1}^{n}\gamma(p_i,p_m)\quad\text{converges.}$$

Since,

$$\sum_{n=1}^{\infty}\delta(p_n,p_{n+1})\prod_{i=1}^{n}\gamma(p_i,p_m) \leqslant \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{k}}}\prod_{i=1}^{n}\gamma(p_i,p_m) \leqslant \frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{k}}};\text{ which is convergent.}$$

Let

$$S=\sum_{n=1}^{\infty}\delta(p_n,p_{n+1})\prod_{i=1}^{n}\gamma(p_i,p_m).$$

$$S_n=\sum_{j=1}^{n}\delta(p_j,p_{j+1})\prod_{i=1}^{j}\gamma(p_i,p_m).$$

Thus for $m>n$ above inequality implies

$\delta(p_n,p_m)\leqslant S_{m-1}-S_{n-1}$.

Letting $n\to\infty$, we conclude that $\{TS(p_n)\}$ is a Cauchy sequence in (X,δ). Since (X,δ) is a complete δ-metric space, there exists $v\in X$ such that $\{TS(p_n)\}\to v$, that is,

$$\lim_{n\to\infty}\delta\ (p_n,v)=0 \tag{40}$$

Now by Lemma 1.9, we have

$$\kappa+F\ (\delta\ (p_{2n+1},Tv)\ )\leqslant\kappa+F\ (\delta\ (Sp_{2n},Tv)\ ) \tag{41}$$

The inequality (2) also holds good for p^*, then we have,

$$\kappa+F\ (\delta\ (p_{2n+1},Tv)\ )\leqslant F\ (\mathcal{A}\ (p_{2n},v)\ ) \tag{42}$$

$$\begin{aligned}\mathcal{A}(p_{2n},v) &= max\left\{\frac{[1+\delta(p_{2n},Sp_{2n})]\delta(v,Tv)}{1+\delta(p_{2n},v)},\frac{\delta(p_{2n},Sp_{2n})\delta(v,Tv)}{1+\delta(p_{2n},v)},\delta(p_{2n},v)\right\}\\ &= \max\left\{\frac{[1+\delta(p_{2n},p_{2n+1})]\delta(v,Tv)}{1+\delta(p_{2n},v)},\frac{\delta(p_{2n},p_{2n+1})\delta(v,Tv)}{1+\delta(p_{2n},v)},\delta(p_{2n},v)\right\}\end{aligned} \tag{43}$$

Taking limit as $n\to\infty$ and using (12), we get,

$$\lim_{n\to\infty}\mathcal{A}(p_{2n},v)=\delta(v,Tv) \tag{44}$$

From (14),

$$\begin{aligned}F(\delta(p_{2n+1},Tv)) &\leqslant F(\mathcal{A}(p_{2n},v))-\kappa\\ &< F(\mathcal{A}(p_{2n},v))\end{aligned} \tag{45}$$

By using F_1, we get $\delta(p_{2n+1},Tv)<\mathcal{A}(p_{2n},v)$. Applying limit as $n\to\infty$ and using (12), we get,

$\delta(v,Tv)<\delta(Tv,v)$,

which is a contradiction. Hence $\delta(v,Tv)=0$.

Similarly using (17) and Lemma 1.9,

$$\kappa+F(\delta(p_{2n+2},Sv))\leqslant\kappa+F(\delta(Tp_{2n+1},Sv))$$

We can prove that $\delta(v,Sv)=0$ or $v=Sv$. Hence S and T have a common fixed point v in X.

Now if $\delta(v,v)\neq 0$ then

$$\begin{aligned}F(\delta(v,v)) &\leqslant F(\delta(Sv,Tv))\\ &\leqslant F(\mathcal{A}(v,v))-\kappa\end{aligned} \tag{46}$$

where,

$$\begin{aligned}\mathcal{A}(v,v) &= \max\left\{\frac{[1+\delta(v,Sv)]\delta(v,Tv)}{1+\delta(v,v)},\frac{\delta(v,Sv)\delta(v,Tv)}{1+\delta(v,v)},\delta(v,v)\right\}\\ &= \max\left\{\frac{[1+\delta(v,v)]\delta(v,v)}{1+\delta(v,v)},\frac{\delta(v,v)\delta(v,v)}{1+\delta(v,v)},\delta(v,v)\right\}\\ &= \delta(v,v)\end{aligned} \tag{47}$$

Therefore from (15), $F(\delta(v,v))\leqslant F(\delta(v,v))-\kappa$, which is a contradiction. Hence $\delta(v,v)=0$. □

Special Cases of the Theorem 4.2: If we take,

(1) $\gamma(p,q)=\gamma(\geqslant 1)$, then above theorem reduces to rational Gupta-Saxena type F-contractions in b-dislocated metric space.
(2) $\gamma(p,q)=1$, then above theorem reduces to rational Gupta-Saxena type F-contractions in dislocated metric space.
(3) $S=T$, then above theorem reduces to single mapping which is also holds good for rational Gupta-Saxena type F-contractions in δ-metric space.
(4) $S=T$ and $\gamma(p,q)=\gamma(\geqslant 1)$, then above theorem reduces to rational Gupta-Saxena type F-contractions in b-dislocated metric space.
(5) $S=T$ and $\gamma(p,q)=1$, then above theorem reduces to rational Gupta-Saxena type F-contractions in dislocated metric space.

Apart from the above special cases, we can establish variety of results as consequences on rational Gupta-Saxena type F-contractions by arranging the below different consecutive values of $\mathcal{A}(p,q)$ in Eq. (28).

Consequences: If we take,

(1) $\mathcal{A}(p,q)=\delta(p,q)$
(2) $\mathcal{A}(p,q)=\frac{\delta(p,Sp)\delta(q,Tq)}{1+\delta(p,q)}$
(3) $\mathcal{A}(p,q)=\frac{[1+\delta(p,Sp)]\delta(q,Tq)}{1+\delta(p,q)}$
(4) $\mathcal{A}(p,q)=\max\left\{\frac{\delta(p,Sp)\delta(q,Tq)}{1+\delta(p,q)},\delta(p,q)\right\}$
(5) $\mathcal{A}(p,q)=\max\left\{\frac{[1+\delta(p,Sp)]\delta(q,Tq)}{1+\delta(p,q)},\frac{\delta(p,Sp)\delta(q,Tq)}{1+\delta(p,q)}\right\}$
(6) $\mathcal{A}(p,q)=\max\left\{\frac{[1+\delta(p,Sp)]\delta(q,Tq)}{1+\delta(p,q)},\delta(p,q)\right\}$

5. An illustrative numerical experiments with 3D surface view

Example 5.1. Let $X=[0,\infty)$. Define a distance function δ on X by $\delta(p,q)=(p+q)^2$ and $\gamma:X\times X\to[1,\infty)$ by $\gamma(p,q)=2p+3q+5$. Then (X,δ) forms a complete δ-metric space. Define the mapping $S,T:X\to X$ as follows:

$$Sp=\begin{cases}\frac{p}{8}, & \text{if } p\in[0,1]\\ (p-1)^2+\frac{1}{8}, & \text{if } p>1\end{cases}$$

$$Tp=\begin{cases}\frac{p}{8}, & \text{if } p\in[0,1]\\ \frac{2p^2-1}{8}, & \text{if } p>1\end{cases}$$

Define the function $F:\mathbb{R}^+\to\mathbb{R}^+$ by $F(\alpha)=\ln(\alpha^2+\alpha)$ for all $\alpha\in\mathbb{R}^+$ and $\kappa>0$.

