Cent. Eur. J. Phys. • 11(10) • 2013 • 1392-1398
DOI: 10.2478/s11534-013-0207-3

VERSITA

Central European Journal of **Physics**

Variational iteration method – a promising technique for constructing equivalent integral equations of fractional order

Research Article

Yi-Hong Wang[1,2*], Guo-Cheng Wu[3,4†], Dumitru Baleanu[5,6,7‡]

1 *Department of Computer Science, Shanghai Normal University Tianhua College, 201815, China*

2 *Department of Mathematics, Zhejiang Forestry & Agriculture University, Linan, 311300, China*

3 *College of Mathematics and Information Science, Neijiang Normal University, Neijiang, 641112, China*

4 *College of Water Resources and Hydropower, Sichuan University, Chengdu, 610065, China*

5 *Department of Mathematics and Computer Sciences, Cankaya University, 06530 Balgat, Ankara, Turkey*

6 *Institute of Space Sciences, Magurele-Bucharest, Romania*

7 *Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia*

Received 31 January 2013; accepted 11 March 2013

Abstract: The variational iteration method is newly used to construct various integral equations of fractional order. Some iterative schemes are proposed which fully use the method and the predictor-corrector approach. The fractional Bagley-Torvik equation is then illustrated as an example of multi-order and the results show the efficiency of the variational iteration method's new role.

PACS (2008): 01.30.-y, 01.30.Xx, 01.30.Tt

Keywords: fractional differential equations • variational iteration method • predictor-corrector • symbolic computation

1. Introduction

Fractional differential equations (FDEs) have been gained much attention recent years. Compared with differential models, they can better describe variety of phenomena

*E-mail: yihongwang2008@yahoo.com.cn
†E-mail: wuguocheng2002@yahoo.com.cn (Corresponding author)
‡E-mail: dumitru@cankaya.edu.tr

such as optical soliton, diffusion phenomena and stability of the control system, etc.

Then, various numerical and analytic methods have been developed from ordinary calculus and play crucial roles when solving FDEs. See, for example, the numerical methods [1–7] and the analytical methods [8–14] which have been extensively applied in fractional calculus. Several excellent monographs [15–17] are available now.

The variational iteration method (VIM) was first applied to FDEs as early as 1998 [13] among which the term with fractional derivatives in the correction functional was assumed as a restricted variation. Recently, the method was extended to other types [14, 18, 19] and the Lagrange multipliers were determined explicitly from Laplace transform. On the other hand, to the best of our knowledge, there is less use of the method in numerical calculus especially in the fractional case. This study reveals the method's new role in seeking integral equations of fractional order. Now various numerical methods for integral equations can be used directly. The famous predicator-corrector formula is considered and some new iterative schemes are newly proposed to solve the FDEs numerically.

2. Preliminaries

2.1. Variational iteration method in fractional calculus

Definition 1 The Riemann-Liouville (R-L) integral [15] is defined as

$$ {}_0I_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau, \quad 0<t,\ 0<\alpha. \tag{1}$$

Definition 2 The Caputo derivative [15] of $u(t)$ is defined by

$$ \begin{aligned} {}_0^CD_t^{\alpha}u(t) &= {}_0I_t^{m-\alpha}u^{(m)}(t) \\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}u^{(m)}(\tau)d\tau, \\ & 0<t,\ m\in Z^+,\ m-1<\alpha\le m. \end{aligned} \tag{2}$$

The variational iteration method was initially developed by He in (1998) [13]. Since then the method became an effective mathematical tool in nonlinear science and often used in fractional calculus. A systemical description of the method and its applications were summarized in some review articles [20, 21]. It can be concluded that the crucial step of the method is to identify the Lagrange multipliers in a more accurate way. In view of this point, several modified versions and new applications were suggested recently, see for example [22–27].

Consider the following often used FDE to illustrate the basic idea of the method,

$$ {}_0^CD_t^{\alpha}u = f(u,t),\ u^{(k)}(0)=c_k,\ k=0,1,\cdots\lceil\alpha\rceil-1 \tag{3}$$

where $f(u,t)$ is a nonlinear term.

