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An Approximate-Analytical Solution to Analyze Fractional View of Telegraph Equations

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ABSTRACT In the present research article, a modified analytical method is applied to solve time-fractional telegraph equations. The Caputo-operator is used to express the derivative of fractional-order. The present method is the combination of two well-known methods namely Mohan transformation method and Adomian decomposition method. The validity of the proposed technique is confirmed through illustrative examples. It is observed that the obtained solutions have strong contact with the exact solution of the examples. Moreover, it is investigated that the present method has the desired degree of accuracy and provided the graphs closed form solutions of all targeted examples. The graphs have verified the convergence analysis of fractional-order solutions to integer-order solution. In conclusion, the suggested method is simple, straightforward and an effective technique to solve fractional-order partial differential equations.

INDEX TERMS Mohand transformation, telegraph equations, Adomian decomposition method, Caputo operator.

I. INTRODUCTION

The fractional calculus (F.C) was created in 1695, with a problem about the value of the half-order derivative. While F.C is the subject that has about 300 years of history. The growth of fractional calculus is a bit slow at early stage and focuses primarily on pure mathematics. At the early growth theory of F.C, it has only discovered in the latest decade that by using fractional differential systems, the behaviors of many structures can be defined, such as fractional kinetics, quantum evolution of complex system, viscoelastic system, dielectric polarization, electrode-electrolyte polarization, colored noise, electromagnetic waves, quantitative finance, phenomenon in power-law regarding fluid and network, applications in biology and ecology allometric laws scaling, the effects of boundary layer in ducts, and so on [1]–[4]. Many authors have been solved different types of fractional models, such as the fractional-order diffusion and Buckmaster's equation are solved by different mathematics [5], fractional-order telegraph equations [6], third-Order dispersive fractional partial differential equations (PDEs) [7], KdV-Kuramoto-Burger equation of fractional-order [8], fractal flow of traffic [9], Drinfeld-Sokolov-Wilson equation [10], time-fractional sub-diffusion and anomalous equations [11], heat equations of fractional-order [9], [12], fractional option pricing problems [13], [14], fractional coupled viscous Burgers' equation [15], hepatitis B virus fractional dynamic model [16], fractional model for tuberculosis [17], pine wilt disease fractional order model [18], fractional diabetes model [19], fractional-order Navier-Stokes equations [20], [21] etc.

In this new era of technology, communication system is considering a powerful system which plays a vital role in many real world problems. For example, engineering problems involved the transmission of a signal from one place to another. So, a specific class in the form of hyperbolic PDEs, which depict the vibrations within objects and the phenomena of wave propagation in medium, is known as telegraph equation [22]. In electrical circuit propagation the study of such type of telegraph equations are used extensively [23].

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The interaction among convection and diffusion process and its reciprocal narrate many non-linear problems that frequently occurs in physical, biological and chemical phenomenons [24]–[26]. In fact, for such branches of science the telegraph equation is more relevant as compared to diffusion reaction phenomena which is modeled of ordinary differential equation. For example, in the field of biology, biologists encounter these mathematical modelings to study the behaviors of pulsatile blood flow in arteries and in one dimensional random mosquito movement along with the shield [27]. Also in the field of physics, the acoustic wave's propagation in Darcy-type porous media [28], and Parallel Maxwell fluid flows [29] are few of the physical processes that have been defined by telegraph models [30]–[32]. The telegraph mathematical models also have been implemented in other fields as well. For instance, the model transport of charged particles is used as a substitute for the diffusion equation [33], [34], the transmission lines with high frequency [2], [35], solar cosmic rays [36], anomalous and chemical diffusion [37], [38], the population dynamics and hydrology [39]. It too has been employed in the theory of hyperbolic heat transfer [40], [41].

In the current article, we have applied the Mohand transformation (M.T) with decomposition procedure for the analytical treatment of fractional telegraph model as

$$\begin{aligned}\frac{\partial^{\beta} u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1^{\beta}} &= \kappa \frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \xi_1^2} + \lambda \frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial {\eta_1}^2} \\ &+ \rho \frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial {\gamma_1}^2} + \$ \frac{\partial u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1} \\ &+ \ell u(\xi_1,\eta_1,\gamma_1,\tau_1), \quad 1 < \beta \leq 2, \end{aligned} \tag{1}$$

with some initial source

$$\begin{aligned} u(\xi_1,\eta_1,\gamma_1,0) &= u(\xi_1,\eta_1,\gamma_1), \\ \frac{\partial}{\partial \tau_1} u(\xi_1,\eta_1,\gamma_1,0) &= u(\xi_1,\eta_1,\gamma_1), \end{aligned}$$

where $\kappa, \lambda, \rho, \ell$ and $\$ > 0$ are real numbers.

The M.T is one of the modern integral transformations used to analyze various physical phenomenon formed by differential equations (DEs), PDEs and fractional partial differential equations (FPDEs). Recently Sudhanshu Aggarwal, Mohand and Aboodh transforms comparatively studied for the mathematical solution of the integer order of DEs. The mathematical outcomes and implementations show the close connection between both the transformations Mohand and Aboodh [42]. The various transformation and their comparison such as Mohand, Laplace, Sumudu and Mahgoub transforms are briefly discussed by Sudhanshu Aggarwal [43]–[45] and similarly explained the Bessel's functions of order zero, one and two with the help of M.T [46]. Using M.T, the exact solution of second kind linear Volterra integral equations is obtained. M.T is said to take little period and has no significant mathematical work [47]. The theory of Bessel,s function is extremely important for the solution of several spherical equations such as wave schemes and heat system. Abel's solution is found by using M.T with the help of an integral equation. The obtained results have confirmed the importance of M.T to handle Abel's solution [48].

II. PRELIMINARY CONCEPTS

This section is concerned with some definitions and preliminaries which are important for the current research work.

A. DEFINITION

M.T was developed by Mohand and Mohgoub in 2017. This transformation is named as M.T. The transformation is defined for a function with exponential order which is defined in the set A of the form

$$\begin{aligned} A = \{f(\tau_1) : \exists M, k_1, k_2 > 0. |f(\tau_1)| < Me^{\frac{|\tau_1|}{k_j}}, \\ if\ \tau_1 \in (-1)^j \times [0,\infty)\}. \end{aligned} \tag{2}$$

where k_1, k_2 are finite or infinite for a factor to be in set A and M must be a finite number define [49], [50].

$$M\{u(x,\tau_1)\} = R(\upsilon) = \upsilon^2 \int_0^{\infty} u(\tau_1) e^{-\upsilon \tau_1} d\tau_1, \quad k_1 \leq \upsilon \leq k_2. \tag{3}$$

The M.T of a $u(\tau_1)$ function is $R(\upsilon)$, so $u(\tau_1)$ is considered the inverse of $R(\upsilon)$.

$$M^{-1}\{R(\upsilon)\} = u(\tau_1), \tag{4}$$

M^{-1} is the inverse operator of Mohand transform.

