Computers and Mathematics with Applications 66 (2013) 838–843

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A new approach for solving a system of fractional partial differential equations

H. Jafari [a,b,*], M. Nazari [b], D. Baleanu [c,d,e], C.M. Khalique [a]

[a] *International Institute for Symmetry Analysis and Mathematical Modeling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, South Africa*

[b] *Department of Mathematics and Computer Science, University of Mazandaran, Babolsar, Iran*

[c] *Department of Chemical and Materials Engineering, Faculty of Engineering, King Abdulaziz University, P.O. Box: 80204, Jeddah, 21589, Saudi Arabia*

[d] *Department of Mathematics and Computer Sciences, Faculty of Art and Sciences, Cankaya University, Ankara, Turkey*

[e] *Institute of Space Sciences, Magurele-Bucharest, Romania*

ARTICLE INFO

Keywords:
Iterative method
Laplace transform method
Fractional partial differential equations

ABSTRACT

In this paper we propose a new method for solving systems of linear and nonlinear fractional partial differential equations. This method is a combination of the Laplace transform method and the Iterative method and here after called the Iterative Laplace transform method. This method gives solutions without any discretization and restrictive assumptions. The method is free from round-off errors and as a result the numerical computations are reduced. The fractional derivative is described in the Caputo sense. Finally, numerical examples are presented to illustrate the preciseness and effectiveness of the new technique.

1. Introduction

Many problems of mathematical physics and engineering such as polymer physics, viscoelastic materials, viscous damping and seismic analysis [1–4] have been successfully modeled in recent years by fractional differential equations (FDEs). So it is very important to find efficient methods for solving FDEs. Various researchers have introduced new methods in the literature. These methods include the Adomian decomposition method (ADM) [5,6], homotopy analysis method (HAM) [7,8], homotopy perturbation method (HPM) [9,10], the variational iteration method (VIM) [6,11,12] and the Laplace decomposition method [13–15].

Recently, a new iterative method was presented by Daftardar-Gejji and Jafari [16,17]. This technique solves many types of nonlinear equations such as ordinary and partial differential equations of integer and fractional order. Jafari et al. [18] applied this method to obtain the solution of linear/nonlinear diffusion and wave fractional equations. Daftardar-Gejji and Bhalekar used it for solving fractional boundary value problem and evolution equations [19,20].

In this paper, we introduce a new method, which we call the Iterative Laplace transform method (ILTM). The suggested ILTM provides the solution in a rapid convergent series which may lead us to the solution in a closed form. This method combines the two powerful methods, namely, the Laplace transform method and the Iterative method, for obtaining the exact solution for the system of fractional partial differential equations. It is worth mentioning that the ILTM is applied without any discretization or restrictive assumptions or transformations and it is free from round-off errors. Also this method provides an analytical solution by using the initial conditions only, unlike the variables separation method, which requires initial and boundary conditions. The boundary conditions can be used to justify the obtained results. The proposed method

* Corresponding author at: Department of Mathematics and Computer Science, University of Mazandaran, Babolsar, Iran. Tel.: +98 9112113567.
E-mail addresses: jafari_h@math.com, jafari@umz.ac.ir (H. Jafari).

0898-1221/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2012.11.014

works efficiently and the results so far are very encouraging and reliable. In this paper we employ the ILTM in solving systems of nonlinear fractional partial differential equations. Several examples are given to verify the reliability and efficiency of the ILTM. The results are then compared with those obtained by other existing methods.

2. Basic definition

In this section, we recall some basic definitions and results dealing with the fractional calculus [2–4] and Laplace transform which are later used in this paper.

Definition 1. A real function $f(t),\ t>0$ is said to be in the space $C_\alpha, \alpha\in\Re$ if there exists a real number $p(>\alpha)$, such that $f(t)=t^p f_1(t)$ where $f_1\in C[0,\infty]$. Clearly $C_\alpha\subset C_\beta$ if $\beta\le\alpha$.