Construct the iteration formula as

$$ u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,\tau)\left({}_0^CD_\tau^{\alpha}u_n(\tau) - f(u_n(\tau),\tau)\right)d\tau.$$

From Laplace transform, we successfully identified the Lagrange multiplier $\lambda(t,\tau)$ and suggested a variational iteration formula [14, 18]

$$ \begin{cases} u_{n+1}(t) = u_n(t) + \displaystyle\int_0^t \lambda(t,\tau)\Big({}_0^CD_t^{\alpha}u_n(\tau) \\ \qquad - f(u_n(\tau),\tau)\Big)d\tau, \\ \lambda(t,\tau) = -\dfrac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \end{cases} \tag{4}$$

which is equal to

$$ u_{n+1}(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} c_k\frac{t^k}{k!} + {}_0I_t^{\alpha}f(u_n,\tau)d\tau. \tag{5}$$

For $n\to\infty$ resulting in $u_n\to u$, the above iteration tends to the Volterra integral equation of fractional order

$$ u(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} c_k\frac{t^k}{k!} + {}_0I_t^{\alpha}f(u,\tau)d\tau. \tag{6}$$

One may note the method indeed provides a potential tool to construct equivalent integral equations. Consider the following FDE

$$ \begin{aligned} &{}_0^CD_t^{\alpha}u(t) + {}_0^CD_t^{\beta}u(t) = f(u,t), \\ &u^{(k)}(0)=c_k,\ k=0,1,\cdots\lceil\alpha\rceil-1,\ 0<\beta<\alpha. \end{aligned} \tag{7}$$

The system covers many famous models of fractional order, for example, the Riccati equations, the Bagley-Torvik equation of fractional order. As is well known, it has a similar equivalent integral equation (6). In this paper, we find some new equivalent equations which can be used to obtain numerical solutions of higher accuracies and this is the main purpose of the present work.

2.2. Construction of integral equations using the VIM

We first consider the construction of the variational iteration formula for the FDE (7). Then we prove the uniform convergence from which the equivalent integral equations can be derived.

Lemma 3 Eq. (7) has an explicit Lagrange multiplier

$$\lambda = -(t-\tau)^{\alpha-1} E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right)$$

and the variational iteration formula is given as

$$\begin{cases} u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,\tau)\left({}_0^C D_\tau^\alpha u_n(\tau) + {}_0^C D_\tau^\beta u_n(\tau) - f\left(u_n(\tau),\tau\right)\right) d\tau, 0<\beta<\alpha, \\ \lambda = -(t-\tau)^{\alpha-1} E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right), \end{cases} \tag{8}$$

where $E_{\alpha,\beta}(-t)$ is the Mittag-Leffler function with two parameters. Readers who feel interested in the details are refereed to [14, 18, 19].

Lemma 4 Suppose $f(u,t)$ satisfies the Lipschitz condition

$$\mid f(x,t) - f(y,t) \mid \leq L \mid x-y \mid \tag{9}$$

where L is the Lipschitz constant and $L>0$. Assume that: $I=[0,T]$, $D := I \times [u(0)-b, u(0)+b]$. If $f(u,t) \in C(D)$ with some $h>0$, and some $b>0$, satisfying the condition of (9) and $LKh^\alpha < 1$, then Eq. (7) has a unique solution $u(t)$. $C(D)$ is the class of all continuous functions defined on D, $h = min[T, (b/MK)^{1/\alpha}, (1/KL)^{1/\alpha}]$, M and K are positive constants.

Proof. The main idea comes from the reference [28]. We assume that $u_k(t)$ is continuous and satisfies $\mid u_k(t) - u_0 \mid \leq b$ for $k = 0, 1, 2, ..., n$.

Taking Laplace transform to both sides of Eq. (8), we can have the variational iteration formula

$$u_{n+1}(t) = u_0(t) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right) f\left(u_n(\tau),\tau\right) d\tau \tag{10}$$

where $u_0 = \mathscr{L}^{-1}\left[\frac{\sum_{k=0}^{[\alpha]-1} c_k s^{\alpha-k-1} + \sum_{k=0}^{[\beta]-1} c_k s^{\beta-k-1}}{s^\alpha + s^\beta}\right]$, s is the complex variable of Laplace transform and $\mathscr{L}^{-1}$ denotes the inverse Laplace transform.

Since $f(u_n, t)$ is continuous on $[0,T]$, we have

$$|u_{n+1}(t) - u_0| = \left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right) f(u_n(\tau),\tau) d\tau\right| \leq Mt^\alpha E_{\alpha-\beta,\alpha+1}\left(-t^{\alpha-\beta}\right) \leq MKh^\alpha \leq b$$

where M and K are positive constants and large enough so that

$$|f\left(u_n(\tau),\tau\right)| \leq M,\ \left|E_{\alpha-\beta,\alpha+1}\left(-t^{\alpha-\beta}\right)\right| \leq K$$

and $u_1(t), ..., u_n(t)$ are continuous on $[0,T]$.

