Applied Mathematical Modelling 37 (2013) 6183–6190

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Variational iteration method for the Burgers' flow with fractional derivatives—New Lagrange multipliers

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ARTICLE INFO

Article history:
Received 24 May 2012
Received in revised form 12 November 2012
Accepted 18 December 2012
Available online 5 January 2013

Keywords:
Fractional differential equation
Caputo derivative
Variational iteration method

ABSTRACT

The flow through porous media can be better described by fractional models than the classical ones since they include inherently memory effects caused by obstacles in the structures. The variational iteration method was extended to find approximate solutions of fractional differential equations with the Caputo derivatives, but the Lagrange multipliers of the method were not identified explicitly. In this paper, the Lagrange multiplier is determined in a more accurate way and some new variational iteration formulae are presented.

1. Introduction

The diffusion process has been observed in many real physical systems such as highly ramified media in porous systems, anomalous diffusion in fractals and heat transfer close to equilibrium. Fractional calculus is a power tool for finding solution of non-linear problems. Some numerical methods [1–6] and analytical methods [7–12] have been developed for fractional differential equations (FDEs).

The variational iteration method (VIM) [13,14] was extended to FDEs and has been one of the methods used most often. Generally speaking, the use of the variational iteration method (VIM) follows the three steps: (a) to establish the correction functional; (b) identification of the Lagrange multipliers; (c) determination of the initial iteration. Obviously, the step (b) is crucial to derive a variational iteration formula.

Noting the applications of the VIM [13,15] only handled the term of fractional derivatives as restricted variations, we consider a more general FDE

$$ {}_0^C D_t^\alpha u + R[u] + N[u] = f(t), \tag{1} $$

where $0_{C\,t}^{D\,\alpha}$ is the Caputo derivative, $R[u]$ is a linear term and $N[u]$ is a nonlinear one. Momani and Inc et al. [16–20] applied the VIM to the above equation and suggested a variational iteration formula

$$ \begin{cases} u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left({}_0^C D_\tau^\alpha u_n + R[u_n] + N[u_n] - f(\tau)\right)d\tau, \quad 0 < t,\ 0 < \alpha \leqslant 1, \\ \lambda(t,\tau) = -1, \end{cases} \tag{2} $$

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0307-904X/$ - see front matter
http://dx.doi.org/10.1016/j.apm.2012.12.018

where the function $\lambda(t,\tau)$ is called the Lagrange multiplier.

This paper gives a new way to identify the Lagrange multiplier and improves the variational iteration formula (2) as

$$\begin{cases} u_{n+1} = u_n + \int_0^t \lambda(t,\tau)({}_0^C D_\tau^\alpha u_n + R[u_n] + N[u_n] - f(\tau))d\tau, \quad 0 < t,\ 0 < \alpha, \\ \lambda(t,\tau) = \frac{(-1)^\alpha (\tau - t)^{\alpha-1}}{\Gamma(\alpha)}. \end{cases} \tag{3}$$

The above iteration formula is also valid for differential equations when α is an arbitrary positive integer.

2. Preliminaries

Definition 2.1. The Caputo derivative is given as

$${}_0^C D_t^\alpha u = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha-m+1}} u^{(m)}(\tau)d\tau, \quad 0 < t, \quad m = [\alpha] + 1. \tag{4}$$

Definition 2.2. The Riemann–Liouville (R–L) integration of u is defined as

$${}_0I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)d\tau, \quad 0 < t, \quad 0 < \alpha. \tag{5}$$

Definition 2.3. Laplace transform of the term ${}_0^C D_t^\alpha u$ is given as

$$L[{}_0^C D_t^\alpha u] = s^\alpha \bar{u}(s) - \sum_{k=0}^{m-1} u^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1 < \alpha \leqslant m, \tag{6}$$

where L is Laplace transform and $\bar{u}(s) = L[u(t)]$.

Assuming $\bar{h}(s) = L[h(t)]$ and $\bar{g}(s) = L[g(t)]$, the convolution theorem is

$$h(t)^* g(t) = \int_0^t h(t-\tau) g(\tau) d\tau, \tag{7}$$

and

$$\bar{h}(s)\bar{g}(s) = L[h(t)^* g(t)]. \tag{8}$$

The detail properties of fractional calculus and Laplace transform can be found in [21–23], respectively.