Definition 2. A function $f(t),\ t>0$ is said to be in the space C_α^m, $m\in N\bigcup\{0\}$, if $f^{(m)}\in C_\alpha$.

Definition 3 ([2]). The left sided Riemann–Liouville fractional integral of order $\mu\ge 0$, of a function $f\in C_\alpha,\ \alpha\ge -1$ is defined as

$$I^\mu f(t)=\begin{cases}\dfrac{1}{\Gamma(\mu)}\displaystyle\int_0^t\frac{f(\tau)}{(t-\tau)^{1-\mu}}d\tau, & \mu>0, t>0,\\ f(t), & \mu=0.\end{cases}\tag{1}$$

Definition 4 ([2]). The left sided Caputo fractional derivative of f, $f\in C_{-1}^m,\ m\in N\cup\{0\}$, is defined as

$$D^\mu f(t)=\frac{\partial^\mu f(t)}{\partial t^\mu}=\begin{cases}I^{m-\mu}\left[\dfrac{\partial^m f(t)}{\partial t^m}\right], & m-1<\mu<m, m\in N,\\ \dfrac{\partial^m f(t)}{\partial t^m}, & \mu=m.\end{cases}\tag{2}$$

Note that

(i) $I_t^\mu f(x,t)=\frac{1}{\Gamma(\mu)}\int_0^t\frac{f(x,t)}{(t-s)^{1-\mu}},\ \mu>0,\ t>0,$

(ii) $D_t^\mu f(x,t)=I_t^{m-\mu}\frac{\partial^m f(x,t)}{\partial t^m},\ m-1<\mu\le m.$

Definition 5. The Mittag-Leffler function $E_\alpha(z)$ with $\alpha>0$ is defined by the following series representation, valid in the whole complex plane:

$$E_\mu(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\mu n+1)},\quad \alpha>0,\ z\in\mathbb{C}.$$

Definition 6. The Laplace transform of $f(t)$ is defined by

$$F(s)=L[f(t)]=\int_0^\infty e^{-st}f(t)dt.\tag{3}$$

Definition 7. The Laplace transform $L[f(t)]$ of the Riemann–Liouville fractional integral is defined as

$$L\{I^\mu f(t)\}=s^{-\mu}F(s).\tag{4}$$

Definition 8. The Laplace transform $L[f(t)]$, of the Caputo fractional derivative is defined as

$$L\{D^\mu f(t)\}=s^\mu F(s)-\sum_{k=0}^{n-1}s^{(\mu-k-1)}f^{(k)}(0),\quad n-1<\mu\le n.\tag{5}$$

3. Iterative Laplace transform method and system of fractional partial differential equations

To illustrate the basic idea of this method, we consider the following system of fractional partial differential equations (FPDEs) with the initial conditions of the form:

$$D_t^{\alpha_i}u_i(\bar{x},t)=A_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t)),\quad m_i-1<\alpha_i\le m_i,\quad i=1,2,\ldots,n,\tag{6}$$

$$\frac{\partial^{(k_i)}u_i(\bar{x},0)}{\partial t^{k_i}}=h_{ik_i}(\bar{x}),\quad k_i=0,1,\ldots,m_i-1,\ m_i\in N,\tag{7}$$

where A_i are nonlinear operators and $u_i(\bar{x},t)$ are unknown functions. Taking the Laplace transform (denoted in this paper by L) on both sides of Eq. (6) we obtain

$$L\left[D_t^{\alpha_i} u_i(\bar{x},t)\right] = L[A_i\,(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))], \quad i=1,2,\ldots,n.$$

In view of Definition 8 and the initial conditions (7) we have

$$s^{\alpha_i} L[u_i(\bar{x},t)] - \sum_{k=0}^{m_i-1} s^{\alpha_i-k-1} u_i^{(k)}(\bar{x},0) = L[A_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))], \quad i=1,2,\ldots,n. \tag{8}$$

