A mathematical model for simulation of a water table profile between two parallel subsurface drains using fractional derivatives

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ABSTRACT

By considering the initial and boundary conditions corresponding to parallel subsurface drains, the linear form of a one-dimensional fractional Boussinesq equation was solved and an analytical mathematical model was developed to predict the water table profile between two parallel subsurface drains. The developed model is a generalization of the Glover–Dumm's mathematical model. As a result, the new model is applicable for both homogeneous and heterogeneous soils. It considers the degree of heterogeneity of soil as a determinable parameter. This parameter was called the heterogeneity index. The laboratory and field tests were conducted to study the performance of the proposed mathematical model in a homogenous soil and in an agricultural soil. The optimal values of parameters of the fractional model developed in this study and Glover–Dumm's model were estimated using the inverse problem method. In the proposed inverse model, a bees algorithm (BA) was used. The results showed that in the homogenous soil, the heterogeneity index was nearly equal to 2 and therefore, the developed mathematical model reduced to the Glover–Dumm's mathematical model. The heterogeneity index of the experimental field soil considered was equal to 1.04; hence, this soil was classified as a very heterogeneous soil. In the experimental field soil, the proposed mathematical model better represented the water table profile between two parallel subsurface drains than the Glover–Dumm's mathematical model. Therefore, it appears that the proposed fractional model presented is a highly general and effective method to estimate the water table profile between two parallel subsurface drains, and the scale effects are robustly reflected by the introduced heterogeneity index.

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1. Introduction

Predicting fluctuations of a water table is very important from an agricultural and environmental perspective. Groundwater flow in an unconfined aquifer can be simulated using the Boussinesq equation. The Boussinesq equation is given by [1]:

$$\frac{\partial}{\partial x}\left(K_x h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h \frac{\partial h}{\partial y}\right) + N = S_y \frac{\partial h}{\partial t}, \tag{1}$$

where K_x is the saturated hydraulic conductivity in the x direction (L/T), K_y is the saturated hydraulic conductivity in the y direction (L/T), h is the hydraulic head (L), S_y is the specific yield (dimensionless), and N is the recharge rate or discharge rate (L/T).

Further applications of this equation to the most non-steady drainage equations were reported (e.g., [2–7]). For more details about Boussinesq and its limitations we refer to [8,9].

The fractional derivatives were used extensively in the last years to improve the existing models describing the porous media.

The Caputo space-fractional derivative of order α for a and $x > a$ is defined as follows [10–12]:

$$\left(D_a^\alpha f\right)(x, t) = \frac{1}{\Gamma(m-\alpha)} \cdot \int_a^x \frac{f^{(m)}(y, t)}{(x-y)^{\alpha-m+1}} dy, \quad m-1 < \alpha \le m, \tag{2}$$

where m is the smallest integer greater than α, and $\Gamma(\cdot)$ is the Gamma function. The fractional Taylor series is a generalization of the Taylor series. The fractional Taylor series of $f(x)$ at the point $x + \Delta x$ in the Caputo sense is defined as [12]:

$$f(x + \Delta x) = f(x) + \frac{\partial^\alpha f(x)}{\partial x^\alpha} \cdot \frac{\Delta x^\alpha}{\Gamma(1+\alpha)} + \frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\alpha f(x)}{\partial x^\alpha}\right) \cdot \frac{\Delta x^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots, \tag{3}$$

where $0 < \alpha \le 1$.

One of the most important properties of fractional derivatives, in contrast to integer-order derivatives, is their property of non-locality (see for more details Refs. [13–19]).

Recently, the fractional derivatives have been used in many diverse fields of science and engineering. In hydrogeology, they appear in the simulation of groundwater flow and solute transport in groundwater (see for example Refs. [17,20–24,18] and the references therein). Wheatcraft and Meerschaert [19] suggested a fractional mass conservation by assuming power-law changes of flux through the control volume and using a fractional Taylor series. Barker [25] proposed a generalized radial flow (GRF) model for hydraulic tests in fractured formations. Barker [25] took into account the flow dimension as a parameter that includes arbitrary values in the range [1, 3]. He [26] presented a generalized Darcy's law using fractional derivatives:

$$v = -K_x \frac{\partial^\eta h}{\partial x^\eta} \quad 0 < \eta \le 1. \tag{4}$$

He [26] did not define the parameters of Eq. (4). One can explain them as follow:

v is the velocity of groundwater flow (L/T), K_x is the fractional hydraulic conductivity (L^η/T), η ($\eta \in (0, 1]$) is the order of differentiation that indicates the degree of heterogeneity (dimensionless), and h is the hydraulic head (L).

