Computers and Mathematics with Applications 66 (2013) 832–837

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A note on stability of sliding mode dynamics in suppression of fractional-order chaotic systems

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ARTICLE INFO

Keywords:
Chaos control
Fractional-order systems
Sliding mode control
Adaptive law
Lyapunov stability theorem

ABSTRACT

We consider a class of fractional-order chaotic systems which undergoes unknown perturbations. We revisit the problem of sliding mode controller design for robust stabilization of chaotic systems using one control input. In the recent works, it was assumed that one of the system equations are perturbed by uncertainties. For this case we show that the sliding mode dynamics are globally stable which is not addressed so far. Next, we allow that *all the system's equations* depend on uncertain terms and provide a theoretical justification for applicability of the existing design. We also determine the least amount of precise information about the chaotic system that is needed to design the controller.

1. Introduction

Fractional calculus has been attracting the attention of scientists and engineers from 300 years ago. However, since the nineties of the last century, fractional calculus has been rediscovered and applied in an increasing number of fields, namely in several areas of physics, control engineering, signal processing and system modeling [1–3].

Since Hartley et al. have shown that there are chaotic solutions in fractional-order systems [4], there has been a surge of interest in control and synchronization of fractional chaotic systems. For example, control and synchronization of fractional Modified Van der Pol–Duffing system is presented in [5], synchronization of different fractional systems via active control is studied in [6] and synchronization of fractional-order Genesio–Tesi system via active control and sliding mode control is reported in [7].

In this paper, we consider the following class of fractional-order chaotic systems

$$\begin{aligned} D^q x &= y \cdot f(x,y,z) + z \cdot \Phi(x,y,z) - \alpha x + \Delta f_1(x,y,z) + d_1(t), \\ D^q y &= g(x,y,z) - \beta y + \Delta f_2(x,y,z) + d_2(t) + u(t), \\ D^q z &= y \cdot h(x,y,z) - x \cdot \Phi(x,y,z) - \gamma z + \Delta f_3(x,y,z) + d_3(t), \end{aligned} \tag{1}$$

where $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and $\Phi(\cdot)$ are smooth nonlinear functions, $\Delta f_i(x,y,z)$, $d_i(t)$, $i = 1, 2, 3$ denote external disturbances and system uncertainties respectively which are bounded by some positive constants i.e. $|\Delta f_i(x,y,z)| \le \Delta_i$ and $|d_i(t)| \le D_i$. It is worth to notice that (1) presents a large class of chaotic system since many chaotic systems can be modeled in this form;

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2012.11.015

see [8]. For the case that there are uncertain terms only in the second equation, sliding mode controller is proposed in [9–11] which results in global stabilization of the chaotic system. However, as we pointed out in our earlier work in [12], there is a drawback in the above methods, that is, the stability of the system being on the sliding surface is not addressed. In fact, for design of a sliding mode controller there are two major steps:

1- Constructing an appropriate sliding surface which represents desired dynamics and
2- Formulate the corresponding control law such that the sliding condition is attained.

Before the second step, it is necessary to show that dynamics of the system being on the sliding surface i.e. sliding mode dynamics is stable. This issue is not addressed in the above papers and here we are about to deal with this problem. We show that under certain conditions, the stability of sliding mode dynamics can be guaranteed globally. We then extend the existing design for the case that all the system equations are perturbed. In such a case, the sliding mode dynamics undergoes unknown perturbation and it is vital to demonstrate its stability. Using the fact that chaotic states will settle onto an attractor, we provide global results for this problem. A striking finding is that there is no need to know the upper bounds of perturbations appeared in sliding mode dynamics. This will theoretically justify the efficiency of sliding mode controller for the case that all the system equations are perturbed.

This paper is organized as follows: Mathematical preliminaries are presented in Section 2. Problem statement is given in Section 3. Main results are included in Section 4. Simulation results are illustrated in Section 5 and concluding remarks are given in Section 6.

2. Mathematical preliminaries

Let us first introduce definitions and theorems which will be used in this paper.

Definition 1 (*[2]*)**.** The Caputo definition q-th order of fractional derivative is given by

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau. \tag{2}$$

In the rest of this paper the notation $D^q(\cdot)$ or $(\cdot)^{(q)}$ denotes the Caputo definition of q-th order fractional derivative. The Laplace transform of q-th order derivative of a function using Caputo's definition is given as follows [2]

$$ \mathrm{L}\left\{D^q f(t)\right\} = s^q F(s) - s^{q-1} f(0), \quad 0 < q < 1. \tag{3}$$

Definition 2 (*[2]*)**.** The Mittag-Leffler function with two parameters is defined as follows

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{4}$$

where $\alpha > 0$ and $\beta > 0$.