B. DEFINITION

M.T of the derivatives of the function $F(\tau_1)$

If $M\{F(\tau_1)\} = R(\upsilon)$ then

$$\begin{aligned} (a) \quad M\{F'(\tau_1)\} &= \upsilon R(\upsilon) - \upsilon^2 F(0) \\ (b) \quad M\{F''(\tau_1)\} &= \upsilon^2 R(\upsilon) - \upsilon^3 F(0) - \upsilon^2 F' \\ (c) \quad M\{F^{(n)}(\tau_1)\} &= \upsilon^n R(\upsilon) - \upsilon^{n+1} F(0) - \upsilon^n F'(0) \\ &\quad - \cdots - \upsilon^2 F^{(n-1)}(0). \end{aligned} \tag{5}$$

C. DEFINITION

Fractional-order derivative in term of Caputo definition [51]

$$D_{\tau_1}^{\beta} f(\tau_1) = \begin{cases} \dfrac{\partial^n f(\tau_1)}{\partial \tau_1^n}, \quad \beta = n \epsilon N, \\ \dfrac{1}{\beta(n-\beta)} \int_0^{\tau_1} (\tau_1 - \phi)^{n-\beta-1} f^{(n)}(\phi) \partial\phi, \\ n-1 < \beta < n. \end{cases} \tag{6}$$

D. DEFINITION

Function of Mittag-Leffler, $E_{\kappa}(x)$ for $\kappa > 0$ is described as

$$E_{\kappa}(x) = \sum_{\tilde{m}=0}^{\infty} \frac{x^{\tilde{m}}}{\Gamma(\kappa \tilde{m} + 1)} \qquad \kappa > 0 \quad x \in \mathbb{C}.$$

III. IMPLEMENTATION OF THE M.T DECOMPOSITION METHOD

In this section we considered the following telegraph equation

$$\frac{\partial^\beta u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1{}^\beta} = \kappa\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \xi_1{}^2} + \lambda\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \eta_1{}^2} + \rho\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \gamma_1{}^2} + \$\frac{\partial u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1} + \ell u(\xi_1,\eta_1,\gamma_1,\tau_1), \quad 1<\beta\le 2, \tag{7}$$

subject to the initial conditions

$$u(\xi_1,\eta_1,\gamma_1,0) = u(\xi_1,\eta_1,\gamma_1),$$
$$\frac{\partial}{\partial \tau_1}u(\xi_1,\eta_1,\gamma_1,0) = u(\xi_1,\eta_1,\gamma_1),$$

where $\kappa, \lambda, \rho, \ell$ and $\$ > 0$ are real numbers.

Applying Mohand transformation on both sides of equation (7), we get

$$M\left\{\frac{\partial^\beta u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1{}^\beta}\right\} = M\{\kappa\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \xi_1{}^2} + \lambda\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \eta_1{}^2} + \rho\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \gamma_1{}^2} + \$\frac{\partial u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1} + \ell u(\xi_1,\eta_1,\gamma_1,\tau_1)\}, \quad 1<\beta\le 2. \tag{8}$$

Using the differential property of Mohand transformation, we have

$$\upsilon^\beta\{R(\upsilon) - \upsilon u(0) - u'(0)\} = M\{\kappa\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \xi_1{}^2} +\} \times\lambda\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \eta_1{}^2} + \rho\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \gamma_1{}^2} + \$\frac{\partial u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1} + \ell u(\xi_1,\eta_1,\gamma_1,\tau_1)\}. \tag{9}$$

After simplification, equation (9) can be written as

$$R(\upsilon) = \upsilon u(0) + u'(0) + \frac{1}{\upsilon^\beta}M\left\{\kappa\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \xi_1{}^2} + \lambda\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \eta_1{}^2} + \rho\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \gamma_1{}^2} + \$\frac{\partial u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1} + \ell u(\xi_1,\eta_1,\gamma_1,\tau_1)\right\}. \tag{10}$$

Taking inverse Mohan transforation of equation (10)

$$u(\xi_1,\eta_1,\gamma_1,\tau_1) = u(0) + \tau_1 u'(0) + M^{-1}\left[\frac{1}{\upsilon^\beta}M\left\{\kappa\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \xi_1{}^2} + \lambda\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \eta_1{}^2} + \rho\frac{\partial^2 u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \gamma_1{}^2} + \$\frac{\partial u(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1} + \ell u(\xi_1,\eta_1,\gamma_1,\tau_1)\}\right]. \tag{11}$$

At the end, we obtained the following recursive formula by using Adomian decomposition method.

$$u_0(\xi_1,\eta_1,\gamma_1,\tau_1) = u(0) + \tau_1\frac{\partial}{\partial \tau_1}u(0), \quad m=0$$
$$u_{m+1}(\xi_1,\eta_1,\gamma_1,\tau_1) = M^{-1}\left[\frac{1}{\upsilon^\beta}M\left\{\kappa\frac{\partial^2 u_m(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \xi_1{}^2}v + \lambda\frac{\partial^2 u_m(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \eta_1{}^2} + \rho\frac{\partial^2 u_m(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \gamma_1{}^2} + \$\frac{\partial u_m(\xi_1,\eta_1,\gamma_1,\tau_1)}{\partial \tau_1} + \ell u_m(\xi_1,\eta_1,\gamma_1,\tau_1)\right\}\right], \quad m\ge 0. \tag{12}$$

IV. APPLICATIONS AND DISCUSSION

Example 1: Consider (1+1) dimensional fractional telegraph model

$$\frac{\partial^\beta u(\xi_1,\tau_1)}{\partial \tau_1{}^\beta} = \frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1{}^2} - 2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1} - u(\xi_1,\tau_1), \quad 1<\beta\le 2, \tag{13}$$

with initial conditions

$$u(x,0) = e^{\xi_1}, \quad u_{\tau_1}(\xi_1,0) = -2e^{\xi_1}.$$

Applying M.T on both sides of equation (13)

$$\upsilon^\beta\{R(\upsilon) - \upsilon u(0) - u'(0)\} = M\{\frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1^2} - 2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1} - u(\xi_1,\tau_1)\}. \tag{14}$$

After simplification

$$R(\upsilon) = \upsilon u(0) + u'(0) + \frac{1}{\upsilon^\beta}\{M\{\frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1} - 2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1} - u(\xi_1,\tau_1)\}\}. \tag{15}$$

Using inverse M.T, we have

$$u(\xi_1,\tau_1) = u(0) + \tau_1 u'(0) + M^{-1}\{\frac{1}{\upsilon^\beta}M\{\frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1} - 2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1} - u(\xi_1,\tau_1)\}\}. \tag{16}$$

Furthermore, by using the recursive system of equation (12), we have

$$u_0(\xi_1,\tau_1)=\upsilon u(0)+u'(0)=e^{\xi_1}-2\tau_1 e^{\xi_1}, \tag{17}$$

$$u_{m+1}(\xi_1,\tau_1) = M^{-1}\left\{\frac{1}{\upsilon^\beta}M\left\{\frac{\partial^2 u_m(\xi_1,\tau_1)}{\partial\xi_1^2} -2\frac{\partial u_m(\xi_1,\tau_1)}{\partial\tau_1}-u_m(\xi_1,\tau_1)\right\}\right\},\quad m=0,1,\cdots \tag{18}$$