Cloot and Botha [27] used the generalized Darcy's law and the law of conservation of mass to derive a new equation for groundwater radial flow. Gehlhausen [28] evaluated a fractional Theis solution with aquifer test data from the Campus Test Site at the University of the Free State, South Africa. The results indicated the necessity of using the fractional Theis equation for one-dimensional flow.

In this paper, an analytical mathematical model is derived by solving the linear fractional Boussinesq equation for the initial and boundary conditions corresponding to parallel subsurface drains. The parameters of the derived mathematical model and Glover–Dumm's mathematical model are estimated using the inverse problem method. In addition, the accuracy of the obtained mathematical model for simulation of the water table profile between two subsurface drains in a homogenous soil and in an agricultural soil is studied and compared with the accuracy of Glover–Dumm's model.

2. Theoretical development

2.1. The model

To develop the fractional Boussinesq equation, consider the fluid mass conservation for the control volume bounded by vertical surfaces at x, $x + \Delta x$, y and $y + \Delta y$ as shown in Fig. 1 [29].

If the variation of h relative to the value of h is infinitesimal, one can consider that the average saturated thickness is equal to a constant value and derive a linear fractional Boussinesq equation [29]:

$$\kappa_x \cdot D \frac{\partial^\nu h}{\partial x^\nu} + \kappa_y \cdot D \frac{\partial^\mu h}{\partial y^\mu} + N = S_y \frac{\partial h}{\partial t}. \tag{5}$$
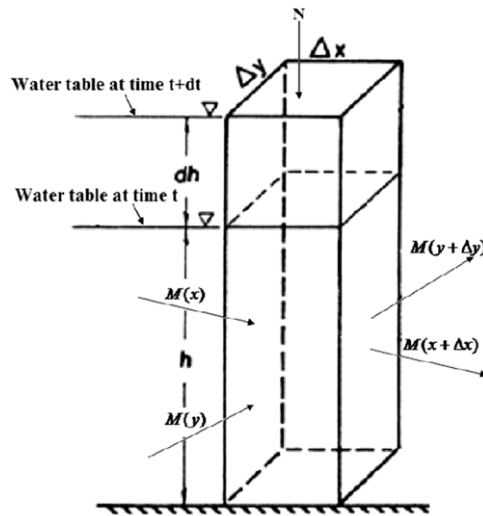
Fig. 1. Control volume in an unconfined aquifer.

Eq. (5) is the linear fractional Boussinesq equation flow in a heterogeneous and an anisotropic aquifer. Because the differentiation order indicates the degree of heterogeneity [16], in this paper, ν is called the heterogeneity index in x direction and μ is called the heterogeneity index in y direction. D is the average saturated thickness (L), $\nu = \alpha_1 + \alpha_2$ and $\nu \in (1, 2]$, $\mu = \beta_1 + \beta_2$ and $\mu \in (1, 2]$. Eq. (5) can be written as [29]:

$$C_\nu \cdot \frac{\partial^\nu h}{\partial x^\nu} + C_\mu \cdot \frac{\partial^\mu h}{\partial y^\mu} + N = \frac{\partial h}{\partial t}, \tag{6}$$

where $C_\nu = \frac{\kappa_x \cdot D}{S_y}$ is the fractional hydraulic dispersion coefficient in the x direction (L^ν/T), and $C_\mu = \frac{\kappa_y \cdot D}{S_y}$ is the fractional hydraulic dispersion coefficient in the y direction (L^μ/T).

In the following, the linear fractional Boussinesq equation is solved for one-dimensional transient flow toward parallel subsurface drains and an analytical mathematical model is presented to simulate the water table profile between two parallel subsurface drains. To this end, the following assumptions are considered:

1. Flow toward subsurface drains is horizontal. To account for the radial flow near the drain, the actual depth of the impermeable layer below drain (D) is replaced by the Hooghoudt's equivalent depth (d_e).
2. Unsaturated flow above the water table is neglected.
3. The initial water table is flat.
4. Recharge occurs instantaneously and the water table rises suddenly.
5. There is a horizontal impermeable layer at a constant depth below drains.
6. The subsurface drains have an equal spacing and lie in a parallel manner above the impermeable layer.