3. Some new Lagrange multipliers

Theorem 3.1. *If the correction functional for* Eq. (1) is *established via the R–L integration*

$$u_{n+1} = u_n + {}_0I_t^\alpha \lambda(t,\tau)\left[{}_0^C D_\tau^\alpha u_n + R[u_n] + N[u_n] - f(\tau)\right], \tag{9}$$

the terms $R[u_n]$ and $N[u_n]$ are restricted variations, the Lagrange multiplier can be identified as

$$\lambda(t,\tau) = -1. \tag{10}$$

Proof. Take Laplace transform (6) on the both sides of Eq. (9)

$$\bar{u}_{n+1}(s) = \bar{u}_n(s) + L\left[{}_0I_t^\alpha \lambda(t,\tau)\left({}_0^C D_\tau^\alpha u_n + R[u_n] + N[u_n] - f(\tau)\right)\right]. \tag{11}$$

Consider the term

$${}_0I_t^\alpha \lambda_0{}^C D_\tau^\alpha u_n = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \lambda(t,\tau) {}_0^C D_\tau^\alpha u_n(\tau) d\tau. \tag{12}$$

Setting the Lagrange multiplier $\lambda(t,\tau) = \lambda(X)/_{X=t-\tau}$, Eq. (12) is the convolution of the function $a(t) = \frac{\lambda(t)t^{\alpha-1}}{\Gamma(\alpha)}$ and the term ${}_0{}^C D_t^\alpha u_n(t)$. The terms $R[u_n]$ and $N[u_n]$ are considered as restricted variations which implies $\delta R[u_n] = 0$ and $\delta N[u_n] = 0$, respectively. Make the correction functional (11) stationary with respect to $\bar{u}_n(s)$ and take the classical variation derivative δ on the both sides of Eq. (11). Then Eq. (11) can be calculated as

$$\delta\bar{u}_{n+1}(s) = \delta\bar{u}_n(s) + \delta\left[\bar{a}(s)s^{\alpha}\bar{u}_n(s) - \sum_{k=0}^{m-1}u^{(k)}(0^+)s^{\alpha-1-k}\right] = (1+\bar{a}(s)s^{\alpha})\delta\bar{u}_n(s). \quad (13)$$

From Eq. (13), we can obtain the equation

$$1+\bar{a}(s)s^{\alpha} = 0, \quad (14)$$

which results in

$$\bar{a}(s) = -\frac{1}{s^{\alpha}} \text{ and } a(t) = -\frac{t^{\alpha-1}}{\Gamma(\alpha)}. \quad (15)$$

□

As a result, the Lagrange multiplier can be identified as

$$\lambda(t,\tau) = a(t-\tau)\Gamma(\alpha)(t-\tau)^{1-\alpha} = -1. \quad (16)$$

The iteration formula (9) reads

$$u_{n+1} = u_n - {}_0I_t^{\alpha}\left[{}_0^CD_{\tau}^{\alpha}u_n + R[u_n] + N[u_n] - f(\tau)\right]. \quad (17)$$

We can check our iteration formula's validness through the relaxation oscillator equation [24]

$${}_0^CD_t^{\alpha}u + \omega^{\alpha}u = 0, \quad u(0)=1, \quad u'(0)=0, \quad 0<t, \quad 1<\alpha\leqslant 2, \omega>0. \quad (18)$$

The iteration formula of Eq. (18) can be given as

$$\begin{cases} u_{n+1} = u_n - {}_0I_t^{\alpha}({}_0^CD_{\tau}^{\alpha}u_n + \omega^{\alpha}u_n), \\ u_0 = 1. \end{cases} \quad (19)$$

As a result, we can obtain the series solution

$$u_0(t) = 1,$$
$$u_1(t) = 1 - \frac{\omega^{\alpha}t^{\alpha}}{\Gamma(1+\alpha)},$$
$$u_2(t) = 1 - \frac{\omega^{\alpha}t^{\alpha}}{\Gamma(1+\alpha)} + \frac{\omega^{2\alpha}t^{2\alpha}}{\Gamma(1+2\alpha)}.$$
$$\cdots$$

For $n\to\infty$, $u_n(t) = \sum_{k=0}^{n}\frac{(-\omega t)^{k\alpha}}{\Gamma(1+k\alpha)}$ rapidly tends to the exact solution $E_{\alpha}(-(\omega t)^{\alpha})$ which is the Mittag-Leffler function.