From Eq. (10), we obtain the following relation

$$\mid u_{n+1}(t) - u_n(t) \mid \ \leq \frac{M}{L}\left(LKh^\alpha\right)^{n+1}.$$

In fact, from (10), for $n=1$ we have

$$\mid u_1(t) - u_0(t) \mid \leq MKh^\alpha.$$

More generally, in the case of n, we can calculate

$$\begin{aligned}|u_n(t)-u_{n-1}(t)| &= \left|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right)\left(f\left(u_{n-1}(\tau),\tau\right)-f\left(u_{n-2}(\tau),\tau\right)\right)d\tau\right| \\ &\le MKh^{\alpha}(LKh^{\alpha})^{n-1} = \tfrac{M}{L}(LKh^{\alpha})^{n}.\end{aligned}$$

Then, using the above equation, we derive

$$\begin{aligned}| u_{n+1}(t)-u_n(t) | &= \left|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right) f\left(u_n(\tau),\tau\right)-f\left(u_{n-1}(\tau),\tau\right)d\tau\right| \\ &\le \left|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right) L\right| u_n-u_{n-1}\,|d\tau| \\ &\le \left|\int_0^t (t-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right) LMKh^{\alpha}\left(LKh^{\alpha}\right)^{n-1}d\tau\right| \\ &= t^{\alpha}E_{\alpha-\beta,\alpha+1}\left(-t^{\alpha-\beta}\right)LMKh^{\alpha}\left(LKh^{\alpha}\right)^{n-1} \\ &\le MKh^{\alpha}\left(LKh^{\alpha}\right)^{n} = \tfrac{M}{L}\left(LKh^{\alpha}\right)^{n+1}\end{aligned}$$

It follows that

$$u_0(t)+\sum_{i=0}^{n-1}\left(u_{n+1}(t)-u_n(t)\right)=u_n(t) \tag{11}$$

is uniformly convergent on $[0,h]$. As a result, we can obtain

$$\lim_{n\to\infty} u_n(t)=u(t).$$

On the other hand, since $f(u,t)$ is uniformly continuous on D, $f(u_n,t)$ is convergent uniformly on $[0,h]$ for $n\to\infty$.
Theorem 5 Eq. (7) has an equivalent integral equation

$$\begin{cases} u(t)=g(t)+\displaystyle\int_0^t (t-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}\left(-(t-\tau)^{\alpha-\beta}\right)f(u(\tau),\tau)d\tau, 0<\beta<\alpha, \\ g(t)=\mathscr{L}^{-1}\left[\frac{\sum_{k=0}^{\lceil\alpha\rceil-1}c_k s^{\alpha-k-1}+\sum_{k=0}^{\lceil\beta\rceil-1}c_k s^{\alpha-k-1}}{s^{\alpha}+s^{\beta}}\right]. \end{cases} \tag{12}$$

The inhomogeneous term $g(t)$ of the above Volterra integral equation can be determined by taking Laplace transform to both sides of Eq. (8).

2.3. The predictor-corrector formula

Firstly, the product trapezoidal quadrature formula is applied to replace the integral of (12). Set $h=T/N$, $t_n=nh$, $n=0,1,\ldots,N\in Z^{+}$. Considering the following integration

$$I_n^{\alpha}=\int_0^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1+k(\alpha-\beta)}f(u(\tau),\tau)d\tau,$$

the above quadrature can be approximated by

$$\int_0^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1+k(\alpha-\beta)}f(u(\tau),\tau)d\tau \approx \int_0^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1+k(\alpha-\beta)}f_{n+1}(u(\tau),\tau)d\tau.$$

