Cent. Eur. J. Phys. • 12(7) • 2014 • 480-489
DOI: 10.2478/s11534-014-0475-6

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Central European Journal of **Physics**

Travelling wave solutions: A new approach to the analysis of nonlinear physical phenomena

Research Article

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Received 14 November 2013; accepted 14 April 2014

Abstract: In this manuscript, a reliable scheme based on a general functional transformation is applied to construct the exact travelling wave solution for nonlinear differential equations. Our methodology is investigated by means of the modified homotopy analysis method which contains two convergence-control parameters. The obtained results reveal that the proposed approach is a very effective. Several illustrative examples are investigated in detail.

PACS (2008): 65N80, 35A09,41 A29,41A21

Keywords: biparametric• homotopy• series solution• convergence• nonlinear differential equation

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1. Introduction

Finding exact or approximate solutions of differential equations is an important part of calculus. Except for a limited number of these equations, we have difficulty in finding their solutions. Therefore, there have been attempts to develop new approaches for obtaining analytical or numerical solutions which reasonably approximate the exact solutions. For more details see [1–6]. Recently, a promising analytical approach called homotopy analysis method (HAM), has successfully been applied to solve many types of linear and nonlinear functional equations [7–12]. HAM was known before the time indicated in [7] and this new scheme can be applied to explore a sector of exact solutions of a given partial differential equation by reducing it to an ordinary differential equation exactly without any approximation, only by assuming an ansatz

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for the solutions [9, 10].
There are many papers that deal with HAM. Abbasbandy et al. [13] applied the Newton-homotopy analysis method to solve nonlinear algebraic equations, Allan [14] constructed the analytical solutions to Lorenz system by the HAM, Bataineh et al. [15, 16] proposed a new reliable modification of the HAM, M. Ganjiani et al [17] constructed the analytical solutions to coupled nonlinear diffusion reaction equations by the HAM, Alomari et al. [18] applied the HAM to study delay differential equations, Chen and Liu. [19] applied the HAM to increase the convergent region of the harmonic balance method. For more details, the reader is advised to consult the results of the research works presented in[21–32].
The investigation of exact traveling wave solutions to nonlinear differential equations plays an important role in the study of nonlinear physical phenomena. This manuscript is concerned with the following nonlinear wave equation:

$$F(u, u_t, u_x, u_{tt}, u_{tx}, ...) = 0. \tag{1}$$

Using a transformation

$$u(x,t) = \Xi\psi(\eta), \;\; \eta = \lambda(x - \omega t), \tag{2}$$

where λ and ω are constants and Ξ is amplitude which will be determined later, we can convert (1) to the following nonlinear ordinary differential equation

$$\begin{aligned} &N[\psi(\eta), \Xi] = \\ &F(\Xi\psi, -\lambda\omega\Xi\psi', \lambda\Xi\psi', \lambda^2\omega^2\Xi\psi'', -\lambda^2\omega\Xi\psi', \ldots) = 0, \end{aligned} \tag{3}$$

where the prime denotes differentiation with respect to η.
In this manuscript, after a short background on a biparametric homotopy [7], we extended the application of this methodology to nonlinear differential equations. Moreover, we proved the convergence of the solution for nonlinear differential equations.

2. Summary of biparametric homotopy and its applications

2.1. The nonlinear beam equation

To illustrate the procedure, we consider the nonlinear beam equation [33] in the form

$$u_{2t} + u_{4x} + \alpha u_{2x} + \beta u + u^2 = 0, \tag{4}$$

where, $u(x,t)$ is the deflection of beam and α and β are parameters.
To solve the beam equation (4), within the following complex transformation

$$\eta = \lambda(x - \omega t),$$

define

$$u(x,t) = \Xi\psi(\eta),$$

where Ξ is amplitude which will determined later, and $\lambda > 0$, ω is arbitrary given constant.
Substituting $u(x,t)$ into equation (4) yields

$$\lambda^4\psi^{(4)} + \lambda^2(\omega^2 + \alpha)\psi'' + \beta\psi + \Xi\psi^2 = 0, \tag{5}$$

Assume now that $\psi \to d_i \exp(-\eta)$, as $\eta \to \infty$ where d_i are arbitrary constants. Now, we assume that the dimensionless solution $\psi(\eta)$ arrives its maximum at the origin. Obviously, $\psi(\eta)$ and its derivatives tend to zero when $\eta \to \infty$. Besides, due to the continuity, the first derivative of $\psi(\eta)$ at crest is zero [8]. Thus, the boundary conditions of our solutions are

$$\psi(0) = 1, \psi'(0) = 0, \psi(\infty) = 0, \psi'(\infty) = 0. \tag{6}$$

substituting $exp(-\eta)$ into (5) we have

$$\begin{aligned} &\lambda^4 exp(-\eta) + \lambda^2(\omega^2 + \alpha)exp(-\eta) + \beta exp(-\eta) \\ &+ \Xi exp(-2\eta) = 0. \end{aligned} \tag{7}$$

