Cent. Eur. J. Phys. • 11(10) • 2013 • 1523-1527
DOI: 10.2478/s11534-013-0209-1

VERSITA

Central European Journal of **Physics**

Homotopy analysis method for solving Abel differential equation of fractional order

Short Communication

Hossein Jafari[1]*, Khosro Sayevand[2]†, Haleh Tajadodi[1]‡, Dumitru Baleanu[3,4,5]§

1 *Department of Mathematics, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran*

2 *Department of Mathematics, Faculty of Basic Sciences, University of Malayer, P.O. Box 65719-95863, Malayer, Iran*

3 *Çankaya University, Faculty of Art and Sciences, Department of Mathematics and Computer Sciences, Ankara, Turkey*

4 *Department of Chemical and Materials Engineering, Faculty of Engineering, King Abdulaziz University, P.O. Box 80204, Jeddah, 21589, Saudi Arabia*

5 *Institute of Space Sciences, MG-23, R 76900, Magurele-Bucharest, Romania*

Received 03 February 2013; accepted 12 March 2013

Abstract: In this study, the homotopy analysis method is used for solving the Abel differential equation with fractional order within the Caputo sense. Stabilityand convergence of the proposed approach is investigated. The numerical results demonstrate that the homotopy analysis method is accurate and readily implemented.

PACS (2008): 02.30.Mv, 04.20.Ex

Keywords: Abel differential equation • fractional derivative • homotopy analysis method

1. Introduction

Liao proposed the homotopy analysis method (HAM) in 1992, [1] and since then it has been used to obtain the analytical, and approximate analytical, solutions of many types ofnonlinear equations and systems of equations. It has also been applied to problems in engineering and science (see for example Refs. [1–7] and the references therein). With this method , we use a certain auxiliary parameter $\hbar$ to control and adjust the rate of convergence and the convergence region of the series solution. The valid regions of $\hbar$ are obtained by using an $\hbar$-curve. The fractional calculus has been used extensively in basic sciences and engineering (see for example Refs. [8–22] and the references therein). A recent application has included numerically determining solutions for various classes of nonlinear fractional differential equations.Many engineering and physical problems have been modelled using frac-

*E-mail: jafari@umz.ac.ir (Corresponding author)
†E-mail: ksayehvand@iust.ac.ir
‡E-mail: tajadodi@umz.ac.ir
§E-mail: dumitru@cankaya.edu.tr

tional differential equations (FDEs) [8–16, 23, 24]. Finding accurate and efficient methods for solving FDEs has been an active research undertaking. Nonlinear FDEs are difficult to solve, especially analytically. Such solutions tong most nonlinear FDEs cannot be found easily, thus approximate analytical and numerical methods must be used. In [6, 13, 15, 21] some numerical methods for solving FDEs were presented.

Different types of definitions of fractional calculus can be found. The Riemann-Liouville integraloperator of order α is defined as [10]

$$(I^{\alpha}f)(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt & \alpha > 0, \quad x > 0, \\ f(x) & \alpha = 0. \end{cases} \tag{1}$$

and the Riemann-Liouville fractional derivative of order α ($\alpha \geq 0$) is also used:

$$\left(\mathcal{D}^{\alpha}_{L-R}f\right)(x) = \left(\frac{d}{dx}\right)^m \left(I^{m-\alpha}f\right)(x), \quad \alpha > 0, \quad m-1 < \alpha \leq m, \quad m \in N. \tag{2}$$

A modification of the Riemann-Liouvill definition, the Caputo fractional derivative [10] is defined as

$$D^{\alpha}f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}dt & (\alpha > 0,\ m-1 < \alpha < m,) \\ \frac{\partial^m f(x)}{\partial x^m}, & \alpha = m \end{cases} \tag{3}$$

where m is an integer. Let $f \in C^m$ and $m-1 < \alpha \leq m$ then

$$(I^{\alpha}D^{\alpha}f)(x) = f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(0^+)}{k!}x^k, \qquad x \geq 0. \tag{4}$$

In this work, the fractional derivative is considered in the Caputo sense because of its applicability to real–worldproblems.