Theorem 3.2. *If the correction functional for* Eq. (1) is *established via the Riemann integration*

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left[{}_0^CD_{\tau}^{\alpha}u_n + R[u_n] + N[u_n] - f(\tau)\right]d\tau, \quad (20)$$

and the terms $R[u_n]$ and $N[u_n]$ are restricted variations, the Lagrange multiplier can be identified as

$$\lambda(t,\tau)) = \frac{(-1)^{\alpha}(\tau-t)^{\alpha-1}}{\Gamma(\alpha)}. \quad (21)$$

From Eq. (17), we can derive

$$u_{n+1} = u_n - {}_0I_t^{\alpha}\left[{}_0^CD_{\tau}^{\alpha}u_n + R[u_n] + N[u_n] - f(\tau)\right] = u_n - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[{}_0^CD_{\tau}^{\alpha}u_n + R[u_n] + N[u_n] - f(\tau)\right]d\tau$$
$$= u_n + \int_0^t \frac{(-1)^{\alpha}(\tau-t)^{\alpha-1}}{\Gamma(\alpha)}\left[{}_0^CD_{\tau}^{\alpha}u_n + R[u_n] + N[u_n] - f(\tau)\right]d\tau.$$

As a result, for Eq. (20), the Lagrange multiplier is derived

$$\lambda(t,\tau) = \frac{(-1)^{\alpha}(\tau-t)^{\alpha-1}}{\Gamma(\alpha)},$$

which completes the proof of (21).

One can check that the following iteration formula (See Eq. (25a) in [25]) for the ordinary differential equation (ODE) $\frac{d^mu}{dt^m} + R[u] + N[u] = f(t)$,

$$\begin{cases} u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left(\frac{d^m u}{d\tau^m} + R[u_n] + N[u_n] - f(\tau)\right)d\tau, \quad 0 < \alpha, \\ \lambda((t,\tau) = \frac{(-1)^m(\tau-t)^{m-1}}{(m-1)!}, \end{cases} \tag{22}$$

is a special case of Eq. (3). We can conclude that our iteration formula is a "uniform" one for both ODEs and FDEs.

Remarks:

(I) The iteration formula is also valid for approximately solving FDEs of arbitrary order in sense of the R–L derivative ${}_0^{RL}D_t^\alpha$ and the sequential derivatives. The differences are the initial iterations.

(II) The Lagrange multiplier presented in (3) is a simplest one. If we consider more terms in $R[u]$ in Eq. (1), more explicit Lagrange multipliers can be identified. For example, consider the FDE

$${}_0^C D_t^\alpha u + \omega^\alpha u + N[u] = f(t), 0 < \alpha \text{ and } {}_0^{RL}D_t^\alpha u + \omega^\alpha u + N[u] = f(t), 0 < \alpha. \tag{23}$$

Similarly, the following variational iteration formulae can be given as

$$\begin{cases} u_{n+1} = u_n + \int_0^t \lambda(t,\tau)({}_0{}^C D_\tau^\alpha u_n + \omega^\alpha u_n + N[u_n] - f(\tau))d\tau, \\ \lambda = -(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\omega^\alpha(t-\tau)^\alpha), \end{cases} \tag{24}$$

and

$$\begin{cases} u_{n+1} = u_n + \int_0^t \lambda(t,\tau)({}_0{}^{RL} D_\tau^\alpha u_n + \omega^\alpha u_n + N[u_n] - f(\tau))d\tau, \\ \lambda = -(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\omega^\alpha(t-\tau)^\alpha). \end{cases} \tag{25}$$