Except for the assumption of soil homogeneity used to develop Glover–Dumm's mathematical model, the other applied assumptions for derivation of mathematical model are the same as those used in Glover–Dumm's mathematical model. For one-dimensional transient flow, the linear fractional Boussinesq equation is in the following form:

$$\frac{\partial h(x, t)}{\partial t} = C_\nu \cdot \frac{\partial^\nu h(x, t)}{\partial x^\nu}, \tag{7}$$

where C_ν is equal to $\frac{\kappa_x \cdot d_e}{S_y}$. As indicated in Fig. 2, the initial and boundary conditions for solving the one-dimensional linear fractional Boussinesq equation in the subsurface drains are as follows:

$$h(x, 0) = h_0, \quad 0 \le x \le L, \tag{8a}$$

$$h(0, t) = 0, \tag{8b}$$

$$h(L, t) = 0. \tag{8c}$$

The analytical solution of Eq. (7) for the initial and boundary conditions as given by Eq. (8a) through Eq. (8c) is obtained using a spectral representation of the fractional derivative [30,31]. To this end, the eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are nonzero constant multiples of $h_n(x) = \sin\left(\frac{n\pi}{L}x\right)$. Therefore, one can write $h(x, t)$ in the following form:

$$h(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{n\pi}{L}x\right). \tag{9}$$
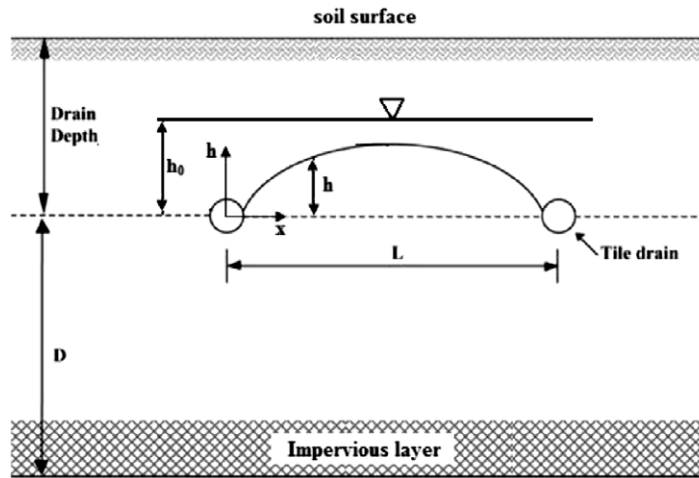
Fig. 2. Considered subsurface drains in this manuscript.

Eq. (9) automatically satisfies the boundary conditions. For function $h(x, t)$, the operator $(-\Delta)^{\frac{\nu}{2}}$ is defined as follows [30]:

$$(-\Delta)^{\frac{\nu}{2}} h(x, t) = -\frac{\partial^\nu h(x, t)}{\partial x^\nu} = \sum_{n=1}^{\infty} b_n(t) \cdot \lambda_n^{\frac{\nu}{2}} \sin\left(\frac{n\pi}{L}x\right). \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (7), we obtained:

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \cdot \sin\left(\frac{n\pi}{L}x\right) = -C_\nu \sum_{n=1}^{\infty} b_n(t) \lambda_n^{\frac{\nu}{2}} \sin\left(\frac{n\pi}{L}x\right). \tag{11}$$

The problem for $b_n(t)$ becomes an ordinary differential equation:

$$\frac{db_n(t)}{dt} + C_\nu \lambda_n^{\frac{\nu}{2}} b_n(t) = 0. \tag{12}$$

The general solution of Eq. (12) is in the following:

$$b_n(t) = b_n(0) \exp\left(-C_\nu (\lambda_n)^{\frac{\nu}{2}} t\right). \tag{13}$$

To obtain $b_n(0)$, the initial condition is used (Eq. (8a)):

$$h(x, 0) = h_0 = \sum_{n=1}^{\infty} b_n(0) \sin\left(\frac{n\pi}{L}x\right). \tag{14}$$

From Eq. (14), it is deduced that:

$$b_n(0) = \frac{2}{L} \int_0^L h_0 \sin\left(\frac{n\pi}{L}\xi\right) d\xi = \frac{2h_0}{n\pi}(1 - \cos(n\pi)). \tag{15}$$

Substituting $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $b_n(0) = \frac{2h_0}{n\pi}(1 - \cos(n\pi))$ into (13), it is obtained that:

$$b_n(t) = \frac{2h_0}{n\pi}(1 - \cos(n\pi)) \exp\left(-C_\nu \left(\frac{n\pi}{L}\right)^\nu t\right). \tag{16}$$