If $f_{n+1}(u(\tau),\tau)d\tau$ is determined by

$$f_{n+1}(u(\tau),\tau)d\tau\,|_{[t_j,t_{j+1}]}=\frac{t_{j+1}-t}{t_{j+1}-t_j}f\left(u\left(t_j\right),t_j\right)+\frac{t-t_j}{t_{j+1}-t_j}f\left(u\left(t_{j+1}\right),t_{j+1}\right), 0\le j\le n,$$

then the fractional trapezoidal formula is derived

$$u(t_{n+1}) = u_0 + \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k(\alpha-\beta)+\alpha)} \frac{h^{k(\alpha-\beta)+\alpha}}{(k(\alpha-\beta)+\alpha)(k(\alpha-\beta)+\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1} f\left(t_j, u\left(t_j\right)\right), \tag{13}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1+k(\alpha-\beta)} - (n-\alpha)(n+1)^{\alpha+k(\alpha-\beta)}, & j=0, \\ (n-j+2)^{\alpha+1+k(\alpha-\beta)} + (n-j)^{\alpha+1+k(\alpha-\beta)} - 2(n-j+1)^{\alpha+1+k(\alpha-\beta)}, & 1 \le j \le n, \\ 1, & j = n+1. \end{cases}$$

Eq. (13) is equal to

$$\begin{aligned} u(t_{n+1}) = u_0(t) &+ \sum_{k=0}^{\infty} \frac{h^{\alpha+k(\alpha-\beta)}(-1)^k}{\Gamma(\alpha+k(\alpha-\beta))(\alpha+k(\alpha-\beta))(\alpha+k(\alpha-\beta)+1)} f\left(u\left(t_{n+1}\right), t_{n+1}\right) \\ &+ \sum_{k=0}^{\infty} \frac{h^{\alpha+k(\alpha-\beta)}(-1)^k}{\Gamma(\alpha+k(\alpha-\beta))(\alpha+k(\alpha-\beta))(\alpha+k(\alpha-\beta)+1)} \sum_{j=0}^{n} a_{j,n+1} f\left(u\left(t_j\right), t_j\right). \end{aligned} \tag{14}$$

The right hand side of system (14) contains the term $u(t_{n+1})$. In order to start the Adams-Moulton iterative method, the solution is accomplished by first "predicting" ($u^p(t_{n+1})$) the result using the explicit Adams-Bashforth formula, and then "correcting" ($u(t_{n+1})$). It is consistent in the predictor-corrector algorithm in [1]. The truncated error estimate is

$$\max_{j=0,1\cdots N} \left| u\left(t_j\right) - u_n\left(t_j\right) \right| = O(h^p)$$

in which $p = \min(2, 1+\alpha)$.
We carry over Deng's technique to get the values $u^p(t_{n+1})$

$$\begin{aligned} \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1+k(\alpha-\beta)} f(u(\tau),\tau) d\tau \approx & \int_0^{t_n} (t_{n+1}-\tau)^{\alpha-1+k(\alpha-\beta)} f(u(\tau),\tau) d\tau \\ &+ \int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1+k(\alpha-\beta)} f(u(t_n), t_n) d\tau. \end{aligned} \tag{15}$$

Similarly, using the standard quadrature technique, the right hand side of (15) can be recast as

$$\begin{aligned} \int_0^{t_n} (t_{n+1}-\tau)^{\alpha-1+k(\alpha-\beta)} f_n(u(\tau),\tau) d\tau &+ \int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1+k(\alpha-\beta)} f\left(u\left(t_n\right), t_n\right) d\tau \\ &= \frac{h^{\alpha+k(\alpha-\beta)}}{(\alpha+k(\alpha-\beta))(\alpha+\alpha k+1)} \sum_{j=0}^{n} b_{j,n+1} f\left(u\left(t_j\right), t_j\right) \end{aligned} \tag{16}$$

where

$$b_{j,n+1} = \begin{cases} \alpha+k(\alpha-\beta)+1, & j=0, \\ a_{j,n+1}, & 1 \le j \le n-1, \\ 2^{\alpha+k(\alpha-\beta)+1}-1, & j=n. \end{cases}$$

Together with (14) and (16), a new predictor corrector approach for solving (7) is derived and it has numerical accuracy $O(h)^p, p = \min(2, 1+2\alpha)$. The present predictor corrector approach has two benefits:

I. The VIM is employed to establish new numerical iterative schemes and memory kernel functions are constructed.
II. For the predictor formula, the numerical approximation is more accurate since the product trapezoidal quadrature rule is used instead of the product rectangle one for the integral in the interval $[0, t_n]$, and almost half of the computational cost is reduced since the most expensive computation$\sum_{j=0}^{n} a_{j,n+1} f(u(t_j), t_j)$ just needs to be computed one time instead two.