Case 1. If $p \in [0,1]$ and $q > 1$, then $Sp = \frac{p}{8}$, $Tq = \frac{2q^2-1}{8}$. Consider,

$$\begin{aligned} F(\delta(Sp, Tq)) &= F\left(\delta\left(\tfrac{p}{8}, \tfrac{2q^2-1}{8}\right)\right) = F\left(\tfrac{p+2q^2-1}{8}\right)^2 \\ &= \ln\left[\left(\tfrac{p+2q^2-1}{8}\right)^4 + \left(\tfrac{p+2q^2-1}{8}\right)^2\right] \end{aligned}$$

Now consider,

$$\begin{aligned} \mathcal{A}(p,q) &= \max\left\{\delta(p,q), \tfrac{\delta(p,\frac{p}{8})}{1+\delta\left(p,\frac{p}{8}\right)}, \tfrac{\delta\left(q,\frac{2q^2-1}{8}\right)}{1+\delta\left(q,\frac{2q^2-1}{8}\right)}, \delta\left(p,\tfrac{p}{8}\right), \delta\left(q,\tfrac{2q^2-1}{8}\right)\right\} \\ &= \max\left\{(p+q)^2, \tfrac{\frac{81}{64}p^2}{1+\frac{81}{64}p^2}, \tfrac{\left(q+\frac{2q^2-1}{8}\right)^2}{1+\left(q+\frac{2q^2-1}{8}\right)^2}, \tfrac{81}{64}p^2, \left(q+\tfrac{2q^2-1}{8}\right)^2\right\} \\ &= \left(q+\tfrac{2q^2-1}{8}\right)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} F(\mathcal{A}(p,q)) &= F\left[\left(q+\tfrac{2q^2-1}{8}\right)^2\right] \\ &= \ln\left[\left(q+\tfrac{2q^2-1}{8}\right)^4 + \left(q+\tfrac{2q^2-1}{8}\right)^2\right] \end{aligned}$$

Consider, $F(\delta(Sp,Tq)) - F(A(p,q)) = \ln\left[\frac{\left(\frac{p}{8}+\frac{2q^2-1}{8}\right)^4+\left(\frac{p}{8}+\frac{2q^2-1}{8}\right)^2}{\left(q+\frac{2q^2-1}{8}\right)^4+\left(q+\frac{2q^2-1}{8}\right)^2}\right]$

Since $p < q, \frac{p}{8} < q$ which implies that $\frac{p}{8} + \frac{2q^2-1}{8} < q + \frac{2q^2-1}{8}$.

Thus
$\left(\frac{p}{8}+\frac{2q^2-1}{8}\right)^4 + \left(\frac{p}{8}+\frac{2q^2-1}{8}\right)^2 < \left(q+\frac{2q^2-1}{8}\right)^4 + \left(q+\frac{2q^2-1}{8}\right)^2.$

Therefore, $\ln\left[\frac{\left(\frac{p}{8}+\frac{2q^2-1}{8}\right)^4+\left(\frac{p}{8}+\frac{2q^2-1}{8}\right)^2}{\left(q+\frac{2q^2-1}{8}\right)^4+\left(q+\frac{2q^2-1}{8}\right)^2}\right] < -\kappa;$ for $\kappa > 0$.

Hence $F(\delta(Sp,Tq)) - F(A(p,q)) < -\kappa$ for $\kappa > 0$. This gives $\kappa + F(\delta(Sp,Tq)) \leqslant F(A(p,q))$.

Case 2. If $p > 1$ and $q \in [0,1]$ then $Sp = (p-1)^2 + \frac{1}{8}$; $Tq = \frac{q}{8}$.

Consider,

$$\begin{aligned} F(\delta(Sp,Tq)) &= F\left(\delta\left((p-1)^2 + \tfrac{1}{8}, \tfrac{q}{8}\right)\right) \\ &= F\left(\left((p-1)^2 + \tfrac{1}{8} + \tfrac{q}{8}\right)^2\right) \\ &= F\left(\left((p-1)^2 + \tfrac{q+1}{8}\right)^2\right) \\ &= \ln\left(\left((p-1)^2 + \tfrac{q+1}{8}\right)^4 + \left((p-1)^2 + \tfrac{q+1}{8}\right)^2\right) \end{aligned}$$

Now consider,

$$\begin{aligned} \mathcal{A}(p,q) &= \max\left\{\delta(p,q), \frac{\delta\left(p,(p-1)^2+\frac{1}{8}\right)}{1+\delta\left(p,(p-1)^2+\frac{1}{8}\right)}, \frac{\delta(q,\frac{q}{8})}{1+\delta(q,\frac{q}{8})}, \right. \\ &\qquad \left. \delta\left(p,(p-1)^2+\tfrac{1}{8}\right), \delta(q,\tfrac{q}{8})\right\} \\ &= \max\left\{(p+q)^2, \frac{\left(p+(p-1)^2+\frac{1}{8}\right)^2}{1+(p+(p-1)^2+\frac{1}{8})^2}, \frac{(q+\frac{q}{8})^8}{1+(q+\frac{q}{8})^2}, \right. \\ &\qquad \left. \left(p+(p-1)^2+\tfrac{1}{8}\right)^2, (q+\tfrac{q}{8})^2\right\} \\ &= \left(p+(p-1)^2+\tfrac{1}{8}\right)^2 \\ &= \left((p-1)^2+(p+\tfrac{1}{8})\right)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} F(\mathcal{A}(p,q)) &= F\left[((p-1)^2+(p+\tfrac{1}{8}))^2\right] \\ &= \ln\left[\left((p-1)^2+(p+\tfrac{1}{8})\right)^4 + \left((p-1)^2+(p+\tfrac{1}{8})\right)^2\right] \end{aligned}$$

