Cent. Eur. J. Phys. • 11(10) • 2013 • 1482-1486
DOI: 10.2478/s11534-013-0203-7

VERSITA

Central European Journal of **Physics**

Fractional sub-equation method for the fractional generalized reaction Duffing model and nonlinear fractional Sharma-Tasso-Olver equation

Research Article

Hossein Jafari[1,*], Haleh Tajadodi[1], Dumitru Baleanu[2,3,4], Abdulrahim A. Al-Zahrani[3], Yahia A. Alhamed[3], Adnan H. Zahid[3]

1 *Department of Mathematics, University of Mazandaran, P.O. Box 47416-95797, Babolsar, Iran*

2 *Çankaya University, Faculty of Art and Sciences, Department of Mathematics and Computer Sciences, Ankara, Turkey*

3 *Department of Chemical and Materials Engineering, Faculty of Engineering, King Abdulaziz University, P.O. Box: 80204, Jeddah, 21589, Saudi Arabia*

4 *Institute of Space Sciences, P.O. Box, MG-23, R 76900, Magurele-Bucharest, Romania*

Received 8 January 2013; accepted 9 March 2013

Abstract: In this paper the fractional sub-equation method is used to construct exact solutions of the fractional generalized reaction Duffing model and nonlinear fractional Sharma-Tasso-Olver equation.The fractional derivative is described in the Jumarie's modified Riemann-Liouville sense. Two illustrative examples are given, showing the accuracy and convenience of the method.

PACS (2008): 45.10.-b, 45.10.Hi

Keywords: fractional sub-equation method • fractional partial differential equation • fractional generalized reaction Duffing model • nonlinear fractional Sharma-Tasso-Olver equation

1. Introduction

Fractional equations, both partial and ordinary ones, have been applied in modeling of many physical, engineering, chemistry, biology, etc in recent years [1–4]. Investigation of the exact travelling wave solutions for nonlinear partial differential equations (PDEs) plays an important role in the study of nonlinear physical phenomena.

Finding exact solutions of most of the fractional PDEs is not easy, so searching and constructing exact solutions for nonlinear fractional partial differential equations is a continuing investigation. Many powerful methods for obtaining exact solutions of nonlinear fractional PDEs have been presented such as, Hirota's bilinear method [5], Bäcklund transformation [6], sine-cosine method [7], tanh-function method [8], Adomian decomposition method [9, 10], vari-

*E-mail: jafari@umz.ac.ir

ational iteration method [11, 12], homotopy perturbation method [13], homotopy analysis method [14], Laplace iterative method [15] and so on.
Recently, the Fan sub-equation method to obtain exact solution of nonlinear partial equations has attracted much attention [16–20]. Moreover there are more wave solutions of nonlinear fractional PDEs that satisfy a general Riccati equation that can be represented as a polynomial. This method is based on the Jumarie's modified Riemann-Liouville derivative [21, 22], on the homogeneous balance principle [17] and on the symbolic computation in order to obtain analytical solutions of fractional PDEs.
In the present manuscript a method is suggested to find exact analytical solutions of some type of nonlinear fractional PDEs with the Jumarie's modified Riemann-Liouville derivative of order α. Here we summarize some useful formulas of Jumarie's modified Riemann-Liouville [21]

$$D_x^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \quad \gamma > 0, \tag{1}$$

$$D_x^\alpha [f(x)g(x)] = g(x) D_x^\alpha f(x) + f(x) D_x^\alpha g(x), \tag{2}$$

$$D_x^\alpha f[g(x)] = f'_g[g(x)] D_x^\alpha g(x) = D_g^\alpha f[g(x)](g'_x)^\alpha, \tag{3}$$

which will be used in the following sections.
In this paper, we applied the fractional sub-equation method [16, 23, 24] to obtain the exact solutions of the fractional generalized reaction Duffing model in the form

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + p \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + qu + ru^2 + su^3 = 0. \tag{4}$$

where p, q, r and s are all constants. Eq. (4) reductions many well-known nonlinear fractional wave equations such as
(i) Fractional Klein-Gordon equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \alpha u + \beta u^3 = 0.$$