Operating with the Laplace inverse on both sides of Eq. (8) we get

$$\begin{aligned} u_i(\bar{x},t) &= L^{-1}\left[\sum_{k=0}^{m_i-1} s^{-k-1} u_i^{(k)}(\bar{x},0)\right] + L^{-1}[s^{-\alpha_i} L[A_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))]] \\ &= f_i + N_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t)), \quad i=1,2,\ldots,n, \end{aligned} \tag{9}$$

which can be rewritten in the form

$$u_i(\bar{x},t) = f_i + N_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t)), \quad i=1,2,\ldots,n, \tag{10}$$

where

$$f_i = L^{-1}\left[\sum_{k=0}^{m_i-1} s^{-k-1} u_i^{(k)}(\bar{x},0)\right], \quad i=1,2,\ldots,n,$$

$$N_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t)) = L^{-1}[s^{-\alpha_i} L[A_i(u_1(\bar{x},t),\ldots,u_n(\bar{x},t))]].$$

We now look for a solution u of Eq. (10) having the series form

$$u_i(\bar{x},t) = \sum_{j=0}^{\infty} u_{ij}(\bar{x},t), \quad i=1,2,\ldots,n. \tag{11}$$

The nonlinear operators N_i can be decomposed as

$$\begin{aligned} N_i\left(\sum_{j=0}^{\infty} u_{1j}(\bar{x},t),\ldots,\sum_{j=0}^{\infty} u_{nj}(\bar{x},t)\right) &= N_i(u_{10}(\bar{x},t),\ldots,u_{n0}(\bar{x},t)) \\ &\quad + \sum_{j=1}^{\infty}\left\{N_i\left(\sum_{k=0}^{j} u_{1k}(\bar{x},t),\ldots,\sum_{k=0}^{j} u_{nk}(\bar{x},t)\right)\right. \\ &\quad \left. - N_i\left(\sum_{k=0}^{j-1} u_{1k}(\bar{x},t),\ldots,\sum_{k=0}^{j-1} u_{nk}(\bar{x},t)\right)\right\}. \end{aligned} \tag{12}$$

In view of Eqs. (11) and (12), Eq. (10) is equivalent to

$$\begin{aligned} \sum_{j=0}^{\infty} u_{ij}(\bar{x}) &= f_i + N_i(u_{10}(\bar{x},t),\ldots,u_{n0}(\bar{x},t)) \\ &\quad + \sum_{j=1}^{\infty}\left\{N_i\left(\sum_{k=0}^{j} u_{1k}(\bar{x},t),\ldots,\sum_{k=0}^{j} u_{nk}(\bar{x},t)\right) - N_i\left(\sum_{k=0}^{j-1} u_{1k}(\bar{x},t),\ldots,\sum_{k=0}^{j-1} u_{nk}(\bar{x},t)\right)\right\}. \end{aligned} \tag{13}$$

We define the recurrence relation

$$\begin{cases} u_{i0}(\bar{x},t) = L^{-1}\left[\sum_{k=0}^{m_i-1} s^{-k-1} u_i^{(k)}(\bar{x},0)\right], \\ u_{i1}(\bar{x},t) = L^{-1}[s^{-\alpha_i} L[A_i(u_{10}(\bar{x},t),\ldots,u_{n0}(\bar{x},t))]], \\ u_{i(m+1)}(\bar{x},t) = L^{-1}[s^{-\alpha_i} L[A_i((u_{10}(\bar{x},t)+\cdots+u_{1m}(\bar{x},t)),\ldots,(u_{n0}(\bar{x},t)+\cdots+u_{nm}(\bar{x},t)))]] \\ \quad - L^{-1}[s^{-\alpha_i} L[A_i((u_{10}(\bar{x},t)+\cdots+u_{1(m-1)}(\bar{x},t)),\ldots,(u_{n0}(\bar{x},t)+\cdots+u_{n(m-1)}(\bar{x},t)))]]. \end{cases} \tag{14}$$

Then

$$u_{i1}(\bar{x},t)+\cdots+u_{im+1}(\bar{x},t) = L^{-1}[s^{-\alpha_i} L[A_i((u_{10}(\bar{x},t)+\cdots+u_{1m}(\bar{x},t)),\ldots,(u_{n0}(\bar{x},t)+\cdots+u_{nm}(\bar{x},t)))]].$$

The n-term approximate solution of (6)–(7) is given by

$$u_i(\bar{x},t) \cong u_{i1}(\bar{x},t)+\cdots+u_{i,n}(\bar{x},t), \quad i=1,2,\ldots,n.$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Jafari and Daftardar-Gejji [16,17].