Consider,
$F(\delta(Sp,Tq)) - F(A(p,q)) = \ln\left[\frac{\left((p-1)^2+\frac{q+1}{8}\right)^4+\left((p-1)^2+\frac{q+1}{8}\right)^2}{\left((p-1)^2+(p+\frac{1}{8})\right)^4+\left((p-1)^2+(p+\frac{1}{8})\right)^2}\right]$

Since $q < p, \frac{q}{8} + \frac{1}{8} < p + \frac{1}{8}$ which yields,

$\left((p-1)^2+\frac{q+1}{8}\right)^4 + \left((p-1)^2+\frac{q+1}{8}\right)^2 < \left((p-1)^2+(p+\frac{1}{8})\right)^4 + \left((p-1)^2+(p+\frac{1}{8})\right)^2$

Thus,

$$\ln\left[\frac{\left((p-1)^2+\frac{q+1}{8}\right)^4+\left((p-1)^2+\frac{q+1}{8}\right)^2}{\left((p-1)^2+(p+\frac{1}{8})\right)^4+\left((p-1)^2+(p+\frac{1}{8})\right)^2}\right] < -\kappa; \ \kappa > 0.$$

Hence $\kappa + F(\delta(Sp,Tq)) \leqslant F(A(p,q))$.

Thus all the conditions of Theorem 3.2 satisfied and '0' is the common fixed point. Moreover $\delta(0,0) = 0$.

Example 5.2. Let $X = [0,1]$. Define a distance function δ over a set X by $\delta(p,q) = (p+q)^2$ and $\gamma : X \times X \to [1,\infty)$ by $\gamma(p,q) = p^2 + q^2 + 2$. Then (X,δ) forms a complete δ-metric space. Define the mappings $S, T : X \to X$ by $Sp = \frac{2p}{15}$ for all $p \in [0,1]$ and

$$Tp = \begin{cases} \frac{p}{5}, & \text{if } p \in [0,1) \\ 0, & \text{if } p = 1. \end{cases}$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(\alpha) = \ln(\alpha^2 + \alpha)$ for all $\alpha > 0$ and $\kappa > 0$.

Case 1. If $0 < p < 1$ and $q = 1$

Consider,

$$\begin{aligned} F(\delta(Sp,Tq)) &= F\left(\delta\left(\tfrac{2p}{15}, 0\right)\right) \\ &= F\left(\tfrac{4p^2}{225}\right) \\ &= \ln\left(\tfrac{16p^4}{50625} + \tfrac{p^24}{225}\right) \end{aligned}$$

Now consider,

$$\begin{aligned} \mathcal{A}(p,q) &= \max\left\{\tfrac{[1+\delta(p,\frac{2p}{15})]\delta(1,0)}{1+\delta(p,1)}, \tfrac{[\delta(p,\frac{2p}{15})]\delta(1,0)}{1+\delta(p,1)}, \delta(p,1)\right\} \\ &= \max\left\{\tfrac{1+(p+\frac{2p}{15})^2}{1+(1+p)^2}, \tfrac{(p+\frac{2p}{15})^2}{1+(1+p)^2}, (p+1)^2\right\} \\ &= (p+1)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} F(\mathcal{A}(p,q)) &= F(p+1)^2 \\ &= \ln[(p+1)^4 + (p+1)^2] \end{aligned}$$

Thus,

$$F(\delta(Sp,Tq)) - F(\boldsymbol{\mathcal{A}}(p,q)) = \ln\left[\frac{\frac{16p^4}{50625}+\frac{4p^2}{225}}{(p+1)^4+(p+1)^2}\right] = -\kappa,$$

for any value of $0 < p < 1$ and $\kappa > 0$

Therefore, $\kappa + F(\delta(Sp,Tq)) \leqslant F(\boldsymbol{\mathcal{A}}(p,q))$.

Case 2. If $p = 1$ and $0 \leqslant q < 1$ then consider, $Sp = \frac{2}{15}, Tq = \frac{q}{5}$.

$$\begin{aligned} F(\delta(Sp,Tq)) &= F\left(\delta\left(\tfrac{2}{15},\tfrac{q}{5}\right)\right) \\ &= F\left(\delta\left(\tfrac{2}{15}+\tfrac{q}{5}\right)^2\right) \\ &= \ln\left[\left(\tfrac{2}{15}+\tfrac{q}{5}\right)^4 + \left(\tfrac{2}{15}+\tfrac{q}{5}\right)^2\right] \end{aligned}$$

Now consider,

$$\begin{aligned} \boldsymbol{\mathcal{A}}(p,q) &= \max\left\{\frac{[1+\delta(1,\frac{2}{15})]\delta(q,\frac{q}{5})}{1+\delta(1,q)}, \frac{\delta(1,\frac{2}{15})\delta(q,\frac{q}{5})}{1+\delta(1,q)}, \delta(1,q)\right\} \\ &= \max\left\{\frac{\left(1+(\frac{17}{15})^2\right)\frac{36q^2}{25}}{1+(1+q)^2}, \frac{\left((\frac{17}{15})^2\right)\frac{36q^2}{25}}{1+(1+q)^2}, (1+q)^2\right\} \\ &= (1+q)^2 \end{aligned}$$

Therefore $F(\boldsymbol{\mathcal{A}}(p,q)) = \ln((1+q)^2) = \ln((1+q)^4 + (1+q)^2)$.

Thus,

$$\begin{aligned} F(\delta(Sp,Tq)) - F(\boldsymbol{\mathcal{A}}(p,q)) &= \ln\left(\frac{(\frac{2}{15}+\frac{q}{5})^4+(\frac{2}{15}+\frac{q}{5})^2}{(1+q)^4+(1+q)^2}\right) \\ &= -\kappa \text{ for any value of } \\ & \quad 0 \leqslant q < 1 \text{ and } \kappa > 0. \end{aligned}$$

Hence $\kappa + F(\delta(Sp,Tq)) \leqslant F(\boldsymbol{\mathcal{A}}(p,q))$.

Thus all the conditions of Theorem 4.2 satisfied and '0' is the common fixed point. Moreover $\delta(0,0) = 0$.

In Fig.1,2 we gave a 3D surface which shows the comparison of the left hand side and right hand side of Eq. 25. Thus the assertions managed by the Theorem 4.2 are gratified, by that considering the stated factors, T has a fixed point and it is unique. (See Table 1,2).

Example 5.3. Define a distance function δ over the set $X = [0,1]$ by $\delta(p,q) = |p| + |q| + \frac{|p|^2}{4} + \frac{|q|^2}{5}$ and $\gamma : X \times X \to [1,\infty)$ by $\gamma(p,q) = |p+q|^2$.