(ii) Fractional Landau-Ginzburg-Higgs equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} - m^2 u + g^2 u^3 = 0.$$

(iii) Fractional φ^4 equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + u - u^3 = 0.$$

(iv) Fractional Duffing equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + bu + cu^3 = 0.$$

(v) Fractional Sine-Gordon equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + u - \frac{1}{6} u^3 = 0.$$

We also consider fractional Sharma-Tasso-Olver equation in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \beta \frac{\partial^\alpha u^3}{\partial x^\alpha} + \frac{3}{2} \beta \frac{\partial^{2\alpha} u^2}{\partial x^{2\alpha}} + \beta \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} = 0. \tag{5}$$

where β is a real constant. Eq. (5) is a double nonlinear dispersive, integrable equation that be called Sharma-Tasso-Olver equation [26]. Recently many physicists have studied equation (5) in [27–29]. In [30] the solitons solutions of Sharma-Tasso-Olver equation are obtained by the tanh method, the extended tanh method.
The rest of this paper is organized as follows. In Section 2, we describe the algorithm for solving exact solutions to nonlinear fractional partial differential equations. In Section 3, consists of a brief conclusion.

2. Exact solutions of some nonlinear fractional PDEs

In this section we will derive the fractional generalized reaction Duffing model

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + p \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + qu + ru^2 + su^3 = 0. \tag{6}$$

Several families of exact solutions including bell-shaped waves and shock waves of Eq. (6) when $\alpha = 1$ have been reported in [25, 31].
To solve Eq. (6) by using fractional sub-equation method, with the assistance of the traveling wave transformation,

$$u(x, t) = u(\xi), \qquad \xi = kx + ct, \tag{7}$$

where ξ is a wave variable and c and k are a wave speed, we reduce nonlinear fractional partial differential equations into nonlinear fractional differential equation (FDE). Substituting Eq. (7) into Eq. (6) yields the following fractional differential equation for $u(\xi)$,

$$c^{2\alpha} D_\xi^{2\alpha} u + pk^{2\alpha} D_\xi^\alpha u + qu(\xi) + ru^2(\xi) + su^3(\xi) = 0. \tag{8}$$

Now we suppose that the Eq. (8) has a solution in the form

$$u(\xi) = \sum_{i=0}^{n} a_i \varphi^i(\xi), \tag{9}$$

where a_i are constants to be determined later and the new variable $\varphi = \varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha \varphi = \sigma + \varphi^2, \qquad 0 < \alpha \leq 1. \tag{10}$$

Substituting Eq. (9) along with Eq. (10) into Eq. (8) and balancing the highest order derivative term $D_\xi^{2\alpha}$ with non-linear term u^3 in Eq. (8) gives $n = 1$, from which we have

$$u(\xi) = a_0 + a_1 \varphi(\xi). \tag{11}$$

Then setting the coefficients of $\varphi^j (j = 0, 1, \dots, 3)$ to zero, we finally obtain a system of algebraic equations

$$\begin{aligned}
&a_0^2 r + a_0^3 s + a_0 q = 0,\\
&2a_1\sigma c^{2\alpha} + 2a_1 p\sigma k^{2\alpha} + 2a_0 a_1 r + 3a_0^2 a_1 s + a_1 q = 0,\\
&a_1^2 r + 3a_0 a_1^2 s = 0,\\
&2a_1 c^{2\alpha} + 2a_1 p k^{2\alpha} + a_1^3 s = 0.
\end{aligned}$$

Solving the set of algebraic equations yields

$$a_0 = -\frac{3q}{2r}, \quad a_1 = \pm\frac{i a_0}{\sqrt{\sigma}}, \quad \sigma = \frac{q}{4(c^{2\alpha} + k^{2\alpha} p)}, \quad r = \pm\frac{3\sqrt{sq}}{\sqrt{2}}, \tag{12}$$

by using five solutions of fractional Riccati equation (10) derived with Zhang et al. [18]