4. Test examples

In this section, we illustrate the applicability of iterative Laplace transform method for solving systems of linear and nonlinear fractional partial differential equations.

Example 1. Consider the following system of linear FPDEs [7]:

$$
\begin{aligned}
&D_t^{\alpha} u - v_x + v + u = 0,\\
&D_t^{\beta} v - u_x + v + u = 0, \quad (0 < \alpha, \beta \le 1),
\end{aligned}
\tag{15}
$$

with initial conditions

$$u(x, 0) = \sinh(x), \qquad v(x, 0) = \cosh(x).$$

The exact solution, when $\alpha = \beta = 1$, is

$$u(x, t) = \sinh(x - t), \qquad v(x, t) = \cosh(x - t).$$

The system of linear FPDEs (15) corresponds to the following Laplace equations:

$$
\begin{aligned}
u(x, t) &= L^{-1}[s^{-1}u(x, 0)] + L^{-1}[s^{-\alpha}L[v_x(x, t) - v(x, t) - u(x, t)]],\\
v(x, t) &= L^{-1}[s^{-1}v(x, 0)] + L^{-1}[s^{-\beta}L[u_x(x, t) - v(x, t) - u(x, t)]].
\end{aligned}
$$

Following the algorithm given in (14) first few terms of $u(x, t)$ and $v(x, t)$ are

$$
\left\{u_0(x, t) = \sinh(x), v_0(x, t) = \cosh(x), \right. \qquad
\left\{
\begin{aligned}
u_1(x, t) &= -\frac{\cosh(x)t^{\alpha}}{\Gamma[\alpha + 1]},\\
v_1(x, t) &= -\frac{\sinh(x)t^{\beta}}{\Gamma[\beta + 1]},
\end{aligned}
\right.
$$

$$
\begin{aligned}
u_2(x, t) &= -\frac{\cosh(x)t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\sinh(x)t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\cosh(x)t^{2\alpha}}{\Gamma(2\alpha + 1)}\\
v_2(x, t) &= -\frac{\sinh(x)t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\cosh(x)t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + \frac{\sinh(x)t^{2\beta}}{\Gamma(2\beta + 1)}.
\end{aligned}
$$

The solution in series form is then given by

$$
\begin{aligned}
u(x, t) \;=\; & u_0(x, t) + u_1(x, t) + u_2(x, t) + \cdots \sinh(x)\left(1 + \frac{t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + \cdots\right)\\
& - \cosh(x)\left(\frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots\right),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
v(x, t) \;=\; & v_0(x, t) + v_1(x, t) + v_2(x, t) + \cdots = \cosh(x)\left(1 + \frac{t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + \cdots\right)\\
& - \sinh(x)\left(\frac{t^{\beta}}{\Gamma[\beta + 1]} + \frac{t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} - \frac{t^{2\beta}}{\Gamma(2\beta + 1)} + \cdots\right).
\end{aligned}
\tag{17}
$$

Setting $\alpha = \beta$ in Eqs. (16) and (17), we reproduce the solution of [7] as follows:

$$u(x, t) = \sinh(x)\left(1 + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots\right) - \cosh(x)\left(\frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots\right), \tag{18}$$

$$v(x, t) = \cosh(x)\left(1 + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots\right) - \sinh(x)\left(\frac{t^{\alpha}}{\Gamma[\alpha + 1]} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots\right). \tag{19}$$

Now setting $\alpha = 1$ in Eqs. (18) and (19), we obtain

$$
\begin{aligned}
u(x, t) &= \sinh(x)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \cosh(x)\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) = \sinh(x - t),\\
v(x, t) &= \cosh(x)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} \cdots\right) - \sinh(x)\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) = \cosh(x - t),
\end{aligned}
$$

which gives us the exact solutions of (15) for $\alpha = \beta = 1$.