In this paper, we employ the homotopy analysis method (HAM) [1] for solving an Abel differential equation with fractional order:

$$D^{\alpha}y(x) = a(x)y^3(x) + b(x)y^2(x) + c(x)y(x) + d(x), \quad 0 < \alpha < 1, \tag{5}$$

where $a(x) \neq 0$, $b(x)$, $c(x)$ and $d(x)$ are meromorphic functions.

This equation has a long history in many areas of pure mathematics and applied mathematics [19, 22]. For solving the following type of nonlinear FDE ,

$$N[y(x)] = 0, \tag{6}$$

using the HAM [1], we firstly construct the zero-order deformation equation as

$$(1-q)L[\phi(x;q) - y_0(x)] = q\,\hbar H(x)N[\phi(x;q)], \tag{7}$$

where $q \in [0,1]$ is the embedding parameter, $\phi(x;q)$ is a mapping of $u(x)$, $\hbar \neq 0$ is an auxiliary parameter, $H(x) \neq 0$ is auxiliary function,$L = D^{\alpha}$ is an auxiliary linear operator,$y_0(x)$ is an initial guess of $y(x)$, and $\phi(x;q)$ is a unknown function, respectively.In the HAM, we assume that the solution can be written as

$$\phi(x;q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)q^m, \tag{8}$$

Substituting (8) in (7) and differentiating it m times with respect to q, setting $q = 0$, we will have the m-th order deformation equation as follows:

$$L[y_m(x) - \chi_m y_{m-1}(x)] = \frac{\hbar H(x)}{(m-1)!}\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0}, \tag{9}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{10}$$

In view of (4), applying the Riemann-Liouville integral operator I^{α} to both sides of Eq. (9), leads to:

$$y_m(x) = \chi_m y_{m-1}(x) - \chi_m \sum_{i=0}^{n-1} y^{(i)}_{m-1}(0^+)\frac{t^i}{i!} + \hbar I^{\alpha}[H(x)\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0}]. \tag{11}$$

It is easily to obtain $u_m(x)$ for $m \geq 1$ using the above recurrent formula.

2. Compact structures of Abel equations with time-fractional derivatives

Hereunder, to demonstrate the efficiency of our scheme, we will implement the homotopy analysis method to construct solutions for compact structures of fractional Abel equations.

Example1 Consider the nonlinear fractional Abel differential equation of the first kind:

$$D_x^\alpha y(x) = y^3(x)\sin x - xy^2(x) + x^2 y(x) - x^3, \quad x \in (0,1], \tag{12}$$

subject to the initial condition $y_0(x) = 0$. For solving Eq. (12) using the homotopy analysis method, with the given initial condition, it is naturalto choose

$$y(0) = 0. \tag{13}$$

We choose the linear operator

$$L[\phi(x;q)] = D_x^\alpha[\phi(x;q)] \tag{14}$$

with the property $L[c] = 0$, where c represents a constant.Now, we define a nonlinear operator as follows

$$N[\phi(x;q)] = D_x^\alpha \phi(x;q) - \phi^3(x;q)\sin x + x\phi^2(x;q) - x^2\phi(x;q) + x^3.$$

We construct the zeroth-order deformation equation assuming that $H(x) = 1$

$$(1-q)L[\phi(x;q) - y_0(x)] = q\hbar N[\phi(x;q)]. \tag{15}$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi(x;0) = y_0(x), \quad \phi(x;1) = y(x). \tag{16}$$

Therefore, the m-th order deformation equation is obtained as follow:

$$L[y_m(x) - \chi_m y_{m-1}(x)] = \hbar R_m(\overrightarrow{y}_{m-1}), \tag{17}$$

keeping in mind that

$$R_m(\overrightarrow{y}_{m-1}) = D_x^\alpha y_{m-1}(x) - \sum_{i=0}^{m-1} y_i \sum_{k=0}^{m-1-i} y_k y_{m-1-i-k}\sin x + x\sum_{i=0}^{m-i} y_i y_{m-i} - x^2 y_{m-1} + (1-\chi_m)(x^3). \tag{18}$$