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma}\ \tanh_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0,\\ -\sqrt{-\sigma}\ \coth_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0,\\ \sqrt{\sigma}\ \tan_\alpha(\sqrt{\sigma}\xi), & \sigma > 0,\\ -\sqrt{\sigma}\ \cot_\alpha(\sqrt{\sigma}\xi), & \sigma > 0,\\ -\dfrac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, \quad \omega = \text{const.}, & \sigma = 0, \end{cases} \tag{13}$$

with the generalized hyperbolic and trigonometric functions and (7)-(12) exact solutions of Eq. (6), namely, generalized hyperbolic function solutions and generalized trigonometric function solutions as follows

$$\begin{aligned}
u_1(x,t) &= -\frac{3q}{2r} \pm \frac{i3q}{2r\sqrt{\sigma}}(\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}(kx+ct))), \quad \sigma < 0,\\
u_2(x,t) &= -\frac{3q}{2r} \pm \frac{i3q}{2r\sqrt{\sigma}}(\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}(kx+ct))), \quad \sigma < 0,\\
u_3(x,t) &= -\frac{3q}{2r} \pm \frac{i3q}{2r\sqrt{\sigma}}(\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}(kx+ct))), \quad \sigma > 0,\\
u_4(x,t) &= -\frac{3q}{2r} \pm \frac{i3q}{2r\sqrt{\sigma}}(\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}(kx+ct))), \quad \sigma > 0.
\end{aligned}$$

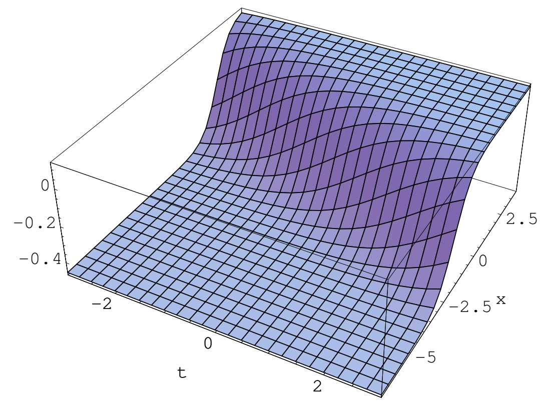

Figure 1. Exact solutions $u_1(x,t)$ for Eq. (6) with $\alpha = 1, p = -1, q = \frac{1}{9}, r = 1, s = -2, c = 0.7, k = 1; \sigma = -1$.

In Figure 1 $u_1(x,t)$ shows exact solutions of Eq. (6) with $\alpha = 1, p = -1, q = \frac{1}{9}, r = 1, s = -2, c = 0.7, k = 1; \sigma = -1$. When $\alpha = 1$ these solutions give the solutions of standard form of the generalized reaction Duffing model [25, 31].

The next step is to consider the nonlinear fractional Sharma-Tasso-Olver equation in the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \beta\frac{\partial^\alpha u^3}{\partial x^\alpha} + \frac{3}{2}\beta\frac{\partial^{2\alpha} u^2}{\partial x^{2\alpha}} + \beta\frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} = 0. \tag{14}$$

To solve Eq. (14), we consider the following traveling wave transformation

$$u = u(\xi), \quad \xi = kx + ct, \tag{15}$$

then Eq. (14) can be reduced to the following nonlinear fractional differential equation (FDE), namely

$$c^\alpha D_\xi^\alpha u + 3\beta k^\alpha u^2 D_\xi^\alpha u + 3\beta((k^\alpha D_\xi^\alpha u)^2 + k^{2\alpha} u D_\xi^\alpha u) + \beta k^{3\alpha} D_\xi^{3\alpha} u = 0. \tag{16}$$