3. Numerical example

The Bagley-Torvik equation of fractional order reads

$$\frac{d^2u}{dt^2} + {}_0^C D_t^\alpha u + u(t) = f(t), 1 < \alpha \leq 2, u(0) = 0, u'(0) = 0, \quad (17)$$

which has the exact solution [15]

$$\int_0^t \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (t-\tau)^{2j+1} E^{(j)}_{2-\alpha,2+\alpha j} \left(-(t-\tau)^{2-\alpha}\right) f(\tau) d\tau. \quad (18)$$

We can obtain the following three variational iteration formulae. The first variational iteration formula is identified as

$$u_{n+1}(t) = u_0(t) + \int_0^t \lambda(t,\tau) \big({}_0^C D_t^\alpha u_n + u_n - f(\tau)\big) d\tau, u_0(t) = 0 \quad (19)$$

where the Lagrange multiplier is

$$\lambda = \tau - t \quad (20)$$

and it comes across as the one for differential equations of second order [23].
This iteration formula leads to the integral equation (VIM-A)

$$u(t) = \int_0^t (\tau - t) \left({}_0^C D_t^\alpha u(\tau) + u(\tau) - f(\tau)\right) d\tau.$$

The comparison between the numerical results by means of the classic Predictor-Corrector method and the exact solution (18) is made which shows the validness of the Lagrange multipliers (20) and the VIM-A in Figure 1.
The second one can be derived as

$$u_{n+1}(t) = u_0(t) + \int_0^t \lambda(t,\tau)(u_n(\tau) - f(\tau)) d\tau, u_0 = 0 \quad (21)$$

where the Lagrange multiplier reads

$$\lambda = (\tau - t) E_{2-\alpha,2} \left(-(t-\tau)^{2-\alpha}\right);$$

The integral equation of second kind (VIM-B) can be obtained accordingly

$$u(t) = \int_0^t (\tau - t) E_{2-\alpha,2} \left(-(t-\tau)^{2-\alpha}\right) (u(\tau) - f(\tau)) d\tau. \quad (22)$$

The third variational iteration formula is given as

$$u_{n+1}(t) = u_0(t) + \int_0^t \lambda(t,\tau)(-f(\tau)) d\tau, u_0(t) = 0, \quad (23)$$

and the Lagrange multiplier is identified as

$$\lambda(t,\tau) = -\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (t-\tau)^{2j+1} E^{(j)}_{2-\alpha,2+\alpha j} \left(-(t-\tau)^{2-\alpha}\right)$$

where $E^{(j)}_{\alpha,\beta}$ is defined as

$$E^{(j)}_{\alpha,\beta}(t) = \frac{d^j}{dt^j} E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{(k+j)!\, t^k}{k!\Gamma(\alpha k + \alpha j + \beta)}.$$

For (23), since the Lagrange multiplier is explicit enough, the exact solution (VIM-C) (18) is obtained within one iteration. The reference [23] suggested the variational iteration formula in the integral forms: the VIM-I, II and III for analytical solutions of ODEs. This study aims at the method's new role in establishing integral equations of fractional order from which various numerical methods and rich numerical schemes can be applied. Then one can choose the best one for numerical solutions.
We set $f(t) = 8$ and apply the numerical algorithm in the Section 2 to the integral equations. The results are shown in Figure 2, respectively. It can be concluded that all of the three numerical schemes are efficient. Furthermore, the more explicit the Lagrange multipliers is, the higher the accuracy of the corresponding numerical solution. Especially for integral equations, the VIM-B and the VIM-C, numerical methods become more convenient since there are no terms containing fractional derivatives.

4. Conclusion

As is well known, the calculation of the analytic solutions of the FDEs becomes tedious even impossible in high order case. In this paper, we consider the combination of

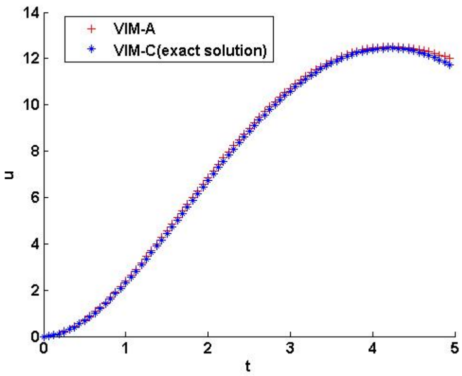

Figure 1. Comparison between the numerical solutions of the VIM-A and VIM-C for $\alpha = 1.5$ and the step size $h = 1/16$.