For $\alpha = 1$ and $\alpha = 2$, Eqs. (24) and (25) reduce to the results in (see the iteration formula in [25])

$$\begin{cases} u_{n+1} = u_n + \int_0^t \lambda(t,\tau)(\frac{du_n}{d\tau} + \omega u_n + N[u_n] - f(\tau))d\tau, \\ \lambda = -(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\omega^\alpha(t-\tau)^\alpha)\big|_{\alpha=1} = -e^{\omega(\tau-t)}, \end{cases}$$

and

$$\begin{cases} u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left(\frac{d^2u_n}{d\tau^2} + \omega^2 u_n + N[u_n] - f(\tau)\right)d\tau, \\ \lambda = -(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\omega^\alpha(t-\tau)^\alpha)\big|_{\alpha=2} = \frac{\sin(\omega(\tau-t))}{\omega}, \end{cases}$$

where $E_{\alpha,\beta}(t)$ is the Mittag-Leffler function with two parameters.

The Lagrange multipliers (24) and (25) lead to approximate solutions of higher accuracies than the results from the variational iteration formula (3). For example, only with one step, one can derive the exact solution of (18).

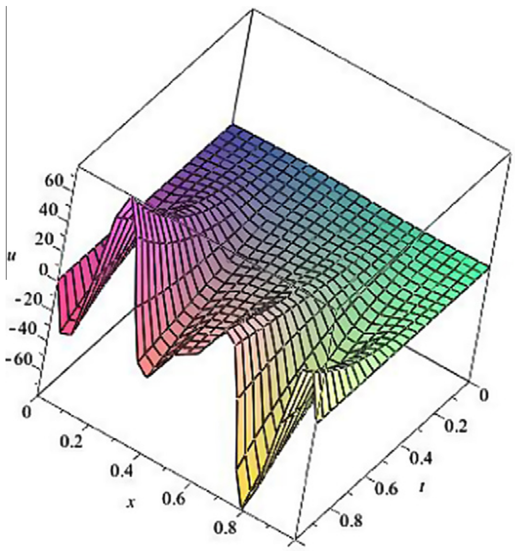

Fig. 1. $g = \sin(2\pi x)$, $\alpha = 0.9$, $\upsilon = 0.1$.

(III) The simplest variational iteration formula (17) can reduce to the Volterra integral equation and the analysis of the convergence and the existence of the solutions can be found in [26].

4. Approximate solutions of the Burgers' flow with fractional derivatives

The classical Burgers equation often appears in traffic flow and gas dynamics. Recently, some researchers considered various fractional Burgers equation to model the diffusion behaviors of the flow through porous medium [27–31]. In this section, the VIM is applied to the time-fractional Burgers equation [10]

$$ {}_0^C D_t^{\alpha} u + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}, \quad 0 < t,\ 0 \leqslant x \leqslant 1,\ 0 < \alpha \leqslant 1, \tag{26} $$

where u is the flow's velocity and v is the viscosity coefficient. It is revealed that the effect of the fractional derivative accumulates slowly to give rise to a significant dissipation.

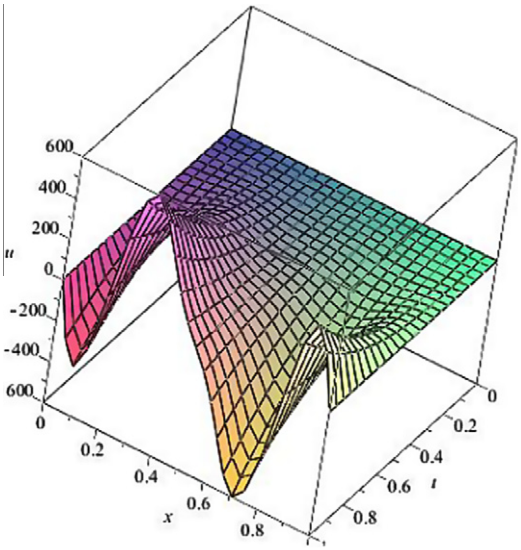

Fig. 2. $g = \sin(2\pi x)$, $\alpha = 0.9$, $v = 0.5$.