Example 2. Consider the system of nonlinear FPDEs [7]:

$$\begin{aligned}
&D_t^{\alpha} u + v_x w_y - v_y w_x = -u,\\
&D_t^{\beta} v + u_x w_y + u_y w_x = v,\\
&D_t^{\gamma} w + u_x v_y + u_y v_x = w, \quad (0 < \alpha, \beta, \gamma \leq 1),
\end{aligned} \tag{20}$$

with initial conditions

$$u(x, y, 0) = e^{x+y}, \qquad v(x, y, 0) = e^{x-y}, \qquad w(x, y, 0) = e^{-x+y}. \tag{21}$$

The exact solution, when $\alpha = \beta = \gamma = 1$, is

$$u(x, y, t) = e^{x+y-t}, \qquad v(x, y, t) = e^{x-y+t}, \qquad w(x, y, t) = e^{-x+y+t}.$$

As in Example 1 above, we construct the following:

$$\begin{aligned}
u(x, y, t) &= L^{-1}[s^{-1}u(x, y, 0)] + L^{-1}[s^{-\alpha}L[-u(x, y, t) - v_x(x, y, t)w_y(x, y, t) + v_y(x, y, t)w_x(x, y, t)]],\\
v(x, y, t) &= L^{-1}[s^{-1}v(x, y, 0)] + L^{-1}[s^{-\beta}L[v(x, y, t) - u_x(x, y, t)w_y(x, y, t) - u_y(x, y, t)w_x(x, y, t)]],\\
w(x, y, t) &= L^{-1}[s^{-1}w(x, y, 0)] + L^{-1}[s^{-\gamma}L[w(x, y, t) - u_x(x, y, t)v_y(x, y, t) - u_y(x, y, t)v_x(x, y, t)]].
\end{aligned}$$

As before the first few terms of $u(x, y, t)$, $v(x, y, t)$ and $w(x, y, t)$ in this case are

$$\begin{cases}
u_0(x, y, t) = e^{x+y},\\
v_0(x, y, t) = e^{x-y},\\
w_0(x, y, t) = e^{-x+y},
\end{cases}$$

$$\begin{cases}
u_1(x, y, t) = -\dfrac{e^{x+y}t^{\alpha}}{\Gamma[\alpha+1]} - \dfrac{e^{x-y}e^{-x+y}t^{\alpha}}{\Gamma[\alpha+1]} + \dfrac{e^{x-y}e^{-x+y}t^{\alpha}}{\Gamma[\alpha+1]} = -\dfrac{e^{x+y}t^{\alpha}}{\Gamma[\alpha+1]},\\
v_1(x, y, t) = \dfrac{e^{x-y}t^{\beta}}{\Gamma[\beta+1]} - \dfrac{e^{x+y}e^{-x+y}t^{\beta}}{\Gamma[\beta+1]} + \dfrac{e^{x+y}e^{-x+y}t^{\beta}}{\Gamma[\beta+1]} = \dfrac{e^{x-y}t^{\beta}}{\Gamma[\beta+1]},\\
w_1(x, y, t) = \dfrac{e^{-x+y}t^{\gamma}}{\Gamma[\gamma+1]} + \dfrac{e^{x+y}e^{x-y}t^{\gamma}}{\Gamma[\gamma+1]} - \dfrac{e^{x+y}e^{x-y}t^{\gamma}}{\Gamma[\gamma+1]} = \dfrac{e^{-x+y}t^{\gamma}}{\Gamma[\gamma+1]},
\end{cases}$$