Now we do the same process, like the previous example. We obtain following system of algebraic equations:

$$\begin{aligned}
&a_1\sigma c^\alpha + 3a_1^2\beta\sigma^2 k^{2\alpha} + 2a_1\beta\sigma^2 k^{3\alpha} + 3a_0^2 a_1\beta\sigma k^\alpha = 0,\\
&6a_0 a_1\beta\sigma k^{2\alpha} + 6a_0 a_1^2\beta\sigma k^\alpha = 0,\\
&a_1 c^\alpha + 12a_1^2\beta\sigma k^{2\alpha} + 8a_1\beta\sigma k^{3\alpha} + 3a_1^3\beta\sigma k^\alpha + 3a_0^2 a_1\beta k^\alpha = 0,\\
&6a_0 a_1\beta k^{2\alpha} + 6a_0 a_1^2\beta k^\alpha = 0,\\
&9a_1^2\beta k^{2\alpha} + 6a_1\beta k^{3\alpha} + 3a_1^3\beta k^\alpha = 0.
\end{aligned} \tag{17}$$

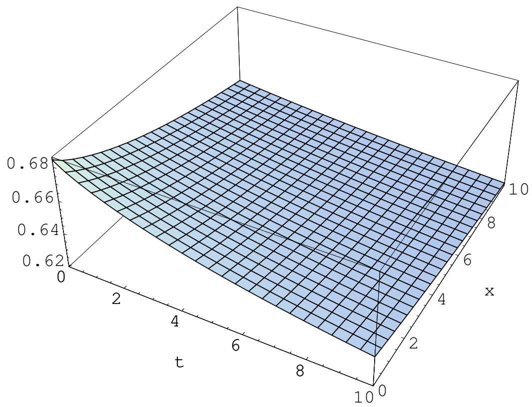

Figure 2. Exact solutions of Eq. (14) ($u_5(x,t)$) for $\alpha = 0.9, \beta = -1, \omega = 10, c = 1, k = 1$.

Solving (17) with the aid of *Mathematica*, we have

$$\begin{cases} a_0 = \pm\dfrac{\sqrt{\beta\sigma k^{3\alpha} - c^\alpha}}{\sqrt{3\beta k^\alpha}}, \qquad a_1 = -k^\alpha, \\ a_0 = \pm\dfrac{i\sqrt{c^\alpha}}{\sqrt{3\beta k^\alpha}}, \qquad a_1 = -k^\alpha, \quad \sigma = 0, \end{cases} \tag{18}$$

Finally, from Eqs. (13), (15)-(18) we obtain the following generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solution of Eq. (14)

$$u_1(x,t) = \pm\frac{\sqrt{\beta\sigma k^{3\alpha} - c^\alpha}}{\sqrt{3}\sqrt{\beta}\sqrt{k^\alpha}} + k^\alpha(\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}(kx+ct)),$$
$$\sigma < 0,$$

$$u_2(x,t) = \pm\frac{\sqrt{\beta\sigma k^{3\alpha} - c^\alpha}}{\sqrt{3}\sqrt{\beta}\sqrt{k^\alpha}} + k^\alpha(\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}(kx+ct)),$$
$$\sigma < 0,$$

$$u_3(x,t) = \pm\frac{\sqrt{\beta\sigma k^{3\alpha} - c^\alpha}}{\sqrt{3}\sqrt{\beta}\sqrt{k^\alpha}} - k^\alpha(\sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}(kx+ct)),$$
$$\sigma > 0,$$

$$u_4(x,t) = \pm\frac{\sqrt{\beta\sigma k^{3\alpha} - c^\alpha}}{\sqrt{3}\sqrt{\beta}\sqrt{k^\alpha}} + k^\alpha(\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}(kx+ct)),$$
$$\sigma > 0,$$

$$u_5(x,t) = \pm\frac{i\sqrt{c^\alpha}}{\sqrt{3}\sqrt{\beta}\sqrt{k^\alpha}} + \frac{k^\alpha\Gamma(1+\alpha)}{(kx+ct)^\alpha + \omega},$$
$$\omega = \text{const}, \qquad \sigma = 0.$$

In Figure 2 $u_5(x,t)$ shows one of exact solutions of Eq. (14) for $\alpha = 0.9, \beta = -1, \omega = 10, c = 1, k = 1$. When $\alpha = 1$, these obtained exact solutions are same with those given solutions in [26].