Then (X,δ) is a complete dislocated extended b-metric space. Define the mapping $T : X \to X$ by

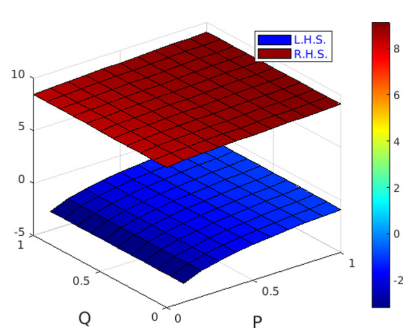

Fig. 1 The value of the correlation of the left hand side and the right hand side of (3) of Theorem 4.2 in Ex.5.2, case1.

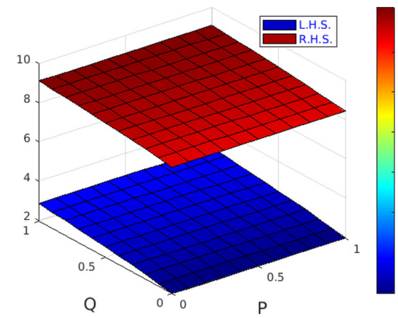

Fig. 2 The value of the correlation of the left hand side and the right hand side of (3) of Theorem 4.2 in Ex.5.2, case2.

Table 1 Numerical comparisons of L.H.S and R.H.S of Ex.5.2 of case-1.

p	q	$\kappa + F(\delta(Sp,Tq))$ where $\kappa = 0.45 > 0$	$F(\boldsymbol{\mathcal{A}}(p,q))$
0.1	1	−8.22952	0.9836
0.2	1	−6.797971164	1.256641153
0.3	1	−5.986152928	1.514269723
0.4	1	−5.41238752	1.758133742
0.5	1	−4.961665805	1.989585213
0.6	1	−4.595077682	2.209767803
0.7	1	−4.284482541	2.41966566
0.8	1	−4.014779659	2.620136599
0.9	1	−3.776229768	2.811935629

Table 2 Numerical comparisons of L.H.S and R.H.S of Ex.5.2 of case-2.

p	q	$\kappa + F(\delta(Sp,Tq))$ where $\kappa = 0.45 > 0$	$F(\boldsymbol{\mathcal{A}}(p,q))$
1	0.0	−3.56239598	0
1	0.1	−3.277043193	0.983612875
1	0.2	−3.025475596	1.256641153
1	0.3	−2.799982794	1.514269723
1	0.4	−0.901528	1.758133742
1	0.5	−0.795566704	1.989585213
1	0.6	−2.233895821	2.209767803
1	0.7	−1.276847905	2.41966566
1	0.8	−1.920349184	2.620136599
1	0.9	−1.777323152	2.811935629

$$Tp = \begin{cases} p^2, & \text{for } \ p \in [0,\frac{1}{2}) \\ \frac{\log(3p)}{2}, & \text{for } \ p \in [\frac{1}{2},1] \end{cases}$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(\alpha) = \log\alpha$ for all $\alpha > 0$ and $\kappa > 0$.

Case 1. If $p,q \in [0,\frac{1}{2})$ then $Tp = p^2, \ Tq = q^2$.

Consider,

$$\begin{aligned} F(\delta(Tp,Tq)) &= F(\delta(p^2,q^2)) \\ &= F(|p^2| + |q^2| + \tfrac{|p|^4}{4} + \tfrac{|q|^4}{5}) \\ &= \log(|p^2| + |q^2| + \tfrac{|p|^4}{4} + \tfrac{|q|^4}{5}) \end{aligned}$$

Now consider,

$$\begin{aligned} F(\mathcal{A}(p,q)) &= F\Big(\max\Big\{\tfrac{[1+\delta(p,Sp)]\delta(q,Tq)}{1+\delta(p,q)}, \tfrac{\delta(p,Sp)\delta(q,Tq)}{1+\delta(p,q)}, \delta(p,q)\Big\}\Big) \\ &= F\Big(\max\Big\{\tfrac{[1+\delta(p,p^2)]\delta(q,q^2)}{1+\delta(p,q)}, \tfrac{\delta(p,p^2)\delta(q,q^2)}{1+\delta(p,q)}, \delta(p,q)\Big\}\Big) \\ &= F\Big(\max\Big\{\tfrac{(1+|p|+|p^2|+\frac{|p|^2}{4}+\frac{|p|^4}{5})(|q|+|q^2|+\frac{|q|^2}{4}+\frac{|q|^4}{5})}{1+|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^4}{5}}, \\ &\tfrac{(|p|+|p^2|+\frac{|p|^2}{4}+\frac{|p|^4}{5})(|q|+|q^2|+\frac{|q|^2}{4}+\frac{|q|^4}{5})}{1+|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^4}{5}}, \\ &|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^4}{5}\Big\}\Big) \\ &= F(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^4}{5}) \\ &= \log(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^4}{5}) \end{aligned}$$

Thus,

$$\begin{aligned} F(\delta(Tp,Tq) - F(\mathcal{A}(p,q)) &= \log(|p|^2+|q|^2+\tfrac{|p|^4}{4}+\tfrac{|q|^4}{5}) \\ &\quad -\log(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}) \\ &= \log\left\{\frac{|p|^2+|q|^2+\frac{|p|^4}{4}+\frac{|q|^4}{5}}{|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}\right\} \\ &= -\kappa \text{ for some } \kappa > 0. \end{aligned}$$

Hence, $F(\delta(Tp,Tq) - F(\mathcal{A}(p,q)) \leqslant -\kappa$ which implies $\kappa + F(\delta(Tp,Tq) \leqslant F(\mathcal{A}(p,q))$.

Case 2. If $p,q \in [\frac{1}{2},1]$ then $Tp = \frac{\log 3p}{2}$, $Tq = \frac{\log 3q}{2}$.