Now applying I^α to both side of (17) leads to

$$y_m(x) = (\chi_m + \hbar)y_{m-1}(x) - (\chi_m + \hbar)y_{m-1}(0) + \hbar I^\alpha\left[-\sum_{i=0}^{m-1} y_i \sum_{k=0}^{m-1-i} y_k y_{m-1-i-k}\sin x + x\sum_{i=0}^{m-i} y_i y_{m-i} + x^2 y_{m-1} + x^3\right].$$

Finally, we assume that

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \tag{19}$$

By using (13) and (2), we obtain

$$\begin{aligned} y_0(x) &= \frac{6\hbar x^{\alpha+3}}{\Gamma(\alpha+4)} \\ y_1(x) &= \frac{6\hbar(\hbar+1)x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{6\hbar^2\Gamma(\alpha+6)x^{2\alpha+5}}{\Gamma(\alpha+4)\Gamma(2\alpha+6)} \\ &\vdots \end{aligned}$$

Hence

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \ldots \tag{20}$$

Figure 1 shows the $\hbar$-curve, and in Figure 2, we show the exact solution together with the solution obtained after 4 iterations. $\hbar = -.7, \alpha = .98$

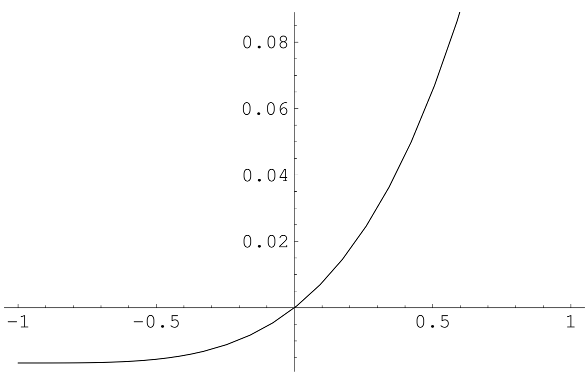

Figure 1. The $\hbar$-curve of $y(0.5)$ based on the fourth-order HAM approximations.

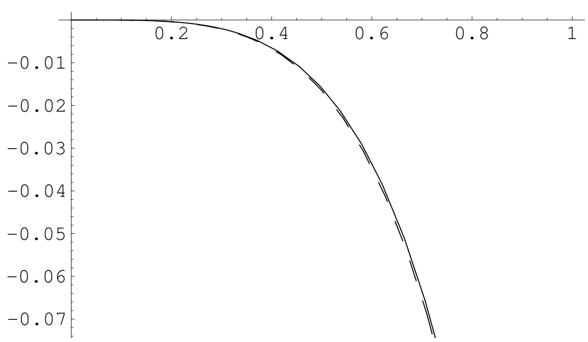

Figure 2. Numerical convergence of the exact solution and the HAM solution.

Example2 Consider a nonlinear fractional Abel differential equation of the first kind:

$$D_x^\alpha y(x) = x^2y^3(x)+xy^2(x)+\sqrt{x}y(x)+tanx, \qquad x\in(0,1], \tag{21}$$

subject to the initial condition $y_0(x)=0$.To use the homotopy analysis method to solve Eq.(21), we choose the initial linear operator as follows:

$$y(0)=0, \qquad L[\phi(x;q)] = D_x^\alpha[\phi(x;q)], \tag{22}$$

with the property $L[c]=0$. The nonlinear operator is

$$N[\phi(x;q)] = D_x^\alpha\phi(x;q) - x^2\phi^3(x;q) - x\phi^2(x;q) - \sqrt{x}\phi(x;q) - tanx.$$

Using the above definition and assuming $H(x)=1$, we construct the zeroth-order deformation equation

$$(1-q)L[\phi(x;q)-y_0(x)] = q\hbar N[\phi(x;q)]. \tag{23}$$

Thus, we obtain the m-th order deformation equation as

$$L[y_m(x)-\chi_m y_{m-1}(x)] = \hbar R_m(\vec{y}_{m-1}), \tag{24}$$

where

$$R_m(\vec{y}_{m-1}) = D_x^\alpha y_{m-1}(x) - x^2\sum_{i=0}^{m-1} y_i \sum_{k=0}^{m-1-i} y_k y_{m-1-i-k} - x\sum_{i=0}^{m-i} y_i y_{m-i} - \sqrt{x}y_{m-1}(x) - (1-\chi_m)(tanx).$$