Eq. (18) implies,
for $m=0$

$$u_1(\xi_1,\tau_1) = M^{-1}\left\{\frac{1}{\upsilon^\beta}M\left\{\frac{\partial^2 u_0(\xi_1,\tau_1)}{\partial\xi_1^2}-2\frac{\partial u_0(\xi_1,\tau_1)}{\partial\tau_1}-u_0(\xi_1,\tau_1)\right\}\right\},$$

$$u_1(\xi_1,\tau_1) = 4e^{\xi_1}\frac{\tau_1^\beta}{\beta!}, \tag{19}$$

for $m=1$

$$u_2(\xi_1,\tau_1)=M^{-1}\left\{\frac{1}{\upsilon^\beta}M\left\{\frac{\partial^2 u_1(\xi_1,\tau_1)}{\partial\xi_1^2} -2\frac{\partial u_1(\xi_1,\tau_1)}{\partial\tau_1}-u_1(\xi_1,\tau_1)\right\}\right\},$$

$$u_2(\xi_1,\tau_1)=-8e^{\xi_1}\frac{(\beta)(\beta-1)!\tau_1^{2\beta-1}}{(2\beta-1)!(\beta)!}, \tag{20}$$

for $m=2$

$$u_3(\xi_1,\tau_1)=M^{-1}\left\{\frac{1}{\upsilon^\beta}M\left\{\frac{\partial^2 u_2(\xi_1,\tau_1)}{\partial\xi_1^2} -2\frac{\partial u_2(\xi_1,\tau_1)}{\partial\tau_1}-u_2(\xi_1,\tau_1)\right\}\right\},$$

$$u_3(\xi_1,\tau_1)=16e^{\xi_1}\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!\tau_1^{3\beta-2}}{(2\beta-1)!(3\beta-2)!(\beta)!}, \tag{21}$$

for $m=3$

$$u_4(\xi_1,\tau_1) = M^{-1}\left\{\frac{1}{\upsilon^\beta}M\left\{\frac{\partial^2 u_3(\xi_1,\tau_1)}{\partial\xi_1^2} -2\frac{\partial u_3(\xi_1,\tau_1)}{\partial\tau_1}-u_3(\xi_1,\tau_1)\right\}\right\},$$

$$u_4(\xi_1,\tau_1) =-32e^{\xi_1}\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!(3\beta-2)(3\beta-3)!\tau_1^{4\beta-3}}{(2\beta-1)!(3\beta-2)!(\beta)!(4\beta-3)!},$$
$$\vdots \tag{22}$$

Thus for the example 1, the M.T result is

$$u(\xi_1,\tau_1)=u_0(\xi_1,\tau_1)+u_1(\xi_1,\tau_1)+u_2(\xi_1,\tau_1) +u_3(\xi_1,\tau_1)+u_4(\xi_1,\tau_1)+\cdots. \tag{23}$$

By putting the corresponding values, we get

$$u(\xi_1,\tau_1) = e^{\xi_1}-2\tau_1e^{\xi_1}+4e^{\xi_1}\frac{\tau_1^\beta}{\beta!}-8e^{\xi_1}\frac{(\beta)(\beta-1)!\tau_1^{2\beta-1}}{(2\beta-1)!(\beta)!} +16e^{\xi_1}\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!\tau_1^{3\beta-2}}{(2\beta-1)!(3\beta-2)!(\beta)!} -32e_1^{\xi}\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!(3\beta-2)(3\beta-3)!\tau_1^{4\beta-3}}{(2\beta-1)!(3\beta-2)!(\beta)!(4\beta-3)!}, \tag{24}$$

For particular case $\beta=2$, the M.T solution become as

$$u(\xi_1,\tau_1)=e^{\xi_1}-2\tau_1e^{\xi_1}+4e^{\xi_1}\frac{\tau_1^2}{2!}-8e^{\xi_1}\frac{\tau_1^3}{3!} +16e^{\xi_1}\frac{\tau_1^4}{4!}-32e^{\xi_1}\frac{\tau_1^5}{5!}+\cdots. \tag{25}$$

The determined result provides the exact solution

$$u(\xi_1,\tau_1)=e^{\xi_1-2\tau_1}. \tag{26}$$

Example 2: Consider (2+1) dimensional linear fractional telegraph model

$$\frac{\partial^\beta u(\xi_1,\eta_1,\tau_1)}{\partial{\tau_1}^\beta} = \frac{\partial^2u(\xi_1,\eta_1,\tau_1)}{\partial{\xi_1}^2}+\frac{\partial^2u(\xi_1,\eta_1,\tau_1)}{\partial{\eta_1}^2} -3\frac{\partial u(\xi_1,\eta_1,\tau_1)}{\partial\tau_1}-2u(\xi_1,\eta_1,\tau_1),\quad 1<\beta\leq 2 \tag{27}$$

with initial conditions

$$u(x,y,0)=e^{\xi_1+\eta_1},\quad u_t(\xi_1+\eta_1,0)=-3e^{\xi_1+\eta_1},$$

the exact solution is

$$u(\xi_1,\eta_1,\tau_1)=e^{\xi_1+\eta_1-3\tau_1}.$$

Applying M.T on both sides of eq. (27)

$$\upsilon^\beta\{R(\upsilon)-\upsilon u(0)-u'(0)\} = M\{\frac{\partial^2u(\xi_1,\eta_1,\tau_1)}{\partial\xi_{\epsilon_1}^2}+\frac{\partial^2u(\xi_1,\eta_1,\tau_1)}{\partial{\eta_1}^2}-3\frac{\partial u(\xi_1,\eta_1,\tau_1)}{\partial\tau_1} -2u(\xi_1,\eta_1,\tau_1)\}. \tag{28}$$

After simplification, equation (28) is modified

$$R(\upsilon)=\upsilon u(0)+u'(0)+\frac{1}{\upsilon^\beta}\{M\{\frac{\partial^2u(\xi_1,\eta_1,\tau_1)}{\partial{\xi_1}^2} +\frac{\partial^2u(\xi_1,\eta_1,\tau_1)}{\partial{\eta_1}^2}-3\frac{\partial u(\xi_1,\eta_1,\tau_1)}{\partial\tau_1} -2u(\xi_1,\eta_1,\tau_1)\}\}, \tag{29}$$

Applying inverse M.T and simplified equation (29) implies

$$u(\xi_1,\eta_1,\tau_1) = u(0)+\tau_1 u'(0)+M^{-1}[\frac{1}{\upsilon^\beta}M\{\frac{\partial^2 u(\xi_1,\eta_1,\tau_1)}{\partial {\xi_1}^2}\}] + \frac{\partial^2 u(\xi_1,\eta_1,\tau_1)}{\partial {\eta_1}^2} - 3\frac{\partial u(\xi_1,\eta_1,\tau_1)}{\partial \tau_1} - 2u(\xi_1,\eta_1,\tau_1)\}\}, \tag{30}$$