The parameter $\lambda > 0$ can be determined by equating the coefficient of $exp(-\eta)$ to be zero

$$\lambda^4 + \lambda^2(\omega^2 + \alpha) + \beta = 0.$$

According to the boundary conditions (6), it is natural to express its solution by a set of base functions,namely

$$\{\exp(-m\eta) | m \geq 1\}, \tag{8}$$

in the form

$$\psi(\eta) = \sum_{m=1}^{\infty} c_m \exp(-m\eta). \tag{9}$$

We choose the auxiliary linear operator as

$$Ł[\psi(\eta)] = [\lambda^4\psi^{(4)}(\eta) + \lambda^2(\omega^2 + \alpha)\psi''(\eta) + \beta],$$

with the property

$$Ł[C_1\exp(-\eta)+C_2\exp(+\eta)+C_3\exp(x_1\eta)+C_4\exp(x_2\eta)]=0,$$

such that $C_i, i=1,2,...,$ are constants and

$$x_{1,2}=\pm\frac{\sqrt{\beta}}{\lambda^2}.$$

For more details about the adopting $Ł[.]$ and its applications, the reader is advised to consult the results of the research work presented in [8].
Now, the next step is to define a nonlinear operator as

$$N[\phi(\eta;q),\Xi(q)]=\lambda^4\frac{d\phi^4(\eta;q)}{d\eta^4}+\lambda^2(\omega^2+\alpha)\frac{d\phi^2(\eta;q)}{d\eta^2} + \beta\phi(\eta;q)+\Xi(q)\phi^2(\eta;q).$$

The homotopy analysis method can be further generalized by means of the zero-order deformation equation in the form

$$(1-q)Ł[\phi(\eta;q)-\psi_0(\eta)]=h_1qN[\phi(\eta;q),\Xi(q)] + h_2q(1-q)[N[\phi(\eta;q),\Xi(q)]-N[\psi_0,\Xi_0]],\ \ q\in[0,1], \tag{10}$$

such that

$$\phi(0;q)=1,\quad \frac{\partial\phi(\eta;q)}{\partial\eta}|_{\eta=0}=0,\quad \phi(\infty;q)=1. \tag{11}$$

Using Taylor's series expansion with respect to the embedding parameter q, we have

$$\phi(\eta;q)=\sum_{m=1}^{\infty}\psi_m(\eta)q^m,\ \ \psi_m(\eta)=\frac{\partial^m\phi(\eta;q)}{m!\partial q^m}|_{q=0}, \tag{12}$$

$$\Xi(q)=\sum_{m=1}^{\infty}\Xi_mq^m,\ \ \Xi_m=\frac{\partial^m\Xi(q)}{m!\partial q^m}|_{q=0}, \tag{13}$$

where Ξ_m and $\psi_m(\eta)$ are functions which should be determined.
By differentiating (10) and (11) m times with respect to q, then dividing the equation by $m!$ and setting $q=0$, the mth-order deformation equation is formulated as follows

$$\begin{cases} Ł[\psi_m(\eta)-\chi_m\psi_{m-1}(\eta)]=\hbar_1\Delta m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1}) \\ \qquad\qquad +\hbar_2\Pi_m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1}), \\ \psi_m(0)=0,\quad \psi_m'(0)=0,\quad ,\psi_m(\infty)=0,\quad m\geq 1, \end{cases} \tag{14}$$

in which

$$\Delta_m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1})=\lambda^4\frac{d\psi_{m-1}^4}{d\eta^4}+\lambda^2(\omega^2+\alpha)\frac{d\psi_{m-1}^2}{d\eta^2} + \beta\psi_{m-1}+\sum_{j=0}^{m-1}\Xi_{m-1-j}\sum_{i=0}^{j}\psi_i\psi_{j-i},$$

$$\begin{aligned}&\Pi_m(\psi_0,\Xi_0,...,\psi_{m-1},\Xi_{m-1})=\\ &-\Upsilon_m\Delta_{m-1}(\psi_0,\Xi_0,...,\psi_{m-2},\Xi_{m-2})\\ &+\chi_m\Delta_m(\psi_0,\Xi_0,...,\psi_{m-1},\Xi_{m-1}),\ \ m\geq 1,\end{aligned}$$

$$\Upsilon_m=\begin{cases}0, & m\leq 2,\\ 1, & m>2,\end{cases} \tag{15}$$

and

$$\chi_m=\begin{cases}0, & m\leq 1,\\ 1, & m>1.\end{cases} \tag{16}$$

The general solution equation (14) is

$$\psi_m(\eta)=\psi_m^*(\eta)+C_1\exp(-\eta)+C_2\exp(+\eta)+C_3\exp(x_1\eta) + C_4\exp(x_2\eta),$$