The Laplace transform of Mittag-Leffler function in two parameters are given as follows

$$ \mathrm{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}, \qquad \mathrm{R}(s) > |\lambda|^{\frac{1}{\alpha}}, \tag{5}$$

where t and s are, respectively, the variables in the time domain and Laplace domain, $\mathrm{R}(s)$ denotes the real part of s and $\lambda \in \mathbf{R}$. We also introduce the following formula which will be used later [2]

$$ \int_0^z t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha) dt = z^\beta E_{\alpha,\beta+1}(-\lambda z^\alpha). \tag{6}$$

Comparison Lemma (*[13]*)**.** *If $D^q x(t) \geq D^q y(t)$, $0 < q < 1$ and $x(0) = y(0)$ then $x(t) \geq y(t)$.*

Stability analysis of fractional-order systems via Lyapunov stability theorem is introduced in [14,15]. However, as mentioned in [16] it is a tedious task to utilize these results to study the stability of nonlinear fractional-order systems. Recently, Zhang et al. have relaxed the stability conditions of nonlinear fractional-order systems by taking into account the relation between asymptotical stability and generalized Mittag-Leffler stability [17]. Therefore, the conditions for stability of a Lyapunov function in [14] are weaken according to the following theorem.

Theorem 1 (*[17]*)**.** *Let $x = 0$ be an equilibrium point of the non-autonomous fractional-order system*

$$ D^q x = f(x,t), \tag{7}$$

and let $D \subset \mathbf{R}^n$ be a domain containing the origin. Assume that there exists a continuously differentiable function $V(t, x(t)) : [t_0, \infty) \times D \to \mathbf{R}_+$ and class-K function α satisfying

$$V(t, x(t)) \geq \alpha(\|x\|) \quad \text{and} \quad D^p V(t, x(t)) \leq 0, \tag{8}$$

where $x \in D$ and $0 < p < 1$. Then $x = 0$ is globally stable.

3. Problem statement

In [9], the authors have considered a fractional-order economical system which is described by the following fractional-order differential equations

$$\begin{aligned} D^{q_1}x &= z + (y - a)x, \\ D^{q_2}y &= 1 - by - x^2 + \Delta f_2(x, y, z) + d_2(t), \\ D^{q_3}z &= -x - cz, \end{aligned} \tag{9}$$

where the perturbation are considered only in the second equation. In order to suppress the chaos governed by the system (9), a control input is introduced in the second equation. In sliding mode controller design, the first and most critical step is to construct an appropriate sliding surface which represents desired dynamics. The following sliding surface is proposed

$$s(t) = D^{q_2-1}y(t) + \int_0^t \left(x^2(\tau) + Ky(\tau)\right) dt, \tag{10}$$

where K is a design parameter. When the system operates in sliding mode we have $s(t) = 0$ and $\dot{s}(t) = 0$. Therefore, the sliding mode dynamics takes the following form

$$\begin{aligned} D^{q_1}x &= z + (y - a)x, \\ D^{q_2}y &= -x^2(t) - Ky(t), \\ D^{q_3}z &= -x - cz. \end{aligned} \tag{11}$$

In [9], without any proof, it is stated that K is a positive constant which can be chosen arbitrarily. In this paper, we provide a theoretical justification for this claim. For this purpose, we consider a large class of fractional-order chaotic systems which the system (9) and several other chaotic systems belong to this class. This class of chaotic systems can be described by the following fractional-order differential equations

$$\begin{aligned} D^{q_1}x &= y \cdot f(x, y, z) + z \cdot \Phi(x, y, z) - \alpha x, \\ D^{q_2}y &= g(x, y, z) - \beta y + \Delta f_2(x, y, z) + d_2(t) + u(t), \\ D^{q_3}z &= y \cdot h(x, y, z) - x \cdot \Phi(x, y, z) - \gamma z. \end{aligned} \tag{12}$$

Sliding mode controller design for the system (12) is studied in [10,11]. The sliding surface is defined as

$$s(t) = D^{q_2-1}y(t) + \int_0^t (x \cdot f(x, y, z) + z \cdot h(x, y, z) + \beta y)\, dt, \tag{13}$$

which yields the following sliding mode dynamics

$$\begin{aligned} D^{q_1}x &= y \cdot f(x, y, z) + z \cdot \Phi(x, y, z) - \alpha x, \\ D^{q_2}y &= -x \cdot f(x, y, z) - z \cdot h(x, y, z) - \beta y, \\ D^{q_3}z &= y \cdot h(x, y, z) - x \cdot \Phi(x, y, z) - \gamma z. \end{aligned} \tag{14}$$

The problem is to show the stability of the sliding mode dynamics (11) or (14) and describe the motivation of choosing the particular structure (10) or (13) for sliding surface. In fact, in case of integer-order system using the Lyapunov function $V = 0.5\left(x^2 + y^2 + z^2\right)$ and taking the time derivative will get $\dot{V} < 0$ which guarantees the asymptotic stability of sliding mode dynamics [8]. In the next section we will address this problem for the case of fractional-order.