Using the recursive system of the equation (12), we get

$$u_0(\xi_1,\eta_1,\tau_1) = u(0)+\tau_1 u'(0) = e^{\xi_1+\eta_1} - 3\tau_1 e^{\xi_1+\eta_1}, \tag{31}$$

and

$$u_{m+1}(\xi_1,\eta_1,\tau_1) = M^{-1}\{\frac{1}{\upsilon^\beta}M\{\frac{\partial^2 u_m(\xi_1,\eta_1,\tau_1)}{\partial {\xi_1}^2} + \frac{\partial^2 u_m(\xi_1,\eta_1,\tau_1)}{\partial {\eta_1}^2} - 3\frac{\partial u_m(\xi_1,\eta_1,\tau_1)}{\partial \tau_1} - 2u_m(\xi_1,\tau_1)\}\}. \tag{32}$$

equation (32) implies that,
for $m=0$

$$u_1(\xi_1,\eta_1,\tau_1) = M^{-1}\{\frac{1}{\upsilon^\beta}M\{\frac{\partial^2 u_0(\xi_1,\eta_1,\tau_1)}{\partial {\xi_1}^2} + \frac{\partial^2 u_0(\xi_1,\eta_1,\tau_1)}{\partial {\eta_1}^2} - 3\frac{\partial u_0(\xi_1,\eta_1,\tau_1)}{\partial \tau_1} - 2u_0(\xi_1,\tau_1)\}\}.$$

$$u_1(\xi_1,\eta_1,\tau_1) = 9e^{\xi_1+\eta_1}\frac{\tau_1^\beta}{\beta!}, \tag{33}$$

for $m=1$

$$u_2(\xi_1,\eta_1,\tau_1) = M^{-1}\{\frac{1}{\upsilon^\beta}M\{\frac{\partial^2 u_1(\xi_1,\eta_1,\tau_1)}{\partial {\xi_1}^2} + \frac{\partial^2 u_1(\xi_1,\eta_1,\tau_1)}{\partial {\eta_1}^2} - 3\frac{\partial u_1(\xi_1,\eta_1,\tau_1)}{\partial \tau_1} - 2u_1(\xi_1,\tau_1)\}\}.$$

$$u_2(\xi_1,\eta_1,\tau_1) = -27e^{\xi_1+\eta_1}\frac{(\beta)(\beta-1)!\tau_1^{2\beta-1}}{(2\beta-1)!(\beta)!}, \tag{34}$$

for $m=2$

$$u_3(\xi_1,\eta_1,\tau_1) = M^{-1}\{\frac{1}{\upsilon^\beta}M\{\frac{\partial^2 u_2(\xi_1,\eta_1,\tau_1)}{\partial {\xi_1}^2} + \frac{\partial^2 u_2(\xi_1,\eta_1,\tau_1)}{\partial {\eta_1}^2} - 3\frac{\partial u_2(\xi_1,\eta_1,\tau_1)}{\partial \tau_1} - 2u_2(\xi_1,\tau_1)\}\}.$$

$$u_3(\xi_1,\eta_1,\tau_1) = 81e^{\xi_1+\eta_1}\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!\tau_1^{3\beta-2}}{(2\beta-1)!(3\beta-2)!(\beta)!}, \tag{35}$$

for $m=3$

$$u_4(\xi_1,\eta_1,\tau_1) = M^{-1}\{\frac{1}{\upsilon^\beta}M\{\frac{\partial^2 u_3(\xi_1,\eta_1,\tau_1)}{\partial {\xi_1}^2} + \frac{\partial^2 u_3(\xi_1,\eta_1,\tau_1)}{\partial {\eta_1}^2} - 3\frac{\partial u_3(\xi_1,\eta_1,\tau_1)}{\partial \tau_1} - 2u_3(\xi_1,\tau_1)\}\}.$$

$$u_4(\xi_1,\eta_1,\tau_1) = -243e^{\xi_1+\eta_1} \times \frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!(3\beta-2)(3\beta-3)!\tau_1^{4\beta-3}}{(2\beta-1)!(3\beta-2)!(\beta)!(4\beta-3)!}, \\ \vdots \tag{36}$$

Thus for the example 2, the M.T solution is

$$u(\xi_1,\eta_1,\tau_1) = u_0(\xi_1,\eta_1,\tau_1) + u_1(\xi_1,\eta_1,\tau_1) + u_2(\xi_1,\eta_1,\tau_1) + u_3(\xi_1,\eta_1,\tau_1) + \cdots. \tag{37}$$

By putting the corresponding values, we get

$$\begin{aligned} u(\xi_1,\eta_1,\tau_1) &= e^{\xi_1+\eta_1} - 3\tau_1 e^{\xi_1+\eta_1} + 9e^{\xi_1+\eta_1}\frac{\tau_1^\beta}{\beta!} \\ &- 27e^{\xi_1+\eta_1}\frac{\beta(\beta-1)!\tau_1^{2\beta-1}}{(2\beta-1)!(\beta)!} \\ &+ 81e^{\xi_1+\eta_1}\frac{\beta(\beta-1)!(2\beta-1)(2\beta-2)!\tau_1^{3\beta-2}}{(2\beta-1)!(3\beta-2)!(\beta)!} \\ &- 243e^{\xi_1+\eta_1} \\ &\times \frac{\beta(\beta-1)!(2\beta-1)(2\beta-2)!(3\beta-2)(3\beta-3)!\tau_1^{4\beta-3}}{(2\beta-1)!(3\beta-2)!(\beta)!(4\beta-3)!}. \end{aligned} \tag{38}$$

Particularly, if $\beta=2$, the M.T solution is

$$u(\xi_1,\eta_1,\tau_1) = e^{\xi_1+\eta_1} - 3\tau_1 e^{\xi_1+\eta_1} + 9e^{\xi_1+\eta_1}\frac{\tau_1^2}{2!} - 27e^{\xi_1+\eta_1}\frac{\tau_1^3}{3!} + 81e^{\xi_1+\eta_1}\frac{\tau_1^4}{4!} - 243e^{\xi_1+\eta_1}\frac{\tau_1^5}{5!} + \cdots. \tag{39}$$

The exact solution provides the closed-form.

$$u(\xi_1,\eta_1,\tau_1) = e^{\xi_1+\eta_1-3\tau_1}. \tag{40}$$

Example 3: Consider (3+1) dimensional fractional telegraph model

$$\frac{\partial^\beta u(\xi_1,\tau_1)}{\partial {\tau_1}^\beta} = \frac{\partial^2 u(\xi_1,\tau_1)}{\partial {\xi_1}^2} + \frac{\partial^2 u(\eta_1,\tau_1)}{\partial {\eta_1}^2} + \frac{\partial^2 u(\gamma_1,\tau_1)}{\partial {\gamma_1}^2} - 2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1} - 3u(\xi_1,\tau_1), \quad 1<\beta\le 2, \tag{41}$$

with initial conditions

$$\begin{aligned} u(\xi_1,\eta_1,\gamma_1,0) &= \sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1), \\ u_\tau(\xi_1,\eta_1,\gamma_1,0) &= -2\sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1), \end{aligned}$$

the exact solution of equation (41) is

$$u(\xi_1,\eta_1,\gamma_1,\tau_1) = e^{-2\tau}\sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1).$$