such that C_1, C_2, C_3 and C_4 are constants and $\psi_m^*(\eta)$ is a special solution of (14). Using the rule of solution expression denoted by (9), we have $C_2=0$, $C_3=0$ and $C_4=0$. The above presented solution automatically fulfills the boundary conditions, therefore the unknown C_1 and Ξ_{m-1} can be found by solving the following linear algebraic equations

$$C_1+\psi_m^*(0)=0,\quad C_1-\psi_m^{*'}(0)=0.$$

2.2. The KdV equation

In the following we are dealing with the fifth order KdV equation, governed by the nonlinear partial differential equation as

$$u_t+\alpha u^2u_x+\beta u_xu_{xx}+u_{5x}=0, \tag{17}$$

with α being a constant [34].
To look for the travelling wave solution of (17), we suppose that the solutions of (17) are the form

$$u(x,t)=\Xi\psi(\eta),\ \ \eta=\lambda(x-\omega t), \tag{18}$$

where $\lambda > 0$ and ω is arbitrary given constants. From (18) one can derive that

$$-\omega\psi + \alpha\Xi^2\psi^3 + \beta\lambda^2\Xi\psi^{'2} + \lambda^4\psi^{(4)} = 0, \tag{19}$$

where Ξ depicts amplitude which will be determined later and the prime is differentiation with respect to η.
Writing $\psi \to d_i \exp(-\eta)$ as $\eta \to \infty$, (d_i are arbitrary constants) and substituting it into (19) and balancing the main term, we obtain $\lambda^4 - \omega = 0$, we consider the positive real value for λ.
We can define $\eta = 0$ so that

$$\psi(0) = 1, \psi^{'}(0) = 0, \psi(\infty) = 0. \tag{20}$$

Assuming that the solutions in (19) can be expressed by a set of base functions

$$\{\exp(-m\eta)|m \geq 1\}, \tag{21}$$

such as

$$\psi(\eta) = \sum_{m=1}^{\infty} c_m \exp(-m\eta). \tag{22}$$

Furthermore, under the rule of solution expression denoted by (22) and by using (19), we choose an auxiliary linear operator

$$Ł[\psi(\eta)] = [\psi^{(4)} + 2\psi^{'''} - \psi^{''} - 2\psi^{'}],$$

with the property

$$Ł[C_1\exp(-\eta) + \exp(-2\eta) + C_3\exp(\eta) + C_4)] = 0,$$

where C_1, C_2, C_3 and C_4 are constants.
Now, from (19), we define the nonlinear operator as

$$N[\phi(\eta;q),\Xi(q)] = -\omega\phi(\eta;q) + \alpha\Xi^2(q)\phi^3(\eta;q) + \beta\lambda^2\Xi(q)\phi(\eta;q)\frac{d^2\phi(\eta;q)}{d\eta^2} + \lambda^4\frac{d^4\phi(\eta;q)}{d\eta^4}.$$

By means of the homotopy properties mentioned in Subsection 2.1, we construct the so-called zero-order deformation equation

$$(1-q)Ł[\phi(\eta;q) - \psi_0(\eta)] = h_1 q N[\phi(\eta;q),\Xi(q)] + h_2 q(1-q)[N[\phi(\eta;q),\Xi(q)] - N[\psi_0,\Xi_0]],\ \ q\in[0,1], \tag{23}$$

such that

$$\phi(0;q) = 1, \quad \frac{\partial\phi(\eta;q)}{\partial\eta}|_{\eta=0} = 0, \quad \phi(\infty;q) = 1. \tag{24}$$

After that, we can differentiate the zeroth-order deformation (23) m times with respect to parameter q, then divide the resulting equation by $m!$ and set $q = 0$, we have the following result

$$\begin{cases} Ł[\psi_m(\eta) - \chi_m\psi_{m-1}(\eta)] = \hbar_1\Delta_m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1}) \\ \quad +\hbar_2\Pi_m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1}), \\ \psi_m(0) = 0, \quad \psi_m^{'}(0) = 0, \quad ,\psi_m(\infty) = 0, \quad m \geq 1, \end{cases} \tag{25}$$

Here we have

$$\Delta_m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1}) = \frac{\partial^{m-1}N[\phi(\eta;q),\Xi(q)]}{(m-1)!\partial q^{m-1}},$$

and

$$\begin{aligned}\Pi_m(\psi_0,\Xi_0,...,\psi_{m-1},\Xi_{m-1}) = \\ -\Upsilon_m\Delta_{m-1}(\psi_0,\Xi_0,...,\psi_{m-2},\Xi_{m-2}) \\ +\chi_m\Delta_m(\psi_0,\Xi_0,...,\psi_{m-1},\Xi_{m-1}),\ \ m\geq 1.\end{aligned}$$