4. Main results

Theorem 2. *Suppose that $\beta > 0$ then the commensurate-ordered sliding mode dynamics as in (14) is globally stable.*

Proof. Choose the following Lyapunov function candidate

$$V = x^2 + y^2 + z^2. \tag{15}$$

Taking q-th order fractional derivative from both sides of (15) and using the Leibniz rule of fractional differentiation [1], one has

$$V^{(q)} = x \cdot x^{(q)} + y \cdot y^{(q)} + z \cdot z^{(q)} + \sum_{j=1}^{\infty} \frac{\Gamma(1+q)}{\Gamma(1+j)\Gamma(1-j+q)} \left(x^{(j)} x^{(q-j)} + y^{(j)} y^{(q-j)} + z^{(j)} z^{(q-j)} \right). \quad (16)$$

Assume that ρ is an arbitrarily large positive constant which is a bound on the rightmost term of (16). Making substitutions from (14) into (16) yields the following expression

$$\begin{aligned} V^{(q)} &\le x \cdot x^{(q)} + y \cdot y^{(q)} + z \cdot z^{(q)} + \rho \\ &= x\,(y \cdot f(x,y,z) + z \cdot \Phi(x,y,z) - \alpha x) + y\,(-x \cdot f(x,y,z) - z \cdot h(x,y,z) - \beta y) \\ &\quad + z\,(y \cdot h(x,y,z) - x \cdot \Phi(x,y,z) - \gamma z) + \rho \\ &= -\alpha x^2 - \beta y^2 - \gamma z^2 + \rho. \end{aligned} \quad (17)$$

Let $\eta = \min\{\alpha, \beta, \gamma\}$ then we have

$$V^{(q)} \le -\eta V + \rho. \quad (18)$$

Next, we solve the fractional-order differential equation $V^{(q)} = -\eta V + \rho$ and use the comparison lemma to show the stability of the sliding mode dynamics. Taking the Laplace transform from both sides of the above equation and from the fact that $V(0) = 0$, it yields

$$V(s) = \frac{\rho}{s(s^q + \eta)}. \quad (19)$$

Taking the inverse Laplace transform of (19) yields

$$V(t) = \int_0^t \rho t^{\alpha-1} E_{\alpha,\alpha}\left(-t^{\alpha}\right) dt. \quad (20)$$

Using Eq. (6) and from the comparison lemma, it follows that

$$V(t) \le t^{\alpha} E_{\alpha,\alpha+1}\left(-\eta t^{\alpha}\right), \quad (21)$$

or equivalently we have

$$V(t) \le \sum_{k=0}^{\infty} \frac{(-1)^k \eta^k t^{\alpha k}}{\Gamma\left((k+1)\alpha + 1\right)}. \quad (22)$$

Using the ratio test it can be concluded that the right hand side of (22) is convergent which implies that the sliding mode dynamic is globally stable. Thus the proof is achieved completely. Notice that this just goes to prove stability in the sense of Lyapunov. □

Next, we consider the case that all the system equations are perturbed by uncertain terms. For this purpose, consider the system (1) and define the sliding surface as in (13). Then we have the following sliding mode dynamics

$$\begin{aligned} D^{q_1} x &= y \cdot f(x,y,z) + z \cdot \Phi(x,y,z) - \alpha x + \Delta f_1(x,y,z) + d_1(t), \\ D^{q_2} y &= -x \cdot f(x,y,z) - z \cdot h(x,y,z) - \beta y, \\ D^{q_3} z &= y \cdot h(x,y,z) - x \cdot \Phi(x,y,z) - \gamma z + \Delta f_3(x,y,z) + d_3(t). \end{aligned} \quad (23)$$

Theorem 3. *Suppose that $\beta > 0$ the commensurate-ordered sliding mode dynamics as in (23) is globally stable.*