Applying M.T on both sides of eq. (41)

$$\upsilon^{\beta}\{R(\upsilon)-\upsilon u(0)-u'(0)\} = M\{\frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1{}^2}+\frac{\partial^2 u(\eta_1,\tau_1)}{\partial \eta_1{}^2}+\frac{\partial^2 u(\gamma_1,\tau_1)}{\partial \gamma_1{}^2} - 2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1}-3u(\xi_1,\tau_1)\}, \quad (42)$$

Equation (42) is modified after further evaluations by M.T as

$$R(\upsilon)=\upsilon u(0)+u'(0)+\frac{1}{\upsilon^{\beta}}M\{\frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1{}^2}+\frac{\partial^2 u(\eta_1,\tau_1)}{\partial \eta_1{}^2}\} + \frac{\partial^2 u(\gamma_1,\tau_1)}{\partial \gamma_1{}^2}-2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1}-3u(\xi_1,\tau_1)\}. \quad (43)$$

By implementing the inverse transform of Mohand, we get

$$u(\xi_1,\eta_1,\gamma_1,\tau_1) = u(0)+u'(0)+M^{-1}\{\frac{1}{\upsilon^{\beta}}M\{\frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1{}^2}+\frac{\partial^2 u(\eta_1,\tau_1)}{\partial \eta_1{}^2} + \frac{\partial^2 u(\gamma_1,\tau_1)}{\partial \gamma_1{}^2}-2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1}-3u(\xi_1,\tau_1)\}\}, \quad (44)$$

Thus, by using the recursive equation (12) scheme, we get it

$$u_0(\xi_1,\eta_1,\gamma_1,\tau_1)=u(0)+\tau_1 u'(0) = \sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1) - 2\tau\sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1), \quad (45)$$

For $m \geq 0$

$$u_{m+1}(\xi_1,\eta_1,\gamma_1,\tau_1) = M^{-1}\{\frac{1}{\upsilon^{\beta}}M\{\frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1{}^2}+\frac{\partial^2 u(\eta_1,\tau_1)}{\partial \eta_1{}^2}+\frac{\partial^2 u(\gamma_1,\tau_1)}{\partial \gamma_1{}^2} - 2\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1}-3u(\xi_1,\tau_1)\}\}, \quad (46)$$

Recursive formula (46),
for $m=0$

$$u_1(\xi_1,\eta_1,\gamma_1,\tau_1)=4\sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1)\frac{\tau_1^{\beta}}{\beta!}, \quad (47)$$

for $m=1$

$$u_2(\xi_1,\eta_1,\gamma_1,\tau_1) = -8\sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1)\frac{(\beta)(\beta-1)!\tau_1^{2\beta-1}}{(2\beta-1)!(\beta)!}, \quad (48)$$

for $m=2$

$$u_3(\xi_1,\eta_1,\gamma_1,\tau_1) = 16\sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1) \times\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!\tau_1^{3\beta-2}}{(2\beta-1)!(3\beta-2)!(\beta)!}, \quad (49)$$

for $m=3$

$$u_4(\xi_1,\eta_1,\gamma_1,\tau_1) = -32\sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1) \times\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!(3\beta-2)(3\beta-3)!\tau_1^{4\beta-3}}{(2\beta-1)!(3\beta-2)!(\beta)!(4\beta-3)!}, \quad \vdots \quad (50)$$

For example 3, the M.T solution is

$$u(\xi_1,\eta_1,\gamma_1,\tau_1) = u_0(\xi_1,\eta_1,\gamma_1,\tau_1)+u_1(\xi_1,\eta_1,\gamma_1,\tau_1) + u_2(\xi_1,\eta_1,\gamma_1,\tau_1)+u_3(\xi_1,\eta_1,\gamma_1,\tau_1)+\cdots. \quad (51)$$

We get the solution as

$$u(\xi_1,\eta_1,\gamma_1,\tau_1) = \sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1)\{1-2\tau_1\sinh(\xi_1)+4\frac{\tau_1^{\beta}}{\beta!} - 8\frac{(\beta)(\beta-1)!\tau_1^{2\beta-1}}{(2\beta-1)!(\beta)!} + 16\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!\tau_1^{3\beta-2}}{(2\beta-1)!(3\beta-2)!(\beta)!} - 32\frac{(\beta)(\beta-1)!(2\beta-1)(2\beta-2)!(3\beta-2)(3\beta-3)!\tau_1^{4\beta-3}}{(2\beta-1)!(3\beta-2)!(\beta)!(4\beta-3)!} + \cdots. \quad (52)$$

In the specific case of $\beta=2$,the M.T solution becomes the same as

$$u(\xi_1,\eta_1,\gamma_1,\tau_1) = \sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1)\{1-2\tau_1+\frac{\tau_1^2}{2!} - 8\frac{\tau_1^3}{3!}+16\frac{\tau_1^4}{4!}-32\frac{\tau_1^5}{5!}+\cdots\}. \quad (53)$$

The exact solution provides the closed-form

$$u(\xi_1,\eta_1,\gamma_1,\tau_1)=e^{-2\tau_1}\sinh(\xi_1)\sinh(\eta_1)\sinh(\gamma_1). \quad (54)$$

Example 4: Consider space-fractional non-linear telegraph model

$$\frac{\partial^{\beta}u(\xi_1,\tau_1)}{\partial \tau_1{}^{\beta}}=\frac{\partial^2 u(\xi_1,\tau_1)}{\partial \xi_1{}^2}+\frac{\partial u(\xi_1,\tau_1)}{\partial \tau_1}-u^2(\xi_1,\tau_1) + \xi_1 u(\xi_1,\tau_1)u_{\xi_1}(\xi_1,\tau_1), \quad 1<\beta\leq 2, \quad (55)$$

with initial conditions

$$u(\xi_1,0)=\xi_1, \quad u_{\tau_1}(\xi_1,0)=\xi_1,$$

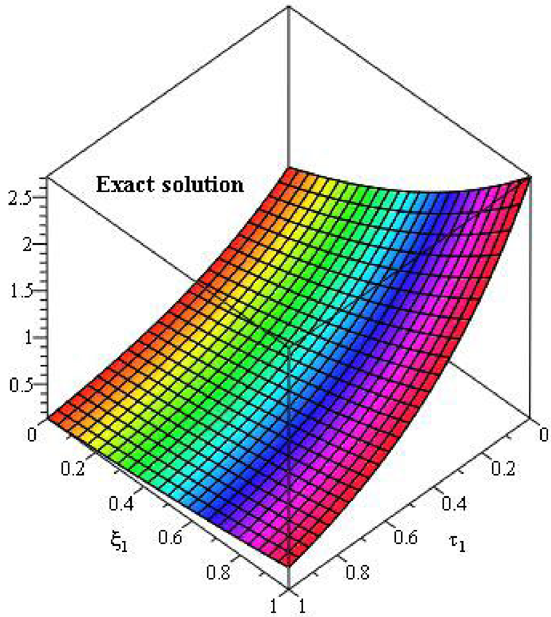

FIGURE 1. Exact solution of example 1 at $\beta = 1$.