We note that ψ_m, Ξ_{m-1} are all unknown, but we have only (25) for ψ_m, thus an additional algebraic equation is required for determining Ξ_{m-1}. According to the property of the auxiliary linear operator L, the solution of the deformation equation contains the so-called term $\eta\exp(-2\eta)$, if the right-hand side of (25) involves the term $exp(-2\eta)$. As a result, we force the coefficient of the term $exp(-2\eta)$ to be zero. Therefore we have additional algebraic equation for determining Ξ_{m-1}.
The general solution equation (25) is

$$\psi_m(\eta) = \psi_m^*(\eta) + C_1\exp(-\eta) + C_2\exp(-2\eta) + C_3\exp(\eta) + C_4,$$

where C_1, C_2, C_3 and C_4 are constants and $\psi_m^*(\eta)$ is a special solution of (25). According to the boundary conditions (25) and the rule of solution expression (22), we have $C_3 = 0$ and $C_4 = 0$. The unknown C_1 and C_2 are obtained by solving the linear algebraic equations

$$C_1 = -2\psi_m^*(0) - \psi_m^{*'}(0), \quad C_2 = \psi_m^*(0) + \psi_m^{*'}(0).$$

3. Convergence

Lemma 3.1.

Write

$$\phi(\eta;q)=\sum_{m=1}^{\infty}\psi_m(\eta)q^m,\ \ \Xi(q)=\sum_{m=1}^{\infty}\Xi_m q^m, \qquad (26)$$

where q is the homotopy-parameter. Let $N[\psi(\eta),\Xi]$, denote nonlinear operators defined in previous sections. It holds that:

$$\begin{aligned}&\Pi_m(\psi_0,\Xi_0,\dots,\psi_{m-1},\Xi_{m-1})=\\&\quad-\Upsilon_m\Delta_{m-1}(\psi_0,\Xi_0,\dots,\psi_{m-2},\Xi_{m-2})\\&\quad+\chi_m\Delta_m(\psi_0,\Xi_0,\dots,\psi_{m-1},\Xi_{m-1}),\ \ m\geq 1,\end{aligned} \qquad (27)$$

Proof. When $m=1,2$, it obviously holds that:

$$\Pi_1(\psi_0,\Xi_0)=0,$$

$$\Pi_2(\psi_0,\Xi_0,\psi_1,\Xi_1)=\Delta_2(\psi_0,\Xi_0).$$

According to Leibnitz's rule for derivatives of product, for $m\geq 3$ it holds

$$\begin{aligned}&\Pi_m(\psi_0,\Xi_0,\dots,\psi_{m-1},\Xi_{m-1})=\frac{\partial^m\left[q(1-q)[N[\phi(\eta;q),\Xi(q)]-N[\psi_0,\Xi_0]]\right]}{\partial q^m}|_{q=0}=\\&\frac{1}{m!}\sum_{j=0}^{m}\frac{m!}{j!(m-j)!}\frac{\partial^j\left[q(1-q)\right]}{\partial q^j}|_{q=0}\frac{\partial^{m-j}\left[N[\phi(\eta;q),\Xi(q)]-N[\psi_0,\Xi_0]\right]}{\partial q^{m-j}}|_{q=0}=\\&\frac{-2\frac{m(m-1)}{2!}\frac{\partial^{m-2}N[\phi(\eta;q)]}{\partial q^{m-2}}\Big|_{q=0}+m(1-2q)\frac{\partial^{m-1}N[\phi(\eta;q)]}{\partial q^{m-1}}\Big|_{q=0}+q(1-q)\frac{\partial^{m}N[\phi(\eta;q)]}{\partial q^{m}}\Big|_{q=0}}{m!}=\\&-\Delta_{m-1}(\psi_0,\Xi_0,\dots,\psi_{m-2},\Xi_{m-2})+\Delta_m(\psi_0,\Xi_0,\dots,\psi_{m-1},\Xi_{m-1}).\end{aligned}$$

□

Lemma 3.2.

Assume that the operator $N[\psi(\eta),\Xi]$ be contraction and the solution series

$$\psi_0(\eta)+\sum_{m=1}^{\infty}\psi_m(\eta), \qquad (28)$$

and

$$\Xi_0+\sum_{j=1}^{\infty}\Xi_m, \qquad (29)$$

converge to $\psi(\eta)$ and Ξ, respectively, then

$$\sum_{m=0}^{\infty}\Pi_m(\psi_0,\Xi_0,\dots,\psi_{m-1},\Xi_{m-1})=0. \qquad (30)$$

Proof. If the solution series

$$\psi_0(\eta)+\sum_{m=1}^{\infty}\psi_m(\eta), \qquad (31)$$

and

$$\Xi_0+\sum_{j=1}^{\infty}\Xi_m, \qquad (32)$$

converge to $\psi(\eta)$ and Ξ, respectively, then the series

$$\sum_{m=0}^{\infty}\Delta_m(\psi_0,\Xi_0,\dots,\psi_{m-1},\Xi_{m-1}), \qquad (33)$$

will converge to $N[\psi(\eta),\Xi]$ (see [35]).
Now, by using Lemma 3.1 we have

$$\sum_{m=0}^{\infty}\Pi_m(\psi_0,\Xi_0,...,\psi_{m-1},\Xi_{m-1}) =$$
$$-\sum_{m=3}^{\infty}\Delta_{m-1}(\psi_0,\Xi_0,...,\psi_{m-2},\Xi_{m-2})$$
$$+\sum_{m=2}^{\infty}\Delta_m(\psi_0,\Xi_0,...,\psi_{m-1},\Xi_{m-1}) =$$
$$N[\psi(\eta),\Xi]-N[\psi(\eta),\Xi]=0. \tag{34}$$

□

Theorem 3.1.