Proof. We again consider the Lyapunov function (15) and use the Leibniz rule of fractional differentiation to arrive at this expression

$$\begin{aligned} V^{(q)} &\le x \cdot x^{(q)} + y \cdot y^{(q)} + z \cdot z^{(q)} + \rho \\ &= x\,(y \cdot f(x,y,z) + z \cdot \Phi(x,y,z) - \alpha x + \Delta f_1(x,y,z) + d_1(t)) + y\,(-x \cdot f(x,y,z) - z \cdot h(x,y,z) - \beta y) \\ &\quad + z\,(y \cdot h(x,y,z) - x \cdot \Phi(x,y,z) - \gamma z + \Delta f_3(x,y,z) + d_3(t)) + \rho \\ &= -\alpha x^2 - \beta y^2 - \gamma z^2 + x\Delta f_1(x,y,z) + x d_1(t) + z\Delta f_3(x,y,z) + z d_3(t) + \rho \\ &\le -\alpha x^2 - \beta y^2 - \gamma z^2 + x(\Delta_1 + D_1) + z(\Delta_3 + D_3) + \rho. \end{aligned} \quad (24)$$

It is worth to notice that the chaotic systems are dissipative. This means that all the system orbits will ultimately be confined to a specific subset of zero volume and the asymptotic motion settles onto an attractor. As a result, all the states

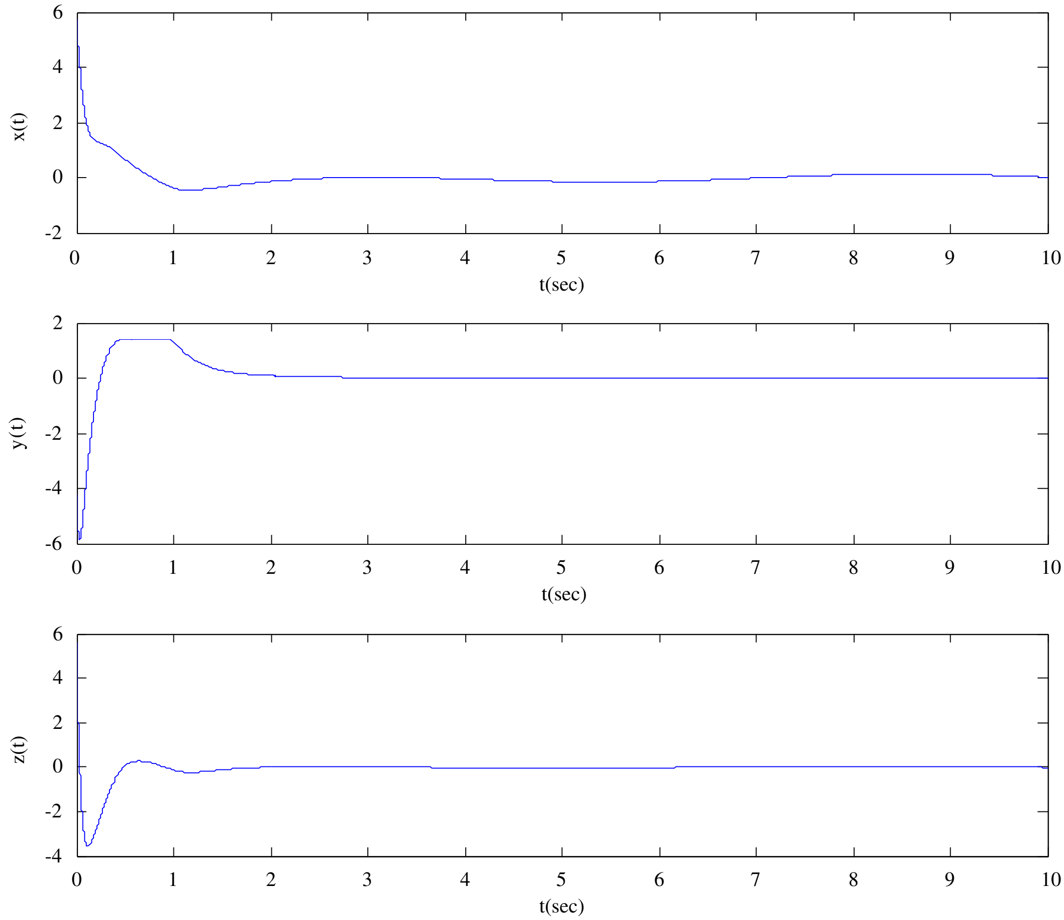

Fig. 1. State trajectories of forced fractional-order Liu system.