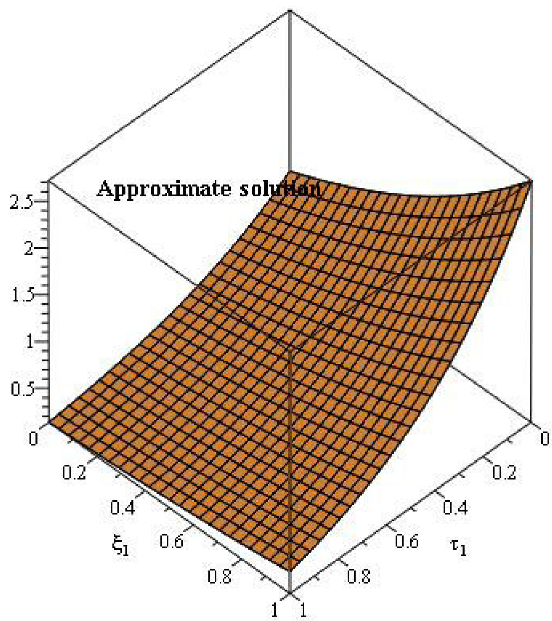

FIGURE 2. Analytical solution of example 1 $\beta = 1$.

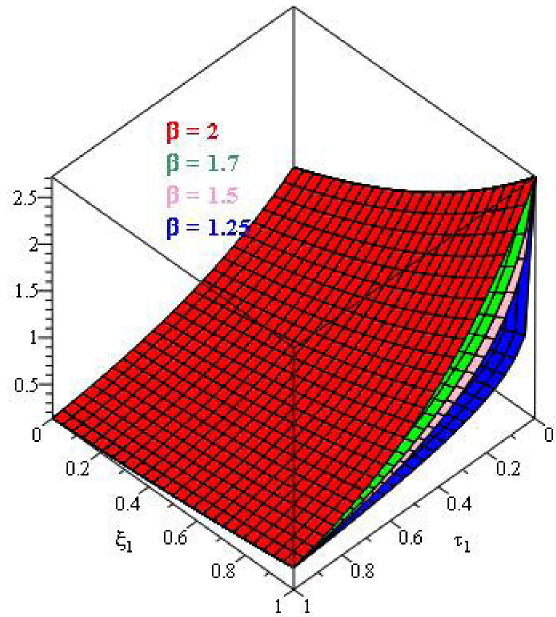

FIGURE 3. The solutions of example 1 at different fractional order β.

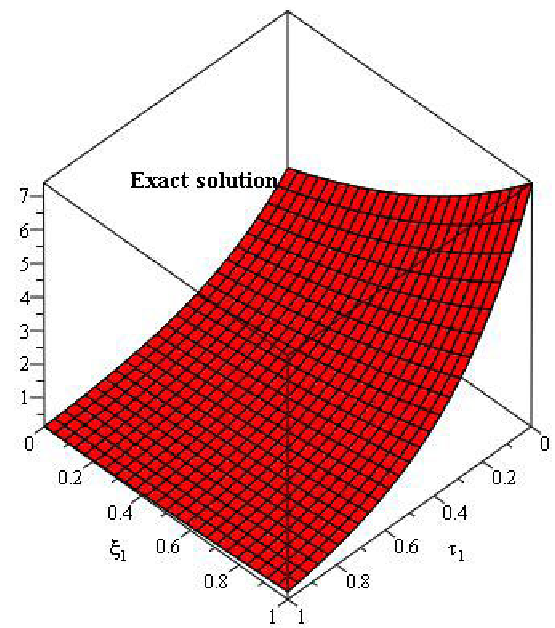

FIGURE 4. Exact solution of example 2 at $\beta = 1$.

Applying M.T on both sides of eq. (55)

$$\upsilon^{\beta}\{R(\upsilon) - \upsilon u(0) - u'(0)\} = M\{\frac{\partial^2 u(\xi_1, \tau_1)}{\partial {\xi_1}^2} + \frac{\partial u(\xi_1, \tau_1)}{\partial \tau_1} - u^2(\xi_1, \tau_1) + \xi_1 u(\xi_1, \tau_1) u_{\xi_1}(\xi_1, \tau_1)\}, \tag{56}$$

Equation (56) is modified after several evaluations as

$$R(\upsilon) = \upsilon u(0) + u'(0) + \frac{1}{\upsilon^{\beta}} M\{\frac{\partial^2 u(\xi_1, \tau_1)}{\partial {\xi_1}^2} + \frac{\partial u(\xi_1, \tau_1)}{\partial \tau_1} - u^2(\xi_1, \tau_1) + \xi_1 u(\xi_1, \tau_1 u_{\xi_1}(\xi_1, \tau_1)\}, \tag{57}$$

By applying the M.T inverse,

$$u(\xi_1, \tau_1) = u(0) + \tau_1 u'(0) + M^{-1}\{\frac{1}{\upsilon^{\beta}} M\{\frac{\partial^2 u(\xi_1, \tau_1)}{\partial {\xi_1}^2} + \frac{\partial u(\xi_1, \tau_1)}{\partial \tau_1} - u^2(\xi_1, \tau_1) + x u(\xi_1, \tau_1) u_{\xi_1}(\xi_1, \tau_1)\}\}, \tag{58}$$

using recursive system of equation (12), we have

$$u_0(\xi_1, \tau_1) = u(0) + \tau_1 u'(0) = \xi_1 + \xi_1 \tau_1, \tag{59}$$

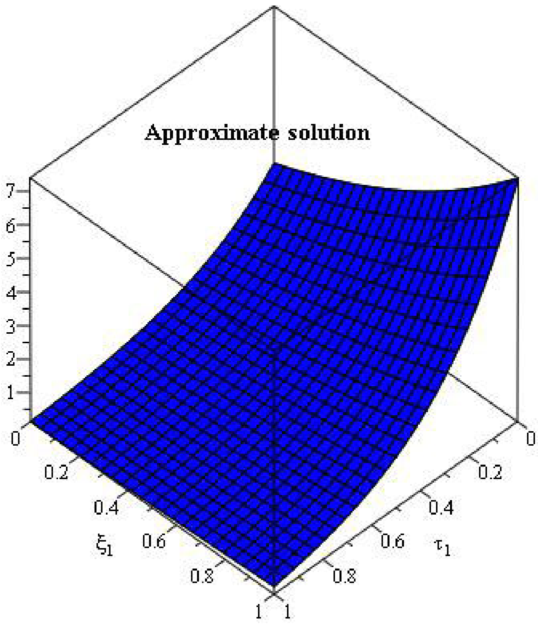

FIGURE 5. Analytical solution of example 2.