Let that the operator $N[\psi(\eta),\Xi]$ be contraction. If the solution series $\psi_0(\eta)+\sum_{m=1}^{\infty}\psi_m(\eta)$ and $\Xi_0+\sum_{j=1}^{\infty}\Xi_m$ are convergent, then they must be the exact solution of Eq. (23).

Proof. Since the solution series

$$\psi_0(\eta)+\sum_{m=1}^{\infty}\psi_m(\eta), \tag{35}$$

is convergent, we have

$$\lim_{m\to\infty}\psi_m(\eta)=0. \tag{36}$$

Using the left-hand side of high-order deformation equations, we have

$$\sum_{m=1}^{\infty}\left[\psi_m(\eta)-\chi_m\psi_{m-1}(\eta)\right]=0. \tag{37}$$

Then, by using Lemma 3.2 we have

$$\hbar_1\sum_{m=1}^{\infty}\Delta_m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1})$$
$$+\hbar_2\sum_{m=1}^{\infty}\Pi_m(\psi_0,\Xi_0,...,\psi_{m-1},\Xi_{m-1}) =$$
$$\hbar_1\sum_{m=1}^{\infty}R_m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1})=0. \tag{38}$$

Since $\hbar_1\neq 0$ then the above equation gives

$$\sum_{m=1}^{\infty}\Delta_m(\psi_0,\Xi_0,\ldots,\psi_{m-1},\Xi_{m-1})=N[\psi(\eta),\Xi]=0. \tag{39}$$

□

4. Estimation of errors and residuals

The Mth-order approximation of the solutions $\psi(\eta)$ and Ξ can be expressed as

$$\psi(\eta)\approx\psi_0(\eta)+\sum_{j=1}^{M}\psi_m(\eta),\quad \Xi\approx\Xi_0+\sum_{j=1}^{M}\Xi_m, \tag{40}$$

which are dependent upon the convergence-control parameters $\hbar_1$ and $\hbar_2$.
Let

$$E_M^n(\hbar_1,\hbar_2)=\frac{1}{n}\sum_{j=0}^{n}(N(\sum_{k=0}^{M}\psi_k(\frac{10j}{n}),\sum_{k=0}^{M}\Xi_k))^2, \tag{41}$$

denote the so-called averaged residual error (ARE) at the Mth order of approximation. At the Mth-order of approximation, the ARE E_M^n is a function of both of $\hbar_1$ and $\hbar_2$. We can gain the "optimal" values of $\hbar_1$ and $\hbar_2$ by solving nonlinear algebraic equations

$$\frac{\partial E_M^n}{\partial\hbar_i}=0,\ \ i=1,2. \tag{42}$$

4.1. Minimum value of E_M^n for nonlinear beam equation

Suppose $\beta=-5$, $\alpha=-5$ and $\omega=1$ hence from $\lambda^4+\lambda^2(\omega^2+\alpha)+\beta=0$, we have $\lambda=\sqrt{5}$. Now for $\hbar_1\neq 0$ and $\hbar_2=0$, E_{10}^{20} has the minimum $1.001E-12$ at the "optimal" values $\hbar_1=-1$ and $\hbar_2=0$. The corresponding ARE E_{10}^{20} has the minimum $1.133E-15$ at the optimal values $\hbar_1=-1.068$ and $\hbar_2=-0.011$. The corresponding approximations converges much faster than those given in case of $\hbar_1=-1$ and $\hbar_2=0$, as shown in Tables 1.

4.2. Minimum value of E_M^n for KdV equation

Suppose $\alpha=2$, $\beta=4$ and $\omega=1$, hence from $\lambda^2-\omega=0$, we have $\lambda=1$. In following, we give minimum value of E_M with different procedures.
The corresponding ARE E_{10}^{20} has the minimum $5.361E-7$ at the "optimal" values $\hbar_1=-0.451$ and $\hbar_2=0$. Also, for $\hbar_1\neq 0$ and $\hbar_2\neq 0$, the corresponding ARE E_{10}^{20} is now a function of both $\hbar_1$ and $\hbar_2$, which has the minimum $3.032E-8$ at the "optimal" values $\hbar_1=-0.487$ and $\hbar_2=-0.029$. In this case, the corresponding homotopy-approximations converges faster than those given in case of $\hbar_1=-0.451$ and $\hbar_2=0$, as shown in Table 1.