Consider,

$$\begin{aligned} F(\delta(Tp,Tq)) &= F(\delta(\tfrac{\log 3p}{2},\tfrac{\log 3q}{2})) \\ &= F(|\tfrac{\log 3p}{2}|+|\tfrac{\log 3q}{2}|+\tfrac{|\log 3p|^2}{16}+\tfrac{|\log 3q|^2}{20}) \\ &= \log(|\tfrac{\log 3p}{2}|+|\tfrac{\log 3q}{2}|+\tfrac{|\log 3p|^2}{16}+\tfrac{|\log 3q|^2}{20}) \end{aligned}$$

Now consider,

$$\begin{aligned} F(\mathcal{A}(p,q)) &= F\Big(\max\Big\{\tfrac{[1+\delta(p,Sp)]\delta(q,Tq)}{1+\delta(p,q)}, \tfrac{\delta(p,Sp)\delta(q,Tq)}{1+\delta(p,q)}, \delta(p,q)\Big\}\Big) \\ &= F\Big(\max\Big\{\tfrac{[1+\delta(p,\frac{\log 3p}{2})]\delta(q,\frac{\log 3q}{2})}{1+\delta(p,q)}, \tfrac{\delta(p,\frac{\log 3p}{2})\delta(q,\frac{\log 3q}{2})}{1+\delta(p,q)}, \delta(p,q)\Big\}\Big) \\ &= F(\max\Big\{\tfrac{(1+|p|+|\frac{\log 3p}{2}|+\frac{|p|^2}{4}+\frac{|\log 3p|^2}{20})(|q|+|\frac{\log 3q}{2}|+\frac{|q|^2}{4}+|\frac{|\log 3q|^2}{20}|}{1+|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}, \\ &\tfrac{(|p|+|\frac{\log 3p}{2}|+\frac{|p|^2}{4}+\frac{|\log 3p|^2}{20})(|q|+|\frac{\log 3q}{2}|+\frac{|q|^2}{4}+|\frac{|\log 3q|^2}{20}|}{1+|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}, \\ &|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}\Big\}) \\ &= F(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^4}{5}) \\ &= \log(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^4}{5}) \end{aligned}$$

Thus,

$$\begin{aligned} F(\delta(Tp,Tq) - F(\mathcal{A}(p,q)) &= \log(|\tfrac{\log 3p}{2}|+|\tfrac{\log 3q}{2}|+\tfrac{|\log 3p|^2}{16}+\tfrac{|\log 3q|^2}{20}) \\ &\quad -\log(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}) \\ &= \log\left\{\frac{|\frac{\log 3p}{2}|+|\frac{\log 3q}{2}|+\frac{|\log 3p|^2}{16}+\frac{|\log 3q|^2}{20}}{|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}\right\} \\ &= -\kappa \text{ for some } \kappa > 0. \end{aligned}$$

Hence, $F(\delta(Tp,Tq) - F(\mathcal{A}(p,q)) \leqslant -\kappa$ which implies $\kappa + F(\delta(Tp,Tq) \leqslant F(\mathcal{A}(p,q))$.

Case 3. If $p \in [0,\frac{1}{2})$ and $q \in [\frac{1}{2},1]$ then $Tp = p^2$, $Tq = \frac{\log 3q}{2}$.

Consider,

$$\begin{aligned} F(\delta(Tp,Tq)) &= F(\delta(p^2,\tfrac{\log 3q}{2})) \\ &= F(|p^2|+|\tfrac{\log 3q}{2}|+\tfrac{|p|^4}{4}+\tfrac{|\log 3q|^2}{20}) \\ &= \log(|p^2|+|\tfrac{\log 3q}{2}|+\tfrac{|p|^4}{4}+\tfrac{|\log 3q|^2}{20}) \end{aligned}$$

Now consider,

$$\begin{aligned} F(\mathcal{A}(p,q)) &= F\Big(\max\Big\{\tfrac{[1+\delta(p,Sp)]\delta(q,Tq)}{1+\delta(p,q)}, \tfrac{\delta(p,Sp)\delta(q,Tq)}{1+\delta(p,q)}, \delta(p,q)\Big\}\Big) \\ &= F\Big(\max\Big\{\tfrac{[1+\delta(p,p^2)]\delta(q,\frac{\log 3q}{2})}{1+\delta(p,q)}, \tfrac{\delta(p,p^2)\delta(q,\frac{\log 3q}{2})}{1+\delta(p,q)}, \delta(p,q)\Big\}\Big) \\ &= F(\max\Big\{\tfrac{(1+|p|+|p^2|+\frac{|p|^2}{4}+\frac{|p|^4}{5})(|q|+|\frac{\log 3q}{2}|+\frac{|q|^2}{4}+|\frac{|\log 3q|^2}{20}|}{1+|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}, \\ &\tfrac{(|p|+|p^2|+\frac{|p|^2}{4}+\frac{|p|^4}{5})(|q|+|\frac{\log 3q}{2}|+\frac{|q|^2}{4}+|\frac{|\log 3q|^2}{20}|}{1+|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}, \\ &|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}\Big\}) \\ &= F(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}) \\ &= \log(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}) \end{aligned}$$

Thus,

$$\begin{aligned} F(\delta(Tp,Tq) - F(\mathcal{A}(p,q)) &= \log(|p^2|+|\tfrac{\log 3q}{2}|+\tfrac{|p|^4}{4}+\tfrac{|\log 3q|^2}{20}) \\ &\quad -\log(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}) \\ &= \log\left\{\frac{|p^2|+|\frac{\log 3q}{2}|+\frac{|p|^4}{4}+\frac{|\log 3q|^2}{20}}{|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}\right\} \\ &= -\kappa \text{ for some } \kappa > 0. \end{aligned}$$

Hence, $F(\delta(Tp,Tq) - F(\mathcal{A}(p,q)) \leqslant -\kappa$ which implies $\kappa + F(\delta(Tp,Tq) \leqslant F(\mathcal{A}(p,q))$.

Case 4. If $p \in [\frac{1}{2},1]$ and $q \in [0,\frac{1}{2})$ then $Tp = \frac{\log 3p}{2}$, $Tq = q^2$.

Consider,

$$\begin{aligned} F(\delta(Tp,Tq)) &= F(\delta(\tfrac{\log 3p}{2},q^2)) \\ &= F(|\tfrac{\log 3p}{2}|+|q^2|+\tfrac{|\log 3p|^2}{16}+\tfrac{|q|^4}{5}) \\ &= \log(|\tfrac{\log 3p}{2}|+|q^2|+\tfrac{|\log 3p|^2}{16}+\tfrac{|q|^4}{5}) \end{aligned}$$