Substituting $b_n(t)$ into Eq. (9), the solution of Eq. (7) is obtained:

$$h(x, t) = \frac{4h_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \exp\left(-C_\nu \left(\frac{n\pi}{L}\right)^\nu t\right) \sin\left(\frac{n\pi}{L}x\right). \tag{17}$$

Eq. (17) is a new equation for predicting the water table profile between two parallel subsurface drains under unsteady state conditions. This equation is applicable for both homogeneous and heterogeneous soils. When $\nu \to 2$, Eq. (17) reduces to Glover–Dumm's mathematical model which was developed by assuming the homogeneity of soil [3]:

$$h(x, t) = \frac{4h_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \exp\left(-C_2 \left(\frac{n\pi}{L}\right)^2 t\right) \sin\left(\frac{n\pi}{L}x\right). \tag{18}$$

2.2. Development of inverse models

Eq. (17) has two parameters, the heterogeneity index (ν) and the fractional hydraulic dispersion coefficient (C_ν), and Eq. (18) has a parameter, the hydraulic dispersion coefficient (C_2). In this paper, the parameters of two mathematical models are estimated using inverse problem method. To achieve this aim, two inverse models are developed. In the inverse models developed, the water table height data between two parallel subsurface drains and the optimization method of the bees algorithm are used.

The bees algorithm (BA) was first presented in [32]. This algorithm imitates the foraging behavior of honeybees to solve an optimization problem. The BA was completely described in [32]. The pseudo code of the basic BA is given as follows [32]:

1. Initialize population with random solutions.
2. Evaluate the fitness of the population.
3. While (stopping criterion not met)
    // forming new population.
4. Select the elite sites for neighborhood search.
5. Select other sites for neighborhood search.
6. Determine the patch size.
7. Recruit bees around selected sites (more bees for the elite sites) and evaluate fitness.
8. Select the fittest bee from each patch.
9. Assign remaining bees to search randomly and evaluate their fitness.
10. End While.

BA requires a number of parameters to be set, namely: number of scout bees (n), number of sites selected out of n visited sites (m), number of elite sites out of m selected sites (e), number of bees recruited for the elite sites (n_e), number of bees recruited for the other ($m - e$) selected sites (n_s), size of patches (n_{gh}) and the stopping criterion.

The proper values of parameters of the BA are determined using trial and error. The objective function, applied in the inverse models, is considered as follows:

$$S = \sum_{i=1}^{N} \left(h_i - h_i' \right)^2 , \qquad (19)$$

where h_i is the observed height of the water table above drain at the ith point, h_i' is the calculated height of the water table above drain at the ith point and N is the number of points used to evaluate the objective function.

3. Material and methods

3.1. Laboratory experiment

The performance of the developed mathematical model for simulation of the water table profile between two parallel subsurface drains in a homogenous soil is investigated. In this spirit, a sand tank with inside dimensions of 200 cm length $\times$ 50 cm width $\times$ 110 cm height was made of 3 mm thick steel sheets. A corrugated plastic drainpipe with inside diameter of 10 cm lengthwise, was installed along one of the narrow ends of the sand tank at a depth of 80 cm below the top of the sand tank. The drainpipe was wrapped with a synthetic envelope of PP450. A control valve was installed on the outside of the drainpipe to stop the exit of water from the drainpipe outlet during the soil saturation. In front of the wall of the sand tank, a set of piezometers with diameter of 1 cm were installed at 7, 22, 37, 52, 67, 86, 105, 124, 143, 162 and 184 cm horizontal spacing from the drainpipe. All the piezometers were inserted up to the middle of the sand tank to remove any seepage effect along the sand tank walls. There were three intake valves at the bottom of the sand tank. A variable head water supply tank fed from a water storage reservoir was connected to the intake valves. Fig. 3 shows the experimental setup.

The sand tank was filled to 100 cm height with very uniform sand in an effort to minimize heterogeneity. The particle size distribution curve of sand used for the laboratory experiment has been shown in Fig. 4.

The sand in the sand tank was saturated slowly from the bottom upward to remove the air in the sand. When the entire sand column was saturated, the intake valves at the bottom of the sand tank were closed and the variable head water supply tank was disconnected from the sand tank. Then, the intake valves at the bottom of the sand tank were opened and the sand tank was allowed to drain. The saturating and draining cycles were repeated several times.

To collect data, the sand in the sand tank was saturated again from the bottom upward. When the water appeared on the sand surface, the variable head water supply tank was cut off. Afterward the drainpipe outlet was opened and the hydraulic heads were recorded using piezometers. Up to 30 min after the start of drainage, the time interval of the measurement of hydraulic heads was 2 min. From 30 min to time 60 min, the hydraulic heads were recorded every three minutes. In the final stage of drainage, from 60 min to 120 min, the measurements were made every thirty minutes. The initial height of the water table above drain (h_0) was 71 cm. From the measured hydraulic head distribution in the sand, the water table profile between two parallel subsurface drains was obtained at a given time. The water table profiles corresponding to times $t = 4, 10, 16, 30, 45, 60, 90$, and 120 min after the start of drainage were used to estimate the parameters of mathematical models.
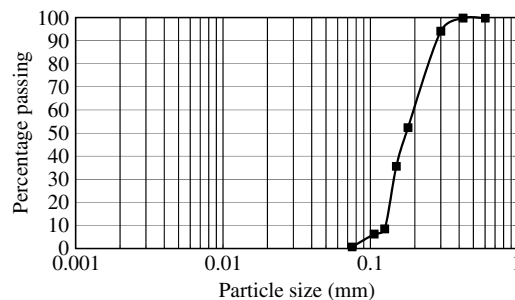
Fig. 3. Experimental setup.

Fig. 4. Particle size distribution curve for the sand used.

Table 1
Drainage system characteristics at experimental field.

Parameter	Field data
Drain depth (m)	1.3
Drain spacing (m)	30
Depth to impervious layer (m)	1.5
Radius of drain (m)	0.1
Drain material	Corrugated PVC drainpipe
Envelope	PP450

3.2. Field experiment

To evaluate the performance of the proposed mathematical model in the field conditions, the water table profiles between two parallel subsurface drains installed in an experimental field in Abadan, Iran (30°1′49.8″ N latitude and 48°29′5.27″ E longitude) were measured. The area of this experimental field was 12 hectares. The soil of the experimental field was silty clay in texture. Brief information on the subsurface drainage system installed in the experimental field is reported in Table 1.

The observation wells, equal to the depth of the drains, were installed to measure the water table height to determine the water table profiles. The water table, in addition to the drain and at mid spacing, was also measured at 0.5, 1.5 and 5 m from the drain. The data from the water table were measured for a period of 10 days at an interval of 1 day. The parameters of mathematical models were estimated using data from the water table at times $t = 2$, 4, and 6 days after the beginning of drainage.

3.3. Evaluation of mathematical models

The water table profiles between two parallel subsurface drains at various times were simulated using the proposed mathematical model and the Glover–Dumm's mathematical model. To evaluate the performance of two mathematical models considered, the graphical displays and the statistical criteria were applied. In this paper, the two methods of graphical

Table 2
Estimated parameters of two mathematical models.

Soil type	Glover–Dumm's mathematical model	Proposed mathematical model	
	C_2 (m^2/s)	v	C_v (m^v/s)
Homogenous soil	1.96×10^{-4}	1.99	1.98×10^{-4}
Experimental field soil	3.83×10^{-6}	1.04	3.28×10^{-6}

display were used for evaluation of two mathematical models considered [33]: (1) comparison of observed and predicted water table profiles; and (2) comparison of matched simulated and observed integrated values. The statistical criteria used to evaluate the performance of two mathematical models were as follows [33]:

Maximum error (ME),

$$\text{ME} = Max \left| h_i - h_i' \right|_{i=1}^{n}. \tag{20}$$

Root mean square error (RMSE),

$$\text{RMSE} = \left[\frac{\sum_{i=1}^{N} (h_i' - h_i)^2}{N} \right]^{\frac{1}{2}} \cdot \frac{100}{\bar{h}}. \tag{21}$$

Coefficient of residual mass (CRM),

$$\text{CRM} = \frac{\sum_{i=1}^{N} h_i' - \sum_{i=1}^{N} h_i}{\sum_{i=1}^{N} h_i'}. \tag{22}$$

Coefficient of determination (CD),

$$\text{CD} = \frac{\sum_{i=1}^{N} \left(h_i - \bar{h} \right)^2}{\sum_{i=1}^{N} \left(h_i' - \bar{h} \right)^2}. \tag{23}$$

Modeling efficiency (EF),

$$\text{EF} = \frac{\sum_{i=1}^{N} (h_i - \bar{h})^2 - \sum_{i=1}^{N} (h_i' - h_i)^2}{\sum_{i=1}^{N} (h_i - \bar{h})^2}, \tag{24}$$

where h_i are the observed values, h_i' are the simulated values, $\bar{h}$ is the mean of the observed data, and N is the number of observational data points.

4. Results and discussion

4.1. Estimation of parameters of mathematical models

The optimal values of the parameters of two mathematical models for two soil types (homogeneous soil and experimental field soil) are shown in Table 2.

The results of the calibration of mathematical models for two soil types indicate that: (1) the heterogeneity index of homogenous soil is very close to 2; (2) the proposed mathematical model reduces to Glover–Dumm's mathematical model for the homogeneous soil; and (3) according to the obtained value for heterogeneity index of the experimental field soil and the classification of Clarke et al. [16], this soil is very heterogeneous.

4.2. Performance of mathematical models

4.2.1. Homogeneous soil

The simulated results of two mathematical models considered in the homogeneous soil are shown in Figs. 5 and 6 and the corresponding values of statistical criteria are listed in Table 3.
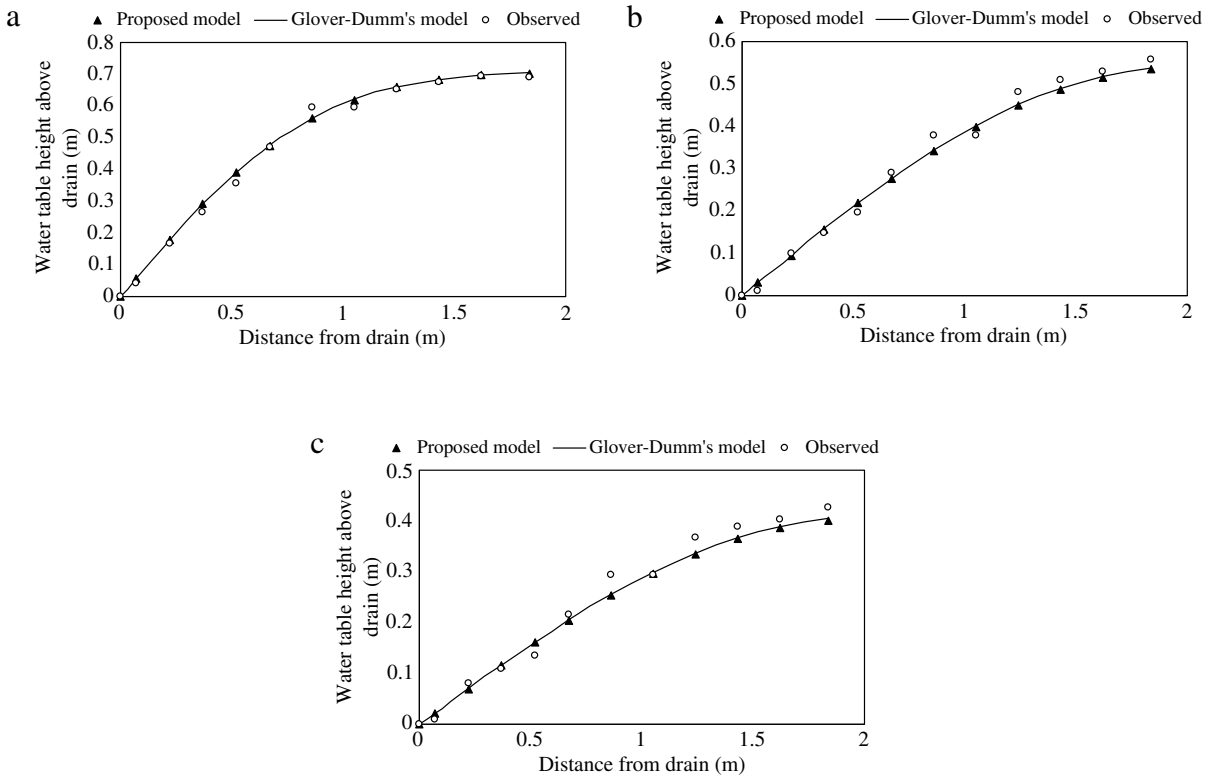
Fig. 5. Comparison of water table profile between two drains simulated by proposed mathematical model and Glover–Dumm's mathematical model at times: (a) $t = 20$ min; (b) $t = 70$ min and (c) $t = 110$ min after beginning of drainage.
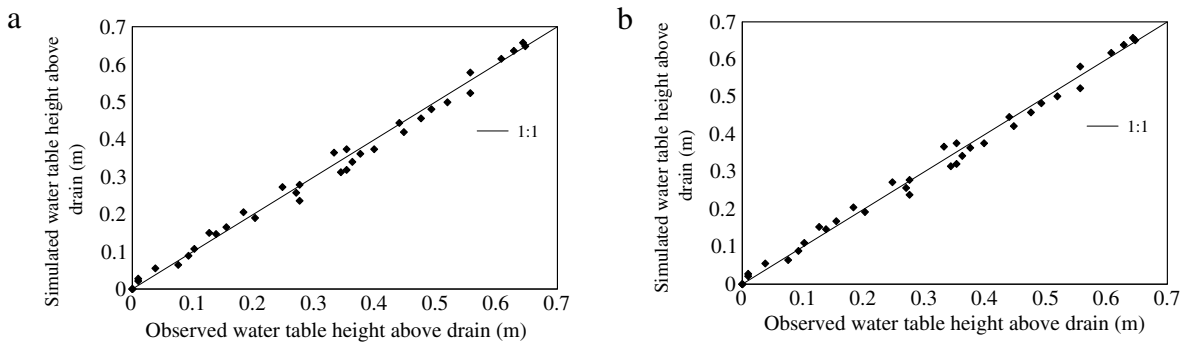

Fig. 6. Observed versus predicted water table height above drain. The water table height above drain was predicted by (a) the proposed mathematical model, and (b) the Glover–Dumm's mathematical model. The line represents the potential 1:1 relationship between the data sets.

Table 3
The statistical criteria values of two mathematical models in the homogeneous soil.

Glover–Dumm's mathematical model					Proposed mathematical model				
ME (cm)	RMSE (%)	CRM	CD	EF	ME (cm)	RMSE (%)	CRM	CD	EF
7.24	5.09	−0.016	1.02	0.99	7.19	5.11	−0.012	1.02	0.99

From Fig. 5, it can be found that the simulated results by both mathematical models are nearly coincident. Fig. 6 indicates that the simulation results of two mathematical models are well correlated with the measured data. The statistical criteria corresponding to two mathematical models also indicate that both mathematical models have a similar performance and a minor error (Table 3).

The similar performance of two mathematical models is due to the homogeneity of soil used in the sand tank. Indeed, the obtained results in the homogenous soil justify practically that the proposed mathematical model reduces to
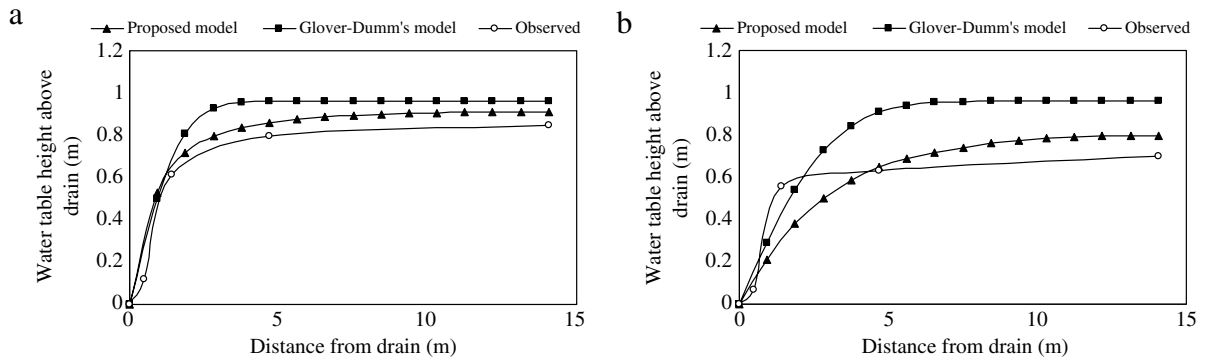
Fig. 7. Comparison of water table profile between two drains simulated by the proposed mathematical model and Glover–Dumm's mathematical model at times: (a) $t = 3$ days and (b) $t = 10$ days after the beginning of drainage.
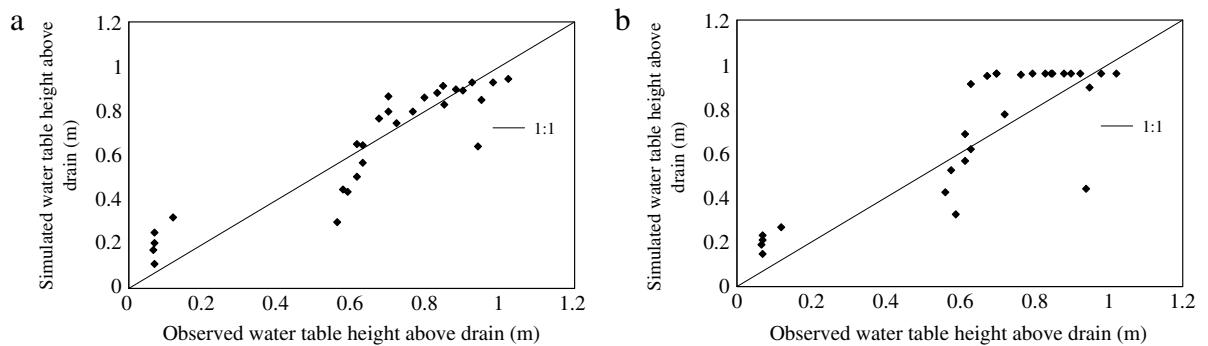

Fig. 8. Observed versus predicted water table height above drain. The water table height above drain was predicted by (a) the proposed mathematical model, and (b) Glover–Dumm's mathematical model. The line represents the potential 1:1 relationship between the data sets.

Table 4
The statistical criteria values of two mathematical models in the experimental field soil.

Glover–Dumm's mathematical model					Proposed mathematical model				
ME (cm)	RMSE (%)	CRM	CD	EF	ME (cm)	RMSE (%)	CRM	CD	EF
27.99	26.63	−0.16	0.78	0.69	25.95	18.54	−0.04	1.12	0.85

Glover–Dumm's mathematical model in the homogenous soil. The satisfactory performance of two mathematical models considered comes out from the validity of most assumptions applied to develop the mathematical models (see Section 2.2).

4.2.2. Experimental field soil

Figs. 7 and 8 illustrate the simulated results by the Glover–Dumm's mathematical model and the proposed mathematical model in the experimental field soil. Moreover, the values of statistical criteria corresponding to two mathematical models are presented in Table 4.

As shown in Figs. 7 and 8, compared to the Glover–Dumm's model, the simulation results of the proposed fractional model are very close to the observed results. The corresponding values of statistical criteria also indicate that the fractional model has a better performance than Glover–Dumm's mathematical model (Table 4).

The better performance of the proposed model comes from the heterogeneity of the experimental field soil. Glover–Dumm's model assumes that the soil is homogenous, while the proposed one considers the degree of heterogeneity of soil as a determinable parameter. In this study, this parameter was called the heterogeneity index (see Section 2.1). Therefore, the fractional model has fewer limitations and better performance than Glover–Dumm's model.

The graphical displays (see Figs. 7 and 8) and the values of statistical criteria (see Table 4) show that both models predict the water table height above the drain with an error. Fig. 8 shows that the simulation results of both models are somewhat larger than measured results. This result can be also indicated by the negative CRM values of mathematical models (Table 4). We believe that the major source of this error is the assumptions used for derivation of the mathematical models (for details, see Section 2.2).

5. Conclusions

An analytical model for simulating the water table profile between two parallel subsurfaces was derived by solving the linear fractional Boussinesq equation for one-dimensional transient flow toward subsurface drains. The developed model is a generalization of Glover–Dumm's mathematical model. The proposed model is applicable for both homogeneous and heterogeneous soils. In the proposed mathematical model, there is a parameter of heterogeneity index (ν) to account for the degree of heterogeneity of the soil. The laboratory and field experiments indicated that: (1) the heterogeneity index of homogenous soil was nearly equal to 2; (2) in the homogenous soil, the proposed fractional model reduced to the Glover–Dumm's mathematical model; (3) the heterogeneity index of experimental field soil considered was 1.04; (4) the value of the heterogeneity index of experimental field soil showed that this soil is very heterogeneous; and (5) in the experimental field soil, the proposed mathematical model could better predict the water table profile between two parallel subsurface drains than Glover–Dumm's model.

Therefore, it appears that the proposed model can be used as a highly general and effective approach to simulate the water table profile between two parallel subsurface drains, and the introduced heterogeneity index robustly reflects the scale effects.

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