have bounded amplitude. Therefore there exist positive constants X and Z such that $|x(t)| \leq X$ and $|z(t)| \leq Z$. Having this in mind we rewrite (24) as follows

$$V^{(q)} \leq -\alpha x^2 - \beta y^2 - \gamma z^2 + \rho_1, \tag{25}$$

where $\rho_1 := X(\Delta_1 + D_1) + Z(\Delta_3 + D_3) + \rho$. From the results of Theorem 2 it can be concluded that the perturbed sliding mode dynamics is globally stable. Thus the proof is achieved completely. □

Notice that Theorem 3 gives a very interesting result, that is, a *prior knowledge* of the bounds on perturbations is not required for guaranteeing the stability. If we pursue the design using the Lyapunov function $V = 0.5s^2$ similar to that of presented in [9–12], it can be concluded that in the design procedure we also do not need to know the upper bounds on the perturbations appeared in the sliding mode dynamics. However, the knowledge of Δ_2 and D_2 is required for guaranteeing the global stability of the closed-loop system.

5. Numerical simulations

In this section sliding mode control of fractional-order Liu system is presented for the case all the system equations are perturbed. Consider the fractional-order Liu system [12]

$$\begin{aligned} D^q x &= -ax - ey^2 + \Delta f_1(x, y, z) + d_1(t), \\ D^q y &= by - kxz + \Delta f_2(x, y, z) + d_2(t) + u(t), \\ D^q z &= -cz + mxy + \Delta f_3(x, y, z) + d_3(t), \end{aligned} \tag{26}$$

where $a = e = 1$, $b = 2.5$, $k = m = 4$ and $c = 5$ yield chaotic trajectory. Define the sliding surface as

$$s(t) = D^{q-1} y + \int_0^t (-exy + mxz + \gamma y)\, dt. \tag{27}$$

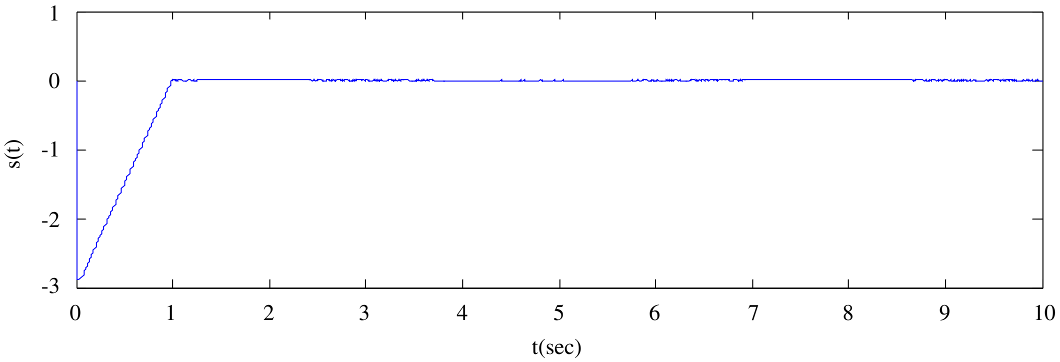

Fig. 2. Sliding surface trajectory.

From Theorem 3 it can be concluded that γ should be a positive constant to ensure the stability. This yields the following sliding mode dynamics

$$\begin{aligned} D^q x &= -ax - ey^2 + \Delta f_1(x, y, z) + d_1(t), \\ D^q y &= exy - mxz - \gamma y, \\ D^q z &= -cz + mxy + \Delta f_3(x, y, z) + d_3(t). \end{aligned} \tag{28}$$

In order to derive an appropriate sliding mode control law, consider the Lyapunov function $V = 0.5s^2$, it is straightforward to derive the following controller

$$u(t) = -(b + \gamma)y + (k - m)xz + exy - \xi \operatorname{sgn}(s), \tag{29}$$

where $\operatorname{sgn}(\cdot)$ is the sign function and ξ is a design parameter which should be chosen as $\xi > \Delta_2 + D_2$ to guarantee the stability. For the purpose of numerical simulations we set $q = 0.98$, $\xi = 3$, $(x_0, y_0, z_0)^T = (5, -3, 4)^T$, $\Delta f(x, y, z) = -0.1\sin(\sqrt{x^2 + y^2 + z^2})$ and $d(t) = 0.2\sin(t)$. Simulation results are provided in Figs. 1 and 2 which shows that the sliding mode controller can stabilize the chaotic system in presence of perturbations in all the system equations. This verifies the obtained theoretic results.

6. Conclusion

This paper extends the earlier work on the sliding mode control of fractional-order chaotic systems in two directions. First, we present global results on the stability of sliding mode dynamics which has not been investigated so far. Second we consider the case a general case, that is, all the system equations are perturbed by uncertainties and disturbances and provide a theoretical justification for the applicability of the existing designs. Simulation studies are presented to verify the theoretical results.

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