Now consider,

$$\begin{aligned} F(\mathcal{A}(p,q)) &= F\Big(\max\Big\{\tfrac{[1+\delta(p,Sp)]\delta(q,Tq)}{1+\delta(p,q)}, \tfrac{\delta(p,Sp)\delta(q,Tq)}{1+\delta(p,q)}, \delta(p,q)\Big\}\Big) \\ &= F\Big(\max\Big\{\tfrac{[1+\delta(p,\frac{\log 3p}{2})]\delta(q,q^2)}{1+\delta(p,q)}, \tfrac{\delta(p,\frac{\log 3p}{2})\delta(q,q^2)}{1+\delta(p,q)}, \delta(p,q)\Big\}\Big) \\ &= F(\max\Big\{\tfrac{(1+|p|+|\frac{\log 3p}{2}|+\frac{|p|^2}{4}+\frac{|\log 3p|^2}{20})(|q|+|q^2|+\frac{|q|}{4}+|\frac{|q|^4}{5}|}{1+|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}, \\ &\tfrac{(|p|+|\frac{\log 3p}{2}|+\frac{|p|^2}{4}+\frac{|\log 3p|^2}{20})(|q|+|q^2|+\frac{|q|}{4}+|\frac{|q|^4}{5}|}{1+|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}, \\ &|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}\Big\}) \\ &= F(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}) \\ &= \log(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}) \end{aligned}$$

Thus,

$$\begin{aligned} F(\delta(Tp,Tq) - F(\mathcal{A}(p,q)) &= \log(|\tfrac{\log 3p}{2}|+|q^2|+\tfrac{|\log 3p|^2}{16}+\tfrac{|q|^4}{5}) \\ &\quad -\log(|p|+|q|+\tfrac{|p|^2}{4}+\tfrac{|q|^2}{5}) \\ &= \log\left\{\frac{|\frac{\log 3p}{2}|+|q^2|+\frac{|\log 3p|^2}{16}+\frac{|q|^4}{5}}{|p|+|q|+\frac{|p|^2}{4}+\frac{|q|^2}{5}}\right\} \\ &= -\kappa \text{ for some } \kappa > 0. \end{aligned}$$

Hence, $F(\delta(Tp,Tq) - F(\mathcal{A}(p,q)) \leqslant -\kappa$ which implies $\kappa + F(\delta(Tp,Tq) \leqslant F(\mathcal{A}(p,q))$.

The numerical experiment is carried out by approximating the fixed point of T in Table 3. Furthermore, the converges behaviour of these iterations is shown in Fig. 3.

6. Applications to the existence of solution for Volterra integral equation via various *F*-contractions

The Volterra integral equations are a special type of integral equations. The theory of Volterra equations plays an important role in the theory of applied mathematics as well as applied sciences. Some times it will be treated as useful mathematical tools in both pure and applied mathematics, and is extensively used in pertinent research. (See for example [48–53]).

6.1. Existence of common fixed point of Volterra integral equation for rational weak F-contractions

As an application, we use Theorem 3.2 to study the existence problem of solution of Volterra integral equation.

Let us consider the following type Volterra integral equations:

$$\Omega_1(u) = \int_0^u g_1(u,v,\Omega_1(v))dv \tag{48}$$

$$\Omega_2(u) = \int_0^u g_2(u,v,\Omega_2(v))dv \tag{49}$$

for all $u \in [0,1]$. We will find the solution of (52) and (55). Let $X = \mathcal{C}([0,1],\mathbb{R}^+)$ be the space of all real continuous functions on $[0,1]$, endowed with the complete δ-metric. For $\Omega_1 \in \mathcal{C}([0,1],\mathbb{R}^+)$, define norm as:

$$||\Omega_1||_\kappa = \max_{u\in[0,1]}\{|\Omega_1(u)|e^{-\kappa u}\}$$

where $\kappa > 0$ is taken as arbitrary. Then, define $\delta_\kappa : X \times X \to [0,\infty)$

$$\begin{aligned}\delta_\kappa(\Omega_1,\Omega_2) &= \max_{u\in[0,1]}\{|\Omega_1(u)+\Omega_2(u)|^2 e^{-\kappa u}\}\\ &= ||\Omega_1+\Omega_2||_\kappa\end{aligned} \tag{50}$$

for all $\Omega_1,\Omega_2 \in \mathcal{C}([0,1],\mathbb{R}^+)$ and $\gamma : X \times X \to [1,\infty)$ by $\gamma(\Omega_1,\Omega_2) = |\Omega_1(i)+|\Omega_2(i)|+1$.

Then (X,δ_κ) becomes a complete dislocated extended *b*-metric space. Now we prove the following theorem to ensure the existence of common solution of Volterra integral equation.

Table 3 Picard iterations.

p_0	$p_0 = 0.05$	$p_0 = 0.45$	$p_0 = 0.75$	$p_0 = 0.98$
p_1	0.00250000	0.20250000	0.4054651081	0.53920479
p_2	0.00000625	0.04100625	0.1644019539	0.24047622
p_3	0.00000000	0.00168151	0.02702800	0.05782881
p_4	0.00000000	0.00000282	0.00073051	0.00334417
p_5	0.00000000	0.00000000	0.00000053	0.00001118
p_6	0.00000000	0.00000000	0.00000000	0.00000000
⋮	⋮	⋮	⋮	⋮

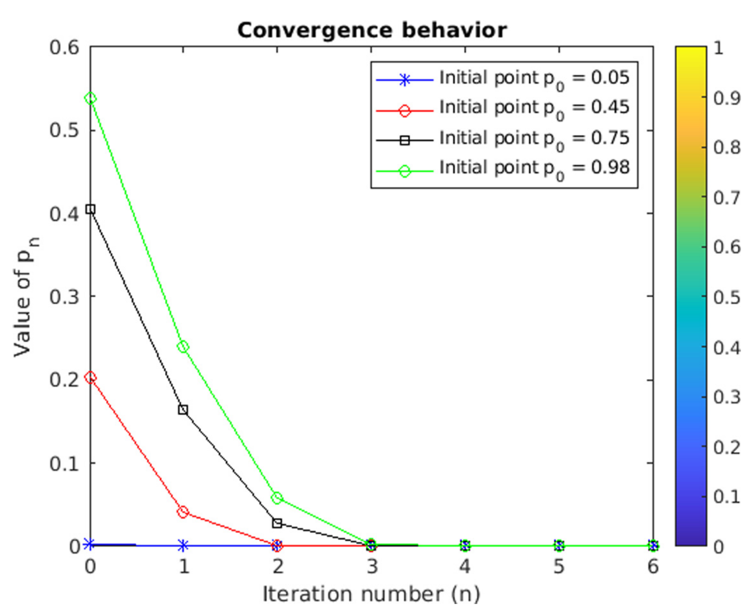

Fig. 3 Convergence behavior for Example 5.3.

