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Analytical Approximate Solutions of (*n* + 1)-Dimensional Fractal Heat-Like and Wave-Like Equations

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Abstract: In this paper, we propose a new type $(n+1)$-dimensional reduced differential transform method (RDTM) based on a local fractional derivative (LFD) to solve $(n+1)$-dimensional local fractional partial differential equations (PDEs) in Cantor sets. The presented method is named the $(n+1)$-dimensional local fractional reduced differential transform method (LFRDTM). First the theories, their proofs and also some basic properties of this procedure are given. To understand the introduced method clearly, we apply it on the $(n+1)$-dimensional fractal heat-like equations (HLEs) and wave-like equations (WLEs). The applications show that this new technique is efficient, simply applicable and has powerful effects in $(n+1)$-dimensional local fractional problems.

Keywords: reduced differential transform method; heat like equation; wave like equation; fractional partial differential equations; local fractional derivative

1. Introduction

The importance of fractional calculus and its popularity have increased during the past four decades, due to its applications in many fields of engineering and applied science. For example, analysis of entropy in fractional dynamical systems, entropy in thermodynamics, control theory of dynamic systems, probability and statistics, electrical networks, signal processing, optics, chemical physics, the electrochemistry of corrosion and so on can all be successfully modelled by fractional order differential equations [1–39].

In thermodynamics, entropy is known as a state function of a thermodynamic system. The production of entropy by fractional calculus was suggested in [4]. The production of entropy rate for fractional diffusion processes was discussed in [6–8]. The analysis of entropy in fractional dynamic systems was proposed in [10]. However, these entropy processes are differentiable. Non-differentiable production of entropy in heat conduction of the fractal temperature field was studied in [13]. The heat conduction equation was discussed by the help of local fractional derivative (LFD) [26]. Some numerical methods are applied to many non-differentiable problems in Cantor sets by using LFD [21–29].

The differential transform method (DTM) which is constructed based on Taylor expansion has been a widely used approximation method in recent years. The DTM was introduced and applied to engineering problems by Zhou [40]. The method was applied to solve linear, nonlinear, ordinary, partial and fractional order differential equation problems in biology, engineering, physics [41–49] and so on. The reduced differential transform method (RDTM) was