Table 1. Comparison of the *ARE* given by different procedures for beam and KdV equations.

beam	$\beta=-5,\ \alpha=-5,\ \omega=1,\ \lambda=\sqrt{5}$		
	M	$\hbar_1=-1, \hbar_2=0$	$\hbar_1=-1.068, \hbar_2=-0.011$
	4	3.707E-5	1.345E-4
	6	6.323E-8	5.998E-8
	8	3.921E-10	6.255E-11
	10	1.001E-12	1.133E-15
KdV	$\alpha=2,\ \beta=4,\ \omega=1,\ \lambda=1$		
	M	$\hbar_1=-0.451, \hbar_2=0$	$\hbar_1=-0.487, \hbar_2=-0.029$
	4	4.337E-3	1.270E-3
	6	5.872E-4	1.923E-4
	8	7.551E-5	1.683E-5
	10	5.361E-7	3.032E-8

Table 2. The Padé approximations of $\psi''(0)$ given by different procedures for the beam and KdV equation.

beam	$\beta=-5,\ \alpha=-5,\ \omega=1,\ \lambda=\sqrt{5}$		
	$\pi_{m,n}$	$\hbar_1=-1, \hbar_2=0$	$\hbar_1=-1.068, \hbar_2=-0.011$
	$\pi_{5,5}$	-1.548178597	-1.548119666
	$\pi_{6,6}$	-1.548147819	-1.548120406
	$\pi_{7,7}$	-1.548137657	-1.548149002
	$\pi_{8,8}$	-1.548135882	-1.548180386
	$\pi_{9,9}$	-1.548135856	-1.548128382
KdV	$\alpha=2,\ \beta=4,\ \omega=1,\ \lambda=1$		
	$\pi_{m,n}$	$\hbar_1=-0.451, \hbar_2=0$	$\hbar_1=-0.487, \hbar_2=-0.029$
	$\pi_{2,2}$	-1.287559772	-1.330695064
	$\pi_{3,3}$	-1.318975383	-1.312329898
	$\pi_{4,4}$	-1.318338968	-1.318338968
	$\pi_{5,5}$	-1.313135052	-1.310289334
	$\pi_{6,6}$	-1.310114082	-1.309617891
	$\pi_{7,7}$	-1.309136177	-1.309384890

Table 3. The Padé approximations of Ξ given by different procedures for the beam and KdV equation.

beam	$\beta=-5,\ \alpha=-5,\ \omega=1,\ \lambda=\sqrt{5}$		
	$\Xi_{m,n}$	$\hbar_1=-1, \hbar_2=0$	$\hbar_1=-1.151, \hbar_2=-1.423E-2$
	$\Xi_{5,5}$	97.00540850	96.96725308
	$\Xi_{6,6}$	97.34805918	97.42577367
	$\Xi_{7,7}$	97.38923412	97.38502313
	$\Xi_{8,8}$	97.38330429	97.38608215
	$\Xi_{9,9}$	97.37863258	97.37975212
KdV	$\alpha=2,\ \beta=4,\ \omega=1,\ \lambda=1$		
	$\Xi_{m,n}$	$\hbar_1=-0.451, \hbar_2=0$	$\hbar_1=-0.487, \hbar_2=-0.029$
	$\Xi_{2,2}$	1.240626839	1.240626839
	$\Xi_{3,3}$	1.271576322	1.236721215
	$\Xi_{4,4}$	1.263398561	1.261675520
	$\Xi_{5,5}$	1.260525504	1.255296180
	$\Xi_{6,6}$	1.256956468	1.256063425
	$\Xi_{7,7}$	1.256081490	1.256591897

5. Padé approximation

Let $\phi(\eta;q) = \psi_0(\eta) + \sum_{j=1}^{\infty} \psi_m(\eta)q^j$ denote the homotopic solution of the nonlinear differential equations in the framework of biparametric homotopy. The Padé approximant $\pi_{m,n}(\eta;q)$ of the function $\phi(\eta;q)$ is a rational function of the form

$$\pi_{m,n}(\eta;q) = \frac{P_m(\eta;q)}{Q_n(\eta;q)} = \frac{a_0(\eta) + a_1(\eta)q + \ldots + a_m(\eta)q^m}{b_0(\eta) + b_1(\eta)q + \ldots + b_n(\eta)q^n},$$

so that the coefficients of q^i , $i \neq 1,2,\ldots,m+n$ of the power series $(\phi Q_n - P_m)(\eta;q)$ vanish. Now the Padé approximant $\pi_{m,n}$ can fit the power series through orders $1, q, q^2, \ldots, q^{m+n}$ with an error of $O(q^{m+n+1})$. Substituting $P_m(\eta;q) = a_0(\eta) + a_1(\eta)q + \ldots + a_m(\eta)q^m$ and $Q_n(\eta;q) = b_0(\eta) + b_1(\eta)q + \ldots + b_n(\eta)q^n$ in $(\phi Q_n - P_m)(\eta;q) = \psi_0(\eta) + \sum_{j=1}^{m+n} \psi_m(\eta)q^j$ and setting $b_0(\eta) = 1$, we have

$$\begin{cases} \psi_j(\eta) + \sum_{i=1}^{\min\{n,j\}} \psi_{j-i}(\eta)b_i(\eta) = a_j(\eta), \\ \quad j = 0,1,2,\ldots,m, \\ \psi_j(\eta) + \sum_{i=1}^{\min\{j,n\}} \psi_{j-i}(\eta)b_i(\eta) = 0, \\ \quad j = m+1, m+2, \ldots, m+n. \end{cases} \tag{43}$$