Theorem 5.1.1. *Let (X, δ_κ) be a complete dislocated extended b-metric space as defined above. Further, assume that the following conditions are satisfied:*

(1) $g_1, g_2 : [0,1] \times [0,1] \times \mathcal{C}([0,1], \mathbb{R}^+) \to \mathbb{R}$
(2) *Define*

$$S\Omega_1(u) = \int_0^u g_1(u, v, \Omega_1(v))dv \tag{51}$$

$$T\Omega_2(u) = \int_0^u g_2(u, v, \Omega_2(v))dv \tag{52}$$

Suppose there exists $\kappa > 0$ such that

$$|g_1(u, v, \Omega_1) + g_2(u, v, \Omega_2|^2 \leqslant \kappa e^{-\kappa} \mathcal{A}(\Omega_1, \Omega_2)$$

for all $u, v \in [0,1]$ and $\Omega_1, \Omega_2 \in \mathcal{C}([0,1], \mathbb{R}^+)$; where

$$\begin{aligned}\mathcal{A}(\Omega_1, \Omega_2) &= \max\Big\{|(\Omega_1 + \Omega_2)^2|, \tfrac{|\Omega_1 + S\Omega_1|^2}{1+|\Omega_1+S\Omega_1|^2}, \tfrac{|\Omega_2+T\Omega_2|^2}{1+|\Omega_2+T\Omega_2|^2}, \\ &\quad |\Omega_1 + S\Omega_1|^2, |\Omega_2 + T\Omega_2|^2\Big\}\end{aligned} \tag{53}$$

Then integral Eqs. (52) and (53) have a common solution.

Proof. For any $\Omega_1, \Omega_2 \in [0,1]$, $u \in [0,1]$. Consider,

$$\begin{aligned}|S\Omega_1(u) + T\Omega_2(u)|^2 &= |\textstyle\int_0^u (g_1(u,v,\Omega_1(v))dv + g_2(u,v,\Omega_2(v)))dv|^2 \\ &\leqslant \textstyle\int_0^u |(g_1(u,v,\Omega_1(v)) + g_2(u,v,\Omega_2(v)))|^2 dv \\ &\leqslant \textstyle\int_0^u [\kappa e^{-\kappa} \mathcal{A}(\Omega_1(v),\Omega_2(v))]dv \\ &\leqslant \kappa e^{-\kappa} \textstyle\int_0^u e^{v\kappa} \max\Big\{|(\Omega_1(v)+\Omega_2(v))|^2 e^{-v\kappa}, \\ &\quad \tfrac{|\Omega_1(v)+S\Omega_1(v)|^2 e^{-2v\kappa}}{[1+|\Omega_1(v)+S\Omega_1(v)|^2]e^{-v\kappa}}, \tfrac{|\Omega_2(v)+T\Omega_2(v)|^2 e^{-2v\kappa}}{[1+|\Omega_2(v)+T\Omega_2(v)|^2]e^{-v\kappa}}, \\ &\quad |\Omega_1(v)+S\Omega_1(v)|^2 e^{-v\kappa}, |\Omega_2(v)+T\Omega_2(v)|^2 e^{-v\kappa}\Big\}dv \\ &\leqslant \kappa e^{-\kappa} \textstyle\int_0^u e^{v\kappa} \max\Big\{\delta_\kappa(\Omega_1,\Omega_2), \tfrac{\delta_\kappa(\Omega_1,S\Omega_1)}{1+\delta_\kappa(\Omega_1,S\Omega_1)}, \\ &\quad \tfrac{\delta_\kappa(\Omega_2,T\Omega_2)}{1+\delta_\kappa(\Omega_2,T\Omega_2)}, \delta_\kappa(\Omega_1,S\Omega_1), \delta_\kappa(\Omega_2,T\Omega_2)\Big\}dv \\ &= \kappa e^{-\kappa} \textstyle\int_0^u e^{v\kappa} \mathcal{A}(\Omega_1(v),\Omega_2(v))dv \\ &\leqslant \kappa e^{-\kappa} \mathcal{A}(\Omega_1(v),\Omega_2(v)) |\tfrac{e^{v\kappa}}{\kappa}|_0^v dv \\ &\leqslant \kappa e^{-\kappa} \mathcal{A}(\Omega_1(v),\Omega_2(v)) \tfrac{e^{u\kappa}}{\kappa} \\ &\leqslant e^{-\kappa(1-u)} \mathcal{A}(\Omega_1(v),\Omega_2(v)).\end{aligned} \tag{54}$$

So we have

$$\begin{aligned}|S\Omega_1(u) + T\Omega_2(u)|^2 e^{-u\kappa} \leqslant e^{-\kappa} \mathcal{A}(\Omega_1(v), \Omega_2(v)) \Rightarrow ||S\Omega_1 + T\Omega_2|| \\ \leqslant e^{-\kappa} \mathcal{A}(\Omega_1(v), \Omega_2(v))\end{aligned}$$

which yields,

$$\delta_\kappa(S\Omega_1, T\Omega_2) \leqslant e^{-\kappa} \mathcal{A}(\Omega_1(v), \Omega_2(v)).$$

Applying logarithms on both sides,

$$\kappa + \ln(\delta_\kappa(S\Omega_1, T\Omega_2)) \leqslant \ln \mathcal{A}(\Omega_1(v), \Omega_2(v)); \quad \forall \Omega_1, \Omega_2 \in X. \tag{55}$$

Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(p) = \ln(p)$, $p > 0$. Then from (103), we get

$$\kappa + F(\delta_\kappa(S\Omega_1, T\Omega_2)) \leqslant F(\mathcal{A}(\Omega_1(v), \Omega_2(v))),$$

where

$$\begin{aligned}\mathcal{A}(\Omega_1(v), \Omega_2(v)) &= \max\Big\{\delta_\kappa(\Omega_1, \Omega_2), \tfrac{\delta_\kappa(\Omega_1, S\Omega_1)}{1+\delta_\kappa(\Omega_1, S\Omega_1)}, \tfrac{\delta_\kappa(\Omega_2, T\Omega_2)}{1+\delta_\kappa(\Omega_2, T\Omega_2)}, \\ &\quad \delta_\kappa(\Omega_1, S\Omega_1), \delta_\kappa(\Omega_2, T\Omega_2)\Big\}\end{aligned} \tag{56}$$

Thus all the conditions of the Theorem 3.2 are satisfied for $F(p) = \ln(p)$, $p > 0$ and $\delta(\Omega_1, \Omega_2) = ||\Omega_1 + \Omega_2||_\kappa$. Hence integral equations given in (52) and (53) have common solution. □

6.2. Existence of common fixed point of Volterra integral equation for rational Gupta-Saxena type F-contractions

Here we present existence of common fixed point for Gupta-Saxena type F-contractions, which yields the existence of common solutions of Volterra type of integral equations as an application.