$$\begin{aligned}
u_2(x, y, t) &= \frac{e^{x+y}t^{2\alpha}}{\Gamma[2\alpha+1]} - \frac{e^{x-y}e^{-x+y}t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} - \frac{e^{x-y}e^{-x+y}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
&\quad - \frac{\Gamma(\gamma+\beta+1)e^{x-y}e^{-x+y}t^{\alpha+\beta+\gamma}}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)} + \frac{e^{x-y}e^{-x+y}t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}\\
&\quad + \frac{e^{x-y}e^{-x+y}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(\gamma+\beta+1)e^{x-y}e^{-x+y}t^{\alpha+\beta+\gamma}}{\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\\
&= \frac{e^{x+y}t^{2\alpha}}{\Gamma[2\alpha+1]},\\
v_2(x, y, t) &= \frac{e^{x-y}t^{2\beta}}{\Gamma(2\beta+1)} - \frac{e^{x+y}e^{-x+y}t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} + \frac{e^{x+y}e^{-x+y}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\\
&\quad + \frac{\Gamma(\gamma+\alpha+1)e^{x+y}e^{-x+y}t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)} + \frac{e^{x+y}e^{-x+y}t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)}\\
&\quad - \frac{e^{x+y}e^{-x+y}t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{\Gamma(\gamma+\alpha+1)e^{x+y}e^{-x+y}t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}\\
&= \frac{e^{x-y}t^{2\beta}}{\Gamma(2\beta+1)},\\
w_2(x, y, t) &= \frac{e^{-x+y}t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{e^{x+y}e^{x-y}t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} - \frac{e^{x+y}e^{x-y}t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}\\
&\quad - \frac{\Gamma(\alpha+\beta+1)e^{x+y}e^{x-y}t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+\gamma+1)} - \frac{e^{x+y}e^{x-y}t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)}\\
&\quad + \frac{e^{x+y}e^{x-y}t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{\Gamma(\alpha+\beta+1)e^{x+y}e^{x-y}t^{\alpha+\beta+\gamma}}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+\gamma+1)}\\
&= \frac{e^{-x+y}t^{2\gamma}}{\Gamma(2\gamma+1)}.
\end{aligned}$$

Therefore, the series solutions can be written in this form

$$u(x,y,t)=e^{x+y}-\frac{e^{x+y}t^{\alpha}}{\Gamma(\alpha+1)}+\frac{e^{x+y}t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots=e^{x+y}\left(1+\sum_{k=1}^{\infty}\frac{(-t^{\alpha})^{k}}{\Gamma(k\alpha+1)}\right)=e^{x+y}E_{\alpha}(-t^{\alpha}),$$

$$v(x,y,t)=e^{x-y}+\frac{e^{x-y}t^{\beta}}{\Gamma(\beta+1)}+\frac{e^{x-y}t^{2\beta}}{\Gamma(2\beta+1)}+\cdots=e^{x-y}\left(1+\sum_{k=1}^{\infty}\frac{(t^{\beta})^{k}}{\Gamma(k\beta+1)}\right)=e^{x-y}E_{\beta}(t^{\beta}),$$

$$w(x,y,t)=e^{y-x}+\frac{e^{y-x}t^{\gamma}}{\Gamma(\gamma+1)}+\frac{e^{y-x}t^{2\gamma}}{\Gamma(2\gamma+1)}+\cdots=e^{-x+y}\left(1+\sum_{k=1}^{\infty}\frac{(t^{\gamma})^{k}}{\Gamma(k\gamma+1)}\right)=e^{y-x}E_{\gamma}(t^{\gamma}).$$

Substituting $\alpha=\beta=\gamma=1$ we obtain

$$u(x,y,t)=e^{x+y}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right)=e^{x+y-t},$$

$$v(x,y,t)=e^{x-y}\left(1+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)=e^{x-y+t},$$

$$w(x,y,t)=e^{-x+y}\left(1+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)=e^{-x+y+t},$$

which gives us the exact solutions of (20) when $\alpha=\beta=\gamma=1$. Also we note that the results obtained here are similar to the solutions obtained by VIM and HAM [7,21].