Solution of the above system gives the unknowns a_i, $i = 1,2,\ldots,m$ and b_j, $j = 1,2,\cdots,n$, and hence by setting $q = 1$ Padé approximant $\pi_{m,n}$ is obtained. In many cases, the Padé approximant $\pi_{m,n}$ does not depend upon the auxiliary parameter h_1 in case of $h_2 = 0$, as pointed out by Liao [7]. Note that the Padé approximant $\pi_{m,n}$ in case of $h_1 \neq 0$ and $h_2 \neq 0$ depend upon the auxiliary parameters h_1 and h_2. But, we cannot give a mathematical proof about it.

The value of $\psi''(0)$ is shown in Table 2. It is obvious that results given by the optimal values $h_1 \neq 0$ and $h_2 \neq 0$ are a little better than those given by the optimal value $h_1 \neq 0$ in case of $h_2 = 0$.

We can apply the padé technique to accelerate the convergence rate of Mth-order approximations of amplitude Ξ. The $\Xi_{m,n}$ homotopy-Padé approximation of amplitude Ξ is formulated by

$$\Xi_{m,n} = \frac{\sum_{j=0}^{m} \Xi_j}{1 + \sum_{i=1}^{n} \Xi_{m+1+i}}. \tag{44}$$

The value of the amplitude is shown in Table 3.

6. Results and discussion

In this section, the comparison of numerical and analytical approximations of our proposed approach for the extended fifth-order Korteweg–de Vries equation [36] and nonlinear beam equation [33] is presented. As shown in Figs. 1–6, the obtained results are in excellent agreement with the references.

Extended fifth-order Korteweg–de Vries equation. Sauceza et al. [36] considered the numerical solution of an extended fifth-order KdV model describing (water) waves

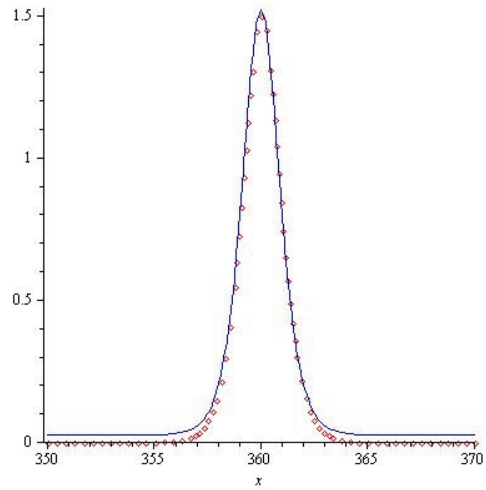

Figure 1. Point: Numerical solution at $t = 300, b = 0$, mesh points 10001,20001,40001 [36]; Solid: Numerical solution with our method ($\hbar_1 = -1.2, \hbar_2 = -0.1$) at $t = 300, b = 0$.

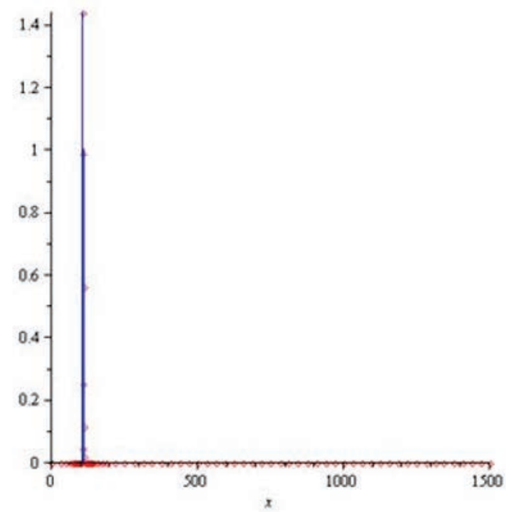

Figure 2. Point: Numerical solution at $t = 90, b = 0$, mesh points 40001 [36]; Solid: Numerical solution with our method ($\hbar_1 = -0.9, \hbar_2 = -0.3$) at $t = 90, b = 0$.