Now the solution of the equations (24) can be obtained with the form

$$y_m(x) = (\chi_m+\hbar)y_{m-1}(x) - (\chi_m+\hbar)y_{m-1}(0) + \hbar I^\alpha[-x^2\sum_{i=0}^{m-1} y_i\sum_{k=0}^{m-1-i} y_k y_{m-1-i-k} - x\sum_{i=0}^{m-i} y_i y_{m-i} - \sqrt{x}y_{m-1}(x) - (1-\chi_m)(tanx)].$$

In view of (8), we have

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \tag{25}$$

The first components of $u(x)$ are obtained from (2) as follows:

$$\begin{aligned}
y_0(x) &= -\frac{\hbar x^{\alpha+1}}{(\alpha^2+\alpha)\Gamma(\alpha)} - \frac{2\hbar x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{16\hbar x^{\alpha+5}}{\Gamma(\alpha+6)}\\
y_1(x) &= -\frac{\hbar^2 x^{\alpha+1}}{(\alpha^2+\alpha)\Gamma(\alpha)} + \frac{\hbar^2\Gamma\left(\alpha+\frac{5}{2}\right)x^{2\alpha+\frac{3}{2}}}{(\alpha^2+\alpha)\Gamma(\alpha)\Gamma\left(2\alpha+\frac{5}{2}\right)}\\
&- \frac{2\hbar^2 x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{16\hbar^2 x^{\alpha+5}}{\Gamma(\alpha+6)} + \frac{2\hbar^2\Gamma\left(\alpha+\frac{9}{2}\right)x^{2\alpha+\frac{7}{2}}}{\Gamma(\alpha+4)\Gamma\left(2\alpha+\frac{9}{2}\right)}\\
&+ \frac{16\hbar^2\Gamma\left(\alpha+\frac{13}{2}\right)x^{2\alpha+\frac{11}{2}}}{\Gamma(\alpha+6)\Gamma\left(2\alpha+\frac{13}{2}\right)} - \frac{\hbar x^{\alpha+1}}{(\alpha^2+\alpha)\Gamma(\alpha)}\\
&- \frac{2\hbar x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{16\hbar x^{\alpha+5}}{\Gamma(\alpha+6)}\\
&\vdots
\end{aligned}$$

Hence

$$y(x) = y_0(x)+y_1(x)+y_2(x)+\ldots. \tag{26}$$

Figure 3 shows a $\hbar$-curve, and in Figure 4, we draw the exact solution, and the solution obtained after 3 iterations.$\hbar=-1.5$, $\alpha=.98$

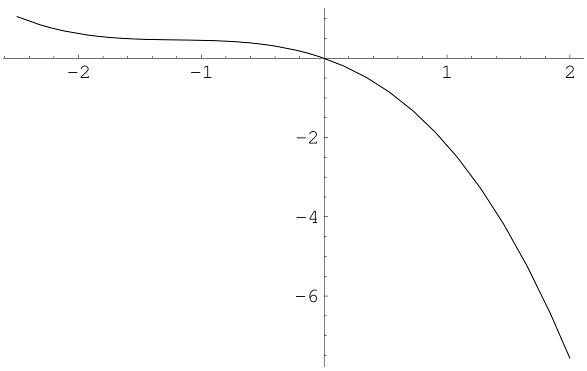

Figure 3. The $\hbar$-curve of $y(0.8)$ based on the fourth-order HAM approximations.

2.1. Convergence and stability analysis

This section is devoted to proving the convergence and stability of solutions to fractional initial value problems, on a finite interval of the complex axis in spaces of continuous functions.

Theorem 1.
If the series$y_i(x,y,t)=\sum_{m=0}^{M} y_{im}(x,y,t)$ converges ($i=1,2,\cdots,n$), where $y_{im}(x,y,t)$ is governed by Eq. (9), under the definition (10), it must be the solution of Eq. (6).