5. Conclusion

In this paper we have presented a new method called the Iterative Laplace transform method and have applied it to derive exact and approximate analytical solutions of fractional partial differential equations. We have shown that this method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high level of accuracy of the numerical results. Also it can be seen that ILTM has a clear advantage over the Adomian decomposition and homotopy analysis methods when solving nonlinear problems as the ILTM does not need to compute Adomian polynomials first. Thus, we conclude that the Iterative Laplace transform method can be considered as a nice refinement in existing numerical techniques and might have wide applications. Finally, two examples were presented and their results, in the special cases, agreed well with the exact solutions.

References

[1] R. Hilfer, Applications of Fractional Calculus in Physics, Academic Press, Orlando, 1999.
[2] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[3] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional Calculus Models and Numerical Methods, in: Series on Complexity, Nonlinearity and Chaos, World Scientific, Singapore, 2012.
[4] A.M. Spasic, M.P. Lazarevic, Electroviscoelasticity of liquid/liquid interfaces: fractional-ordermodel, J. Colloid Interface Sci. 282 (2005) 223–230.
[5] H. Jafari, V. Daftardar-Gejji, Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition, Appl. Math. Comput. 180 (2006) 488–497.
[6] Z. Odibat, S. Momani, Numerical methods for nonlinear partial differential equations of fractional order, Appl. Math. Model. 32 (1) (2008) 28–39.
[7] H. Jafari, S. Seifi, Solving system of nonlinear fractional partial differention equations by homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1962–1969.
[8] M. Zurigat, S. Momani, Z. Odibat, A. Alawneh, The homotopy analysis method for handling systems of fractional differential equations, Appl. Math. Model. 34 (1) (2010) 24–35.
[9] S. Momani, O. Abdulaziz, I. Hashim, Application of homotopy-perturbation method to fractional IVPs, J. Comput. Math. Appl. 216 (2008) 574–584.
[10] J. Singh, P.K. Gupta, K.N. Rai, CIMS-DST, Homotopy perturbation method to space time fractional solidification in a finite slab, Appl. Math. Model. 35 (4) (2011) 1937–1945.
[11] Z. Odibat, S. Momani, Application of variational iteration method to Nonlinear differential equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul. 7 (1) (2006) 27–34.
[12] H. Jafari, A. Kadem, D. Baleanu, T. Yilmaz, Solutions of the fractional Davey–Stewartson equations with variational iteration method, Rom. Rep. Phys. 64 (2) (2012) 337–346.
[13] H. Jafari, C.M. Khalique, M. Nazari, Application of Laplace decomposition method for solving linear and nonlinear fractional diffusion—wave equations, Appl. Math. Lett. 24 (11) (2011) 1799–1805.
[14] S.A. Khuri, A Laplace decomposition algorithm applied to class of nonlinear differential equations, J. Math. Appl. (2001) 141–155.
[15] M. Khan, M. Hussain, Application of Laplace decomposition method on semi-infinite domain, Numer. Algorithms 56 (2011) 211–218.
[16] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, J. Math. Anal. Appl. 316 (2006) 753–763.
[17] H. Jafari, Iterative Methods for solving system of fractional differential equations, Ph.D. Thesis, Pune University, 2006.
[18] H. Jafari, S. Seifi, A. Alipoor, M. Zabihi, An iterative method for solving linear and nonlinear fractional diffusion-wave equation, J. Nonlinear Fract. Phenom. Sci. Eng. (2007).
[19] S. Bhalekar, V. Daftardar-Gejji, Solving evolution equations using a new iterative method, Numer. Methods Partial Differential Equations 26 (4) (2010) 906–916.
[20] V. Daftardar-Gejji, S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, Comput. Math. Appl. 59 (2010) 1801–1809.
[21] A.M. Wazwaz, The variational iteration method for solving linear and nonlinear systems of PDEs, Comput. Math. Appl. 54 (2007) 895–902.