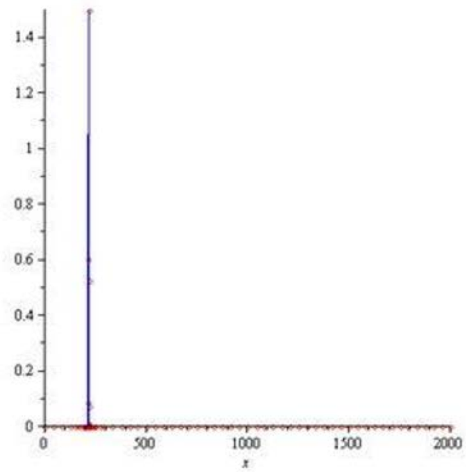

Figure 3. Point: Numerical solution at $t = 180, b = 0$, mesh points 40001 [36]; Solid: Numerical solution with our method ($\hbar_1 = -1.3, \hbar_2 = -0.13$) at $t = 180, b = 0$.

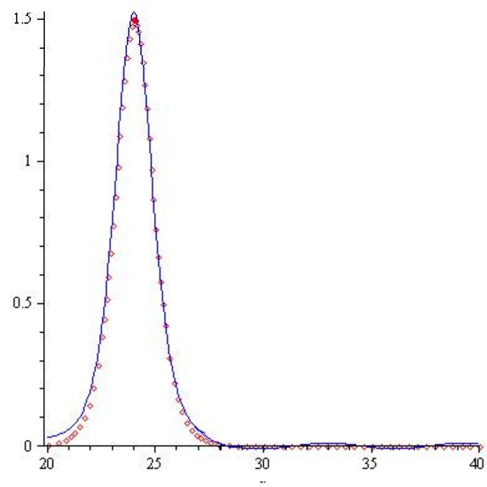

Figure 4. Point: Numerical solution at $t = 20, b = 0$, mesh points 1001 [36]; Solid: Numerical solution with our method ($\hbar_1 = -.85, \hbar_2 = -0.09$) at $t = 20, b = 0$.

and solitons in the presence of surface tension by using the finite difference method. Their model is described by

$$u_t + \frac{5}{12} u_{5x} + (x - b) u_{3x} + (3u + 2\mu u_{xx}) u_x = 0. \quad (45)$$

Nonlinear beam equation. As mentioned before, in [33]

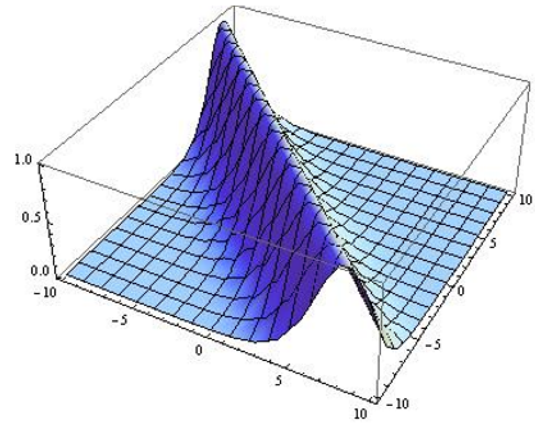

Figure 5. Describe the solution of Eq. (46) where $x, t \in [-10, 10]$ and $\hbar_1 = \hbar_2 = -1, \alpha = 2, \beta = -0.66$.

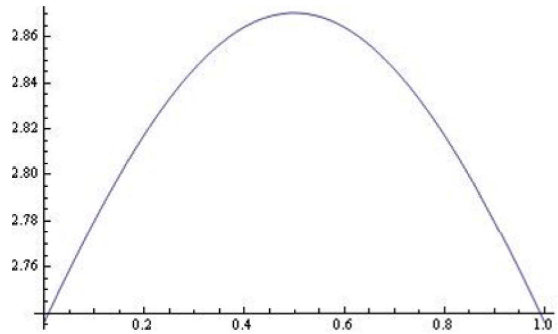

Figure 6. Describe the solution of Eq. (47) where $x \in [0, 1]$ and $\hbar_1 = \hbar_2 = -1, \alpha = -9, \beta = -4$.

Zahra et al. considered the analytical solution of nonlinear beam equation by using the Exp-function method.Their models are described by

$$u_{tt} + u_{4x} + \alpha u_{xx} + \beta u + u^2 = 0, \quad (46)$$

$$u_{4x} + \alpha u_{xx} + +\beta u + u^3 = 0. \quad (47)$$

Remark 1: In Figs. 1–6 the numerical convergence of our solution (for Eq. (45)) and the solution obtained in [36] is presented.
Remark 2: In Figs. 6–6 analytical solution of our approach for Eqs. (46, 47) is presented. The obtained results are exactly same with the results obtained in [33].

7. Conclusion

In this manuscript, we studied the application of a bi-parametric homotopy for solving the nonlinear differential equations and its application in physical phenomena. The present homotopy adds a new parameter to the convergence region that increases the convergence region of the series solutions and generalizes the homotopy analysis method for a wider range of nonlinear problems. All given examples reveal that the present homotopy yields a very effective and convenient technique to the approximate solutions of nonlinear differential equations.

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