Consider the below Volterra type integral equations which is in the form of

$$\left.\begin{aligned}\theta_1(u) &= \textstyle\int_0^u Z_1(u, v, \theta_1(v))dv \\ \theta_2(u) &= \textstyle\int_0^u Z_2(u, v, \theta_2(v))dv\end{aligned}\right\} \tag{57}$$

$u \in [0, T]$, where $T > 0$ and $Z_1, Z_2 : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}$.

Let $X = \mathcal{C}([0, T], \mathbb{R})$ be the set of all continuous functional on $[0, T]$ endowed with the complete δ-metric space. For $\theta_1 \in X$, we define the supremum norm as,

$$||\theta_1|| = sup_{u \in [0,T]}\{|\theta_1(u)|e^{-\kappa u}\},$$

where $\kappa > 0$, and $\theta_1 : [0, T] \to \mathbb{R}$ equipped with Bielecki's norm.

Now define $\delta : X \times X \to \mathbb{R}$ by

$$\begin{aligned}\delta(\theta_1, \theta_2) &= sup_{u \in [0,T]}\{|\theta_1(u) + \theta_2(u)|^2 e^{-\kappa u}\} \\ &= ||\theta_1 + \theta_2||\end{aligned}$$

for all $\theta_1, \theta_2 \in X = \mathcal{C}([0, T], \mathbb{R})$ and $\gamma : X \times X \to [1, \infty)$ by $\gamma(\theta_1, \theta_2) = |\theta_1(u) + \theta_2(u) + 1|$ with these scenario's, $(\mathcal{C}([0, T], \mathbb{R}), \delta)$ becomes an δ-metric space. In order to obtain our claims, we will need the following settings:

$(\mathcal{A}_1)$ The functions Λ, Θ are continuous.
$(\mathcal{A}_2)$ Define,

$$S\theta_1 u = \int_0^u Z_1(u, v, \theta_1(v))dv$$

$$T\theta_2 u = \int_0^u Z_2(u, v, \theta_2(v))dv$$

$(\mathcal{A}_3)$ Suppose there exists $\kappa > 0$,such that

$$|Z_1(u, v, \theta_1(v)) + Z_2(u, v, \theta_2(v))|^2 \leqslant \frac{\kappa \mathcal{A}(\theta_1, \theta_2)}{\kappa ||\mathcal{A}(\theta_1, \theta_2)|| + 1}$$

$\forall u, v \in [0, T], \theta_1, \theta_2 \in \mathcal{C}([0, T], \mathbb{R})$,

$$where, \ \mathcal{A}(\theta_1, \theta_2) = \max\left\{\frac{[1 + |\theta_1 + S\theta_1|^2]|\theta_2 + T\theta_2|^2}{1 + |\theta_1 + \theta_2|^2}, \frac{[|\theta_1 + S\theta_1|^2]|\theta_2 + T\theta_2|^2}{1 + |\theta_1 + \theta_2|^2}, |\theta_1 + \theta_2|^2\right\}$$

Now we prove the following theorem to ensure the existence of unique solution of Volterra type integral equation.

Theorem 5.2.1. *Let (X, δ) be an δ-metric space as notified above. If $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ are satisfied by S, T, then the integral Eq. (1) has a unique solution.*

$$\begin{aligned}|S\theta_1(u)+T\theta_2(u)|^2 &= |(\int_0^u Z_1(u,v,\theta_1(v))dv+\int_0^u Z_1(u,v,\theta_1(v))dv)|^2\\ &= |(\int_0^u Z_1(u,v,\theta_1(v))+\int_0^u Z_1(u,v,\theta_1(v))dv)dv|^2\\ &= \int_0^u(|Z_1(u,v,\theta_1(v))+Z_1(u,v,\theta_1(v))|)^2dv\\ &\leqslant \int_0^u \frac{\kappa\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)}{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}dv\\ &;\leqslant \int_0^u \frac{\kappa}{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)e^{-\kappa j}dv\\ &\leqslant \int_0^u \frac{\kappa}{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||e^{\kappa v}dv\\ &\leqslant \frac{\kappa}{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||\int_0^u e^{\kappa v}dv\\ &< \frac{\kappa}{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||\frac{e^{\kappa u}}{\kappa}\\ &< \frac{||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||}{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}e^{\kappa u}\end{aligned} \tag{58}$$

which implies,

$$|S\theta_1(u)+T\theta_2(u)|^2e^{-\kappa u}\leqslant\frac{||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||}{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}$$

$$\Rightarrow ||S\theta_1(u)+T\theta_2(u)||\leqslant\frac{||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||}{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}$$

$$\Rightarrow \frac{\kappa||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||+1}{||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||}\leqslant\frac{1}{||S\theta_1(u)+T\theta_2(u)||}$$

$$\Rightarrow \kappa+\frac{1}{||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||}\leqslant\frac{1}{||S\theta_1(u)+T\theta_2(u)||}$$

$$\Rightarrow \kappa-\frac{1}{||S\theta_1(u)+T\theta_2(u)||}\leqslant-\frac{1}{||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||}$$

Thus, we will get,

$$\kappa-\frac{1}{\delta(\theta_1,\theta_2)}\leqslant-\frac{1}{||\boldsymbol{\mathcal{A}}(\theta_1,\theta_2)||} \tag{59}$$

Define $F:\mathbb{R}\to\mathbb{R}$ *by* $F(p)=-\frac{1}{p};\quad p>0$*; then,*

$$\kappa+F(\delta(\theta_1,\theta_2))\leqslant F(\boldsymbol{\mathcal{A}}(\theta_1,\theta_2))$$

Hence all the conditions of the Theorem 4.2 are satisfied for $F(p)=-\frac{1}{p};\quad p>0$*. Hence Volterra integral equation for weak F-contractions have a common solution.*

7. Conclusion

Dislocated extended b-metric spaces are introduced and proved related fixed point theorems. We have conducted a numerical experiment for approximating the fixed point. Thereafter, we proposed simple and efficient solution for a Volterra integral equation by using the technique of fixed point in the setting of new abstract space: dislocated extended b-metric space. Many researchers have connected fixed point technique and classical Volterra integral equations in various abstract spaces such as metric space, b-metric space and partial metric space. We also follow same method in new abstract space: dislocated extended b-metric space. Our obtained applications are an extension and/or generalization of many existing classical Volterra integral equations in the literature. The observed results of this paper open new framework research avenues for:

- Fixed point method for Volterra-Fredholm integral equation in dislocated extended b-metric space
- Hyers-Ulam-Rassias stability of nonlinear integral equation in dislocated extended b-metric space
- Collocation-type method for Volterra-Hammerstein integral equations in dislocated extended b-metric space

Authors contributions

All authors read and approved the final manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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