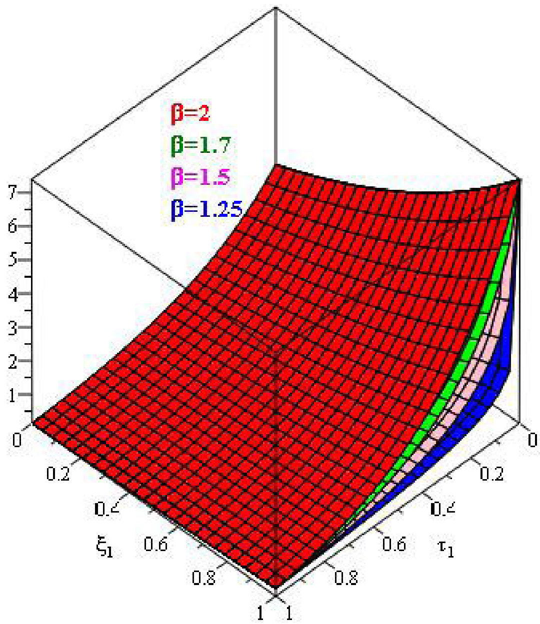

FIGURE 6. The solutions of example 2 at different fractional order β.

$$u_{m+1}(\xi_1,\tau_1) = M^{-1}\{\frac{1}{\upsilon^{\beta}}M\{\frac{\partial^2 u_m(\xi_1,\tau_1)}{\partial {\xi_1}^2} + \frac{\partial u_m(\xi_1,\tau_1)}{\partial \tau_1} - u_{m^2}(\xi_1,\tau_1) + \xi u_m(\xi_1,\tau_1)\frac{\partial u_m(\xi_1,\tau_1)}{\partial \xi_1}\}\}\},$$
$$m = 0, 1, \cdots \tag{60}$$

Recursive formula (60),
for $m = 0$

$$u_1(\xi_1,\tau_1) = \xi_1 \frac{\tau_1^{\beta}}{\Gamma(\beta+1)}, \tag{61}$$

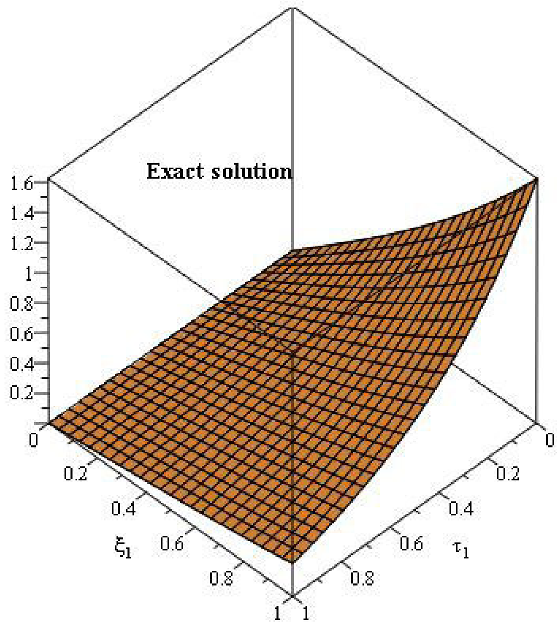

FIGURE 7. Exact solution of example 3 at $\beta = 1$.

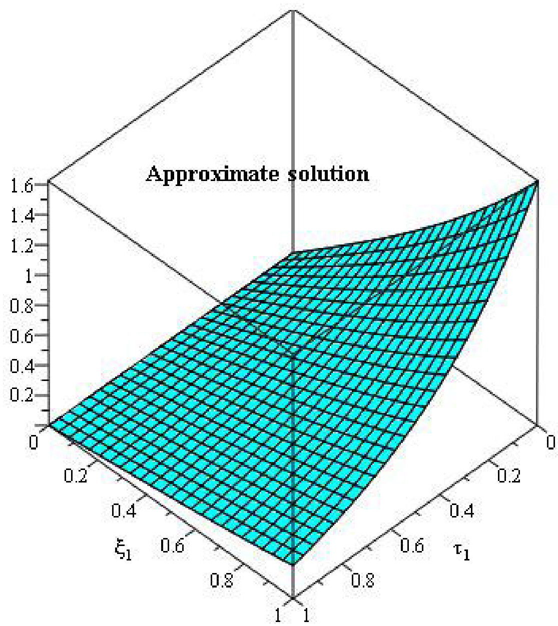

FIGURE 8. Analytical solution of example 3.

for $m = 1$

$$u_2(\xi_1,\tau_1) = \xi_1 \frac{\tau_1^{\beta+1}}{\Gamma(\beta+2)}, \tag{62}$$

for $m = 2$

$$u_3(\xi_1,\tau_1) = \xi_1 \frac{\tau_1^{\beta+2}}{\Gamma(\beta+3)}, \tag{63}$$

for $m = 3$

$$u_4(\xi_1,\tau_1) = \xi_1 \frac{\tau^{\beta+3}}{\Gamma(\beta+4)},$$
$$\vdots \tag{64}$$

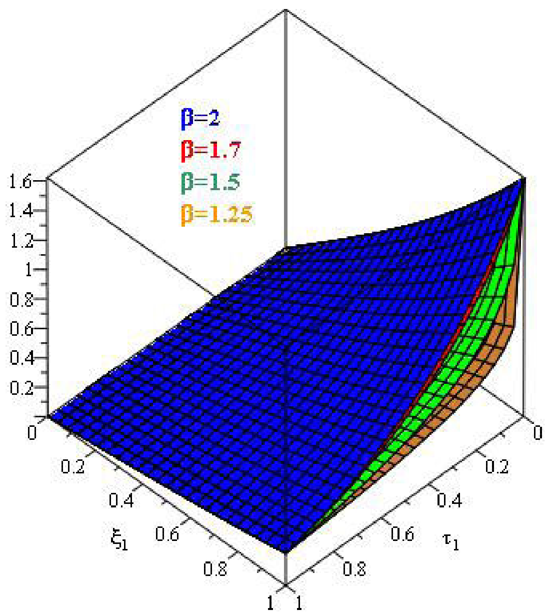

FIGURE 9. The solutions of example 3 at different fractional order β.

TABLE 1. The comparison of exact and analytical solutions of different values of β at $\tau_1 = 1$.