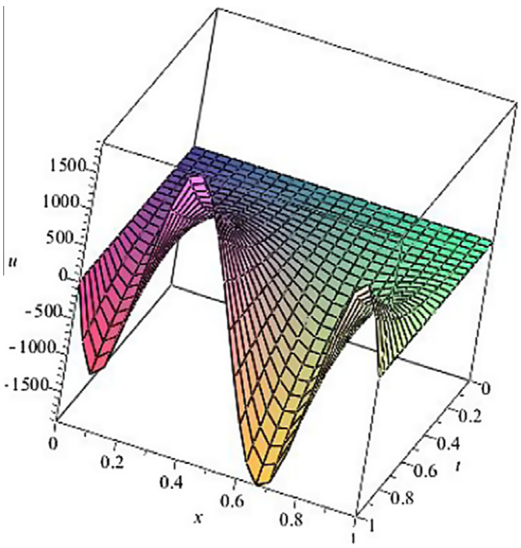

Fig. 3. $g = \sin(2\pi x)$, $\alpha = 0.9$, $v = 0.9$.

We consider the Burgers equations with the initial condition $u(x,0) = g(x)$. From Eq. (17), we can have

$$\begin{cases} u_{n+1} = u_n - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left({}_0^C D_\tau^\alpha u_n + u_n \frac{\partial u_n}{\partial x} - v \frac{\partial^2 u_n}{\partial x^2} \right) d\tau, \quad 0 < \alpha \leqslant 1, \\ u_0 = u(x,0) = g(x), \quad u(0,t) = u(1,t) = 0. \end{cases} \tag{27}$$

More generally, for the time-fractional couple Burgers equations of fractional order in [32]

$$\begin{cases} {}_0^C D_t^\alpha u + 2u \frac{\partial u}{\partial x} - \frac{\partial^2 (uv)}{\partial x^2} = 0, 0 < \alpha, \\ {}_0^C D_t^\beta v + 2v \frac{\partial v}{\partial x} - \frac{\partial^2 (uv)}{\partial x^2} = 0, 0 < \beta, \end{cases} \tag{28}$$

the variational iteration formula can be given as

$$\begin{cases} u_{n+1} = u_n + \int_0^t \lambda_1(t,\tau) \left({}_0^C D_\tau^\alpha u_n + 2u_n \frac{\partial u_n}{\partial x} - \frac{\partial^2 u_n v_n}{\partial x^2} \right) d\tau, \quad \lambda_1(t,\tau) = \frac{(-1)^\alpha (\tau-t)^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha, \\ v_{n+1} = v_n + \int_0^t \lambda_2(t,\tau) \left({}_0^C D_\tau^\beta v_n + 2v_n \frac{\partial v_n}{\partial x} - \frac{\partial^2 u_n v_n}{\partial x^2} \right) d\tau, \quad \lambda_2(t,\tau) = \frac{(-1)^\beta (\tau-t)^{\beta-1}}{\Gamma(\beta)}, \quad 0 < \beta, \end{cases} \tag{29}$$

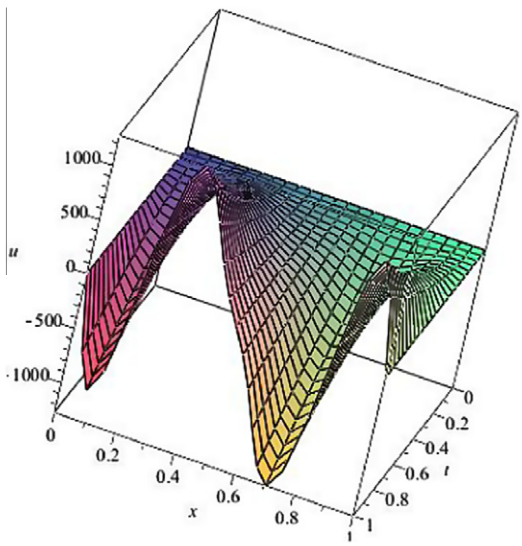

Fig. 4. $g = \sin(2\pi x)$, $\alpha = 0.5$, $v = 0.5$.