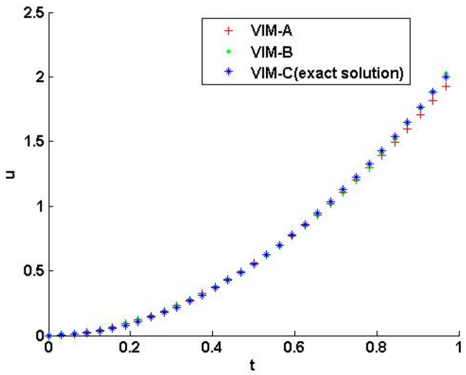

Figure 2. Comparison among the numerical solutions of the VIM-A (the classical predictor-corrector method), VIM-B (our predictor-corrector method) and VIM-C for $\alpha = 1.9$ and the step size $h = 1/32$.

the VIM and numerical methods. The VIM plays a crucial role in constructing new integral equations and the Adams-Moulton formula is then used to establish a new iterative scheme which is accurate and stable with enough "memorized" values. This is consistent with the non-local structure of the fractional differential operators. On the other hand, this also makes the VIM more powerful since the numerical method developed here can overcome the difficulty arising in the analytical calculus when the Lagrange multiplier is complicated and analytical solutions become impossible.

5. Acknowledgment

This work is financially supported by the Zhejiang Natural Science Foundation (Grant Number: LQ12A01010) and the key program of the NSFC (Grant Number: 51134018).

References

[1] K. Diethelm, N. J. Ford, Appl. Math. Comput. 154, 621 (2004)
[2] F. W. Liu, V. Anh, I. Turner, J. Comput. Appl. Math. 166, 209 (2004)
[3] C. Tadjeran, M. M. Meerschaert, H. P. Scheffler, J. Comput. Phys. 213, 205 (2006)
[4] W. H. Deng, J. Comput. Phys. 227, 1510 (2007)
[5] X. J. Li, C. J. Xu, SIAM J. Numer. Anal. 47, 2108 (2009)
[6] C. P. Li, F. H. Zeng, Int. J. Bifur. Chao. 22, 1230014 (2012)
[7] H. G. Sun, W. Chen, Y. Q. Chen, Physica A 388, 4586 (2009)
[8] N. T. Shawagfeh, Appl. Math. Comput. 131, 517 (2002)
[9] H. Jafari, V. Daftardar-Gejji, Appl. Math. Comput. 180, 488 (2006)
[10] J. S. Duan, R. Rach, D. Baleanu, A. M. Wazwaz, Commun. Frac. Calc. 3, 73 (2012)
[11] Q. Wang, Chaos, Soliton. Fractal. 35, 843 (2008)
[12] S. Momani, Z. Odibat, Phys. Lett. A 365, 345 (2007)
[13] J. H. He, Comput. Meth. Appl. Mech. Engr. 167, 57 (1998)
[14] G. C. Wu, D. Baleanu, Appl. Math. Model. 37, 6183 (2013)
[15] I. Podlubny, Fractional Differential Equations (Academic Press, New York, 1999)
[16] A. A. Kilbas, H. M. Srivastav, J. J. Trujillo, Theory and applications of fractional differential equations (Elsevier, New York, 2006)
[17] D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, Fractional calculus models and numerical methods (World Scientific, Singapore, 2012)
[18] G. C. Wu, D. Baleanu, Adv. Diff. Equa. 2013, 18 (2013)
[19] G. C. Wu, D. Baleanu, Adv. Diff. Equa. 2013, 21 (2013)
[20] J. H. He, Int. J. Mod. Phys. B 20, 1141 (2006)
[21] J. H. He, Abstr. Appl. Anal. 2012, 916793 (2012)
[22] A. Ghorbani, Comput. Meth. Appl. Mech. Engr. 197, 4173 (2007)
[23] J. H. He, X. H. Wu, Comput. Math. Appl. 54, 881 (2007)
[24] J. H. He, J. Comput. Appl. Math. 207, 3 (2007)
[25] S. A. Khuri, A. Sayfy, Appl. Math. Lett. 25, 2298 (2012)
[26] H. Jafari, M. Saeidy, D. Baleanu, Cent. Eur. J. Phys. 10, 76 (2012)
[27] H. Jafari, C. M. Khalique, Commun. Frac. Calc. 3, 38 (2012)
[28] K. Diethelm, The Analysis of Fractional Differential Equations (Springer-Verlag, New York, 2004)