	MADM	MADM	MADM	Exact	Absolute Error
ξ_1	$\beta = 1.55$	$\beta = 1.75$	$\beta = 2$		$\beta = 2$
1	1.010576722	1.010196638	1.010050002	1.010050167	1.65E-07
2	2.021153444	2.020393276	2.020100004	2.020100334	3.30E-07
3	3.031730166	3.030589914	3.030150006	3.030150501	4.95E-07
4	4.042306888	4.040786552	4.040200008	4.040200668	6.60E-07
5	5.052883610	5.050983190	5.050250010	5.050250835	8.25E-07
6	6.063460332	6.061179828	6.060300012	6.060301002	9.90E-07
7	7.074037054	7.071376466	7.070350014	7.070351169	1.15E-06
8	8.084613776	8.081573104	8.080400016	8.080401336	1.32E-06
9	9.095190498	9.091769742	9.090450018	9.090451503	1.48E-06
10	10.10576722	10.10196638	10.10050002	10.10050167	1.65E-06

Hence for the example 4, the M.T solution is

$$u(\xi_1, \tau_1) = u_0(\xi_1, \tau_1) + u_1(\xi_1, \tau_1) + u_2(\xi_1, \tau_1) + u_3(\xi_1, \tau_1) + u_4(\xi_1, \tau_1) + \cdots \tag{65}$$

By putting the corresponding values,

$$u(\xi_1, \tau_1) = \xi_1[1 + \tau_1 + \frac{\tau_1^{\beta}}{\Gamma(\beta+1)} + \frac{\tau_1^{\beta+1}}{\Gamma(\beta+2)} + \frac{\tau_1^{\beta+2}}{\Gamma(\beta+3)} + \frac{\tau_1^{\beta+3}}{\Gamma(\beta+4)} + \cdots .]$$

The exact solution of equation (55) is $u(\xi_1, \tau_1) = \xi_1\, e^{\tau_1}$.

V. RESULTS AND DISCUSSION

The exact and analytical solutions of example 1 are represented in Figures 1 and 2. It is observed that the solution

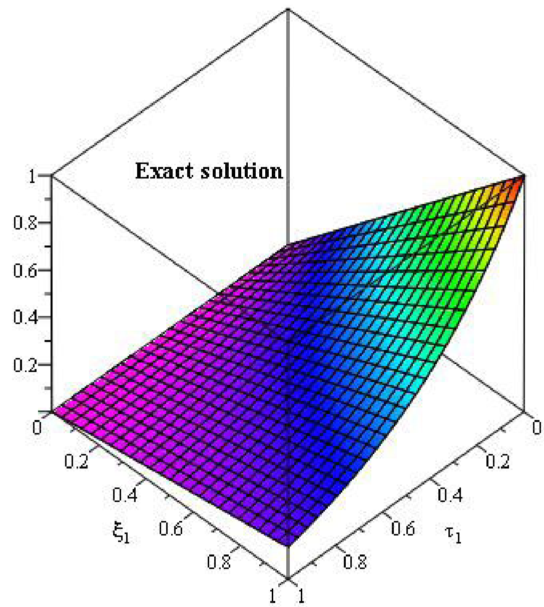

FIGURE 10. Exact solution of example 4.

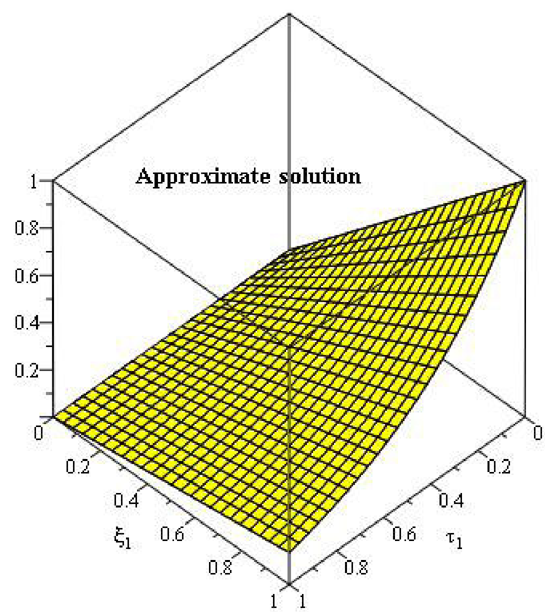

FIGURE 11. Analytical solution of example 4.

obtained by the proposed technique is in good contact with the exact solution of the problem. The solutions of problem 1 at fractional-orders $\beta = 2, 1.7, 1.5$ and 1.25 are plotted in Figure 3. It is observed that the solution fractional-order is approached toward the solutions of integer-order. Similarly, Figure 4 and 5 show the exact and approximate solutions of example 2. In Figure 6, we displayed the fractional-order solutions of problem 4.2. The similar representation of example 3 has been expressed with help of Figures 7,8 and 9. At the end problem 4 represents the non-linear telegraph equations, where the exact and approximate solutions are expressed in Figures 10 and 11. Figures 12 has provided the fractional-order solutions of problem 4.4 at $\beta = 0.6$ and

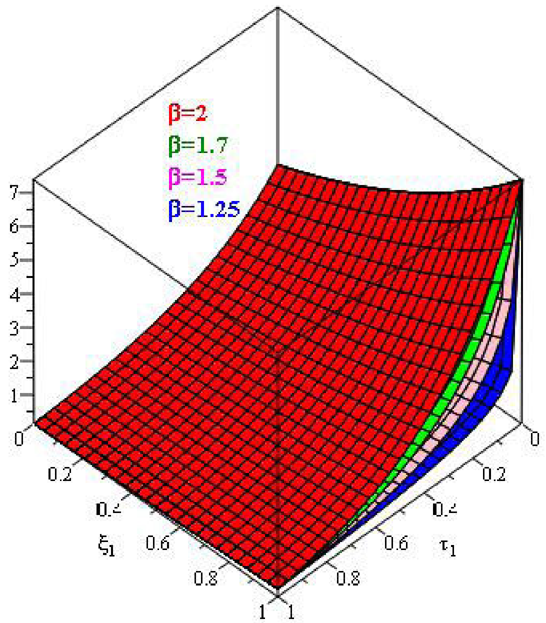

FIGURE 12. The solution of example 4 at different fractional order β.

$\beta = 0.3$ respectively. In conclusion, the graphical representations have confirmed the accuracy and reliability of the suggested technique.

VI. CONCLUSION

It is concluded that M.T with decomposition method is a new developed technique to solve various fractional telegraph models. The procedure of the proposed method is quite new and effective for both linear and nonlinear fractional differential equations. To investigate the validity of the presented method, few examples of telegraph equation are considered for analytical solutions. It is observed that the current method provides the closed form solutions, which are fastly convergent toward the exact solution of the problems. Moreover, the suggested method has a relatively small number of calculations as compared to the other contemporary techniques available in literature. Thus it may be considered one of the best analytical tools to solve fractional partial differential equations.

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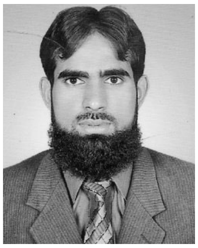

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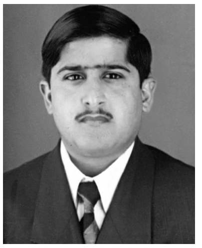

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