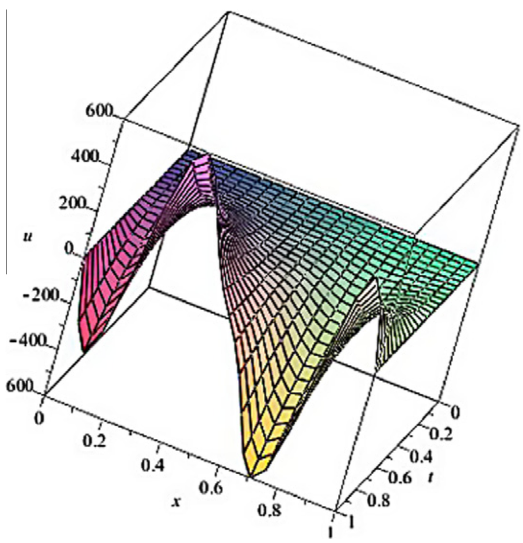

Fig. 5. $g = \sin(2\pi x)$, $\alpha = 0.9$, $v = 0.5$.

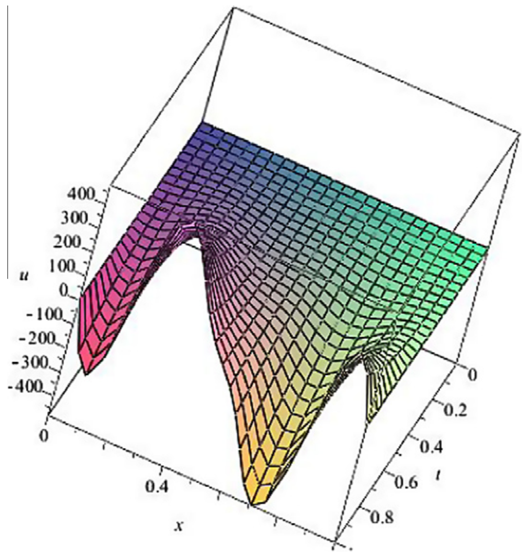

Fig. 6. $g = \sin(2\pi x)$, $\alpha = 0.99$, $v = 0.5$.

For Eq. (26), the successive approximate solutions can be obtained

$$\begin{aligned}
u_0 &= g(x) = g,\\
u_1 &= g - \left(gg' - vg^{(2)}\right)\frac{t^\alpha}{\Gamma(\alpha+1)},\\
u_2 &= g - \left(gg' - vg^{(2)}\right)\frac{t^\alpha}{\Gamma(1+\alpha)} + \left(2gg'^2 + g^2g^{(2)} - 2vgg^{(3)} - 4vg'g^{(2)} + v^2g^{(4)}\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\\
&\quad -\left(gg' - vg^{(2)}\right)\left(g'^2 + gg^{(2)} - vg^{(3)}\right)\frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\vdots
\end{aligned}$$

where g' denotes $\frac{dg}{dx}$ and $g^{(m)} = \frac{d^mg}{dx^m}$.

We get the approximate solution u_2 as the second term approximation. For $g(x) = \sin(2\pi x)$ and the fractional order $\alpha = 0.9$, Figs 1–3 show the velocity of the flow with various viscosity coefficients v. For $g(x) = \sin(2\pi x)$ and the viscosity coefficient $v = 0.5$, Figs 4–6 illustrate the velocity at different fractional orders.

5. Conclusions

FDEs have been proven to be a useful tool to describe the nonlocal diffusion of the flow in porous media. As one of the analytical methods in FDEs, the existing applications of the VIM in FDEs handled the terms of fractional derivatives as restricted variations or directly employed the one for ODEs. So the Lagrange multipliers determined in that way are not good enough to obtain the approximate solutions of high accuracies. The main reason is that it is difficult for one to use the integration by parts to derive the Lagrange multipliers explicitly.

In this study, the Lagrange multiplier is identified by Laplace transform and such a drawback is overcomed. The VIM for FDEs is completed now.

Furthermore, with the result in this study, the following aspects can be considered in future work:

- New analytical methods employing other linearized techniques, for example, the Adomian decomposition series, which can handle the nonlinear terms of FDEs and improve the accuracies of approximate solutions;
- To develop numerical algorithms of fractional partial differential equations based on the VIM which can fully use the merits of the method;
- To consider other applications of the VIM in new non-classical models such as fractional fuzzy equations, fractional time-delay models, the fractional q-difference equations and other dynamical equations on time scales.

Acknowledgment

The first author would like to feel grateful to the referees' suggestions. This work is financially supported by the Key Program of NSFC (No. 51134018).

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