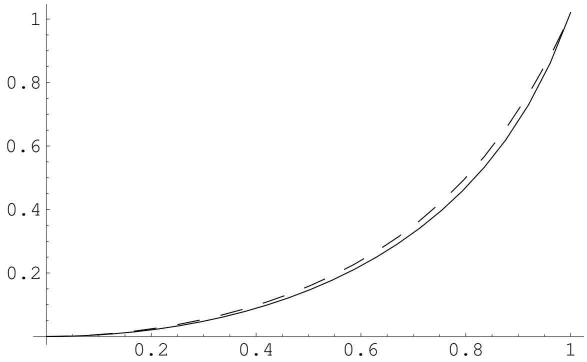

Figure 4. Numerical convergence of the exact solution and the HAM-solution.

Proof. This proof is similar to Theorem 3.1. in [2]. A clear conclusion can be drawn from the numerical results and Theorem 1 that our approach provides highly accurate numericalsolutions without spatial discretization of the problems. Overall, results show that the proposed approach isunconditionally stable and convergent. In other words, we can always find a proper value of the convergence control parameter$\hbar$ to ensure the convergent series solution, and our approximate results agree well with numerical ones. It should bepointed out that the response and stability of this type of problem, in general, can also be studied in a similar way. □

3. Conclusion

In this manuscript the homotopy analysis method is used to solve non-linear Abel differential equations with fractional order. The method provides a simple way to control the convergence region of the solution by introducing an auxiliary parameter $\hbar$ and auxiliary function $H(x)$. This is an obvious advantage of the HAM. It is also proved that homotopy perturbation and Adomian decomposition methods are only a special cases of the homotopy analysis method. This work illustrates the potential and the validity of the homotopy analysis method for solving nonlinear fractional differential equations. In this paper *Mathematica* has been used for computations and programming.

References

[1] S. Liao, Ph.D thesis, Shanghai Jiao Tong University (1992)
[2] S. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method (Chapman & Hall/CRC Press, Boca Raton, 2003)
[3] O. Abdulaziz, A. S. Bataineh, I. Hashim, Journal of Applied Mathematics and Computing 33, 61 (2010)
[4] S. Liao, Int. J. Nonlinear Mech. 30, 37180 (1995)
[5] S. Liao, Commun. Nonlinear Sci. 2, 95 (1997)
[6] H. Xu, S. Liao, X. C. You, Commun. Nonlinear Sci. 14, 1152 (2009)
[7] H. Jafari, S. Seifi, Commun. Nonlinear Sci. 14, 2006 (2009)
[8] K. B. Oldham, J. Spanier, The Fractional Calculus (Academic Press, Now York and London, 1974)
[9] I. Podlubny, Fractional Differential Equation (Academic Press, San Diego, 1999)
[10] I. Podlubny, Fract. Calculus Appl. Anal. 5, 367 (2002)
[11] S. G. Samko, A. A. Kilbas, O. Igorevich, Fractional Integrals and Derivatives, Theory and Applications (Gordon and Breach, Yverdon, 1993)
[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations (Elsevier Science, Amsterdam, 2006)
[13] D. Băleanu, K. Diethelm, E. Scalas, J. J. Trujillo, Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos (World Scientific, Boston 2012)
[14] J. G. Lu, G. Chen, Chaos Sol. Fract. 27, 685 (2006)
[15] H. Jafari, S. Momani, Phys. Lett. A 370, 388 (2007)
[16] H. Jafari, V. Daftardar-Gejji, J. Comput. Appl. Math. 196, 644 (2006)
[17] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics (Springer, Berlin, 1997)
[18] E. Hilfer (Ed.), Applications of Fractional Calculus in Physics (World Scientific, Singapore, 2000)
[19] J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado (Springer, The Netherlands, 2007)
[20] V. Lakshmikantham, A. S. Vatsala, Nonlinear Anal. 69, 2677 (2008)
[21] H. Weitzner, G. M. Zaslavsky, Commun. Nonlinear Sci. 8, 273 (2003)
[22] J. Gine, X. Santallusia, J. Math. Anal. Appl. 370, 187 (2010)
[23] H. Jafari, S. Das , H. Tajadodi, Journal of King Saud University Science 23, 151 (2010)
[24] H. Jafari, N. Kadkhoda, H. Tajadodi, S. A. Hosseini Matikolai, Int. J. Nonlin. Sci. Num. 11, 271 (2010)