3. Conclusion

In this paper, based on a fractional sub-equation method, we obtained the exact solutions of the generalized fractional reaction Duffing equation and the nonlinear fractional Sharma-Tasso-Olver equation. The results show that the fractional sub-equation method is accurate and effective. These solutions may be useful for describing certain nonlinear physical phenomena. This method can be applied to other nonlinear fractional PDEs in mathematical physics that are worth studying.
Mathematica has been used for computations and programming in this paper.

Acknowledgments

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under grant No. Gr/34/11. The last four authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] I. Podlubny, Fractional Differential Equation, (San Diego, Academic Press, 1999)
[2] S.G. Samko, A. A. Kilbas, O. Igorevich, Fractional Integrals and Derivatives, Theory and Applications, (Yverdon, Gordon and Breach, 1993)
[3] A.A.Kilbas, H.M.Srivastava, J.J.Trujillo, Theory and Applications of Fractional Differential Equations, (Amsterdam, Elsevier Science, 2006)
[4] D. Băleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos, (Boston, World Scientific, 2012)
[5] R. Hirota , Phys. Rev. Lett. 27, 1192 (1971)
[6] M.R. Miurs, (Springer, Berlin, 1978)
[7] C.T. Yan, Phys. Lett. A 224, 77 (1996)
[8] Z.S. Lü, H.Q. Zhang, Commun. Theor. Phys. 39, 405 (2003)
[9] V. Daftardar-Gejji, H. Jafari, J. Math. Anal. Appl. 301, 508 (2005)
[10] H. Jafari, V. Daftardar-Gejji, Appl. Math. Comput. 181, 598 (2006)
[11] H. Jafari, H. Tajadodi, International Journal of Differential Equations 2010, 764738 (2010)
[12] H. Jafari, A. Kadem, D. Baleanu, T. Yilmaz, Rom. Rep. Phys. 64, 337 (2012)
[13] H. Jafari, N. Kadkhoda, H. Tajadodi, S. A. Hosseini Matikolai, Int. J. Nonlin. Sci. Num. 11, 271 (2010)

[14] H. Jafari, S. Das, H. Tajadodi, Journal of King Saud University Science 23, 151 (2011)

[15] H. Jafari, M. Nazari, D. Baleanu, C.M. Khalique, Comput. Math. Appl. [In Press] (2012)

[16] E.G. Fan, Y. Hon, Chaos Soliton. Fract. 15, 599 (2003)

[17] M. Wang, Phys. Lett. A 199, 169 (1995)

[18] S. Zhang, Q.A. Zong, D. Liu, Q. Gao, Commun. Fract. Calc. 1, 48 (2010)

[19] Y. Zhou, M. Wang, Y. Wang, Phys. Lett. A 308, 31 (2003)

[20] H. Jafari, M. Ghorbani, C.M. Khalique, Abstr. Appl. Anal. 2012, 962789 (2012).

[21] G. Jumarie, Comput. Math. Appl. 51, 1367 (2006)

[22] G. Jumarie, Appl. Math. Lett. 23, 1444 (2010)

[23] Y.C. Hon, E.G. Fan, Chaos Soliton. Fract. 19, 515 (2004)

[24] S. Zhang, H. Zhang, Phys. Lett. A 375, 1069 (2011)

[25] B. Tian, Y.T. Gao, Z. Naturforsch. A 57, 39 (2002)

[26] Elsayed M.E. Zayed, Appl. Math. Comput. 218, 3962 (2011)

[27] Z. Lian, S.Y. Lou, Nonlinear Anal. 63, 1167 (2005)

[28] Z. Yan, Chaos, MM Res. 22, 302 (2003)

[29] S. Wang, X. Tang, S.Y. Lou, Chaos Soliton. Fract. 21, 231 (2004)

[30] A.M. Wazwaz, Appl. Math. Comput. 188, 1205 (2007)

[31] J.J. Kim, W.P. Hong, Z. Naturforsch. 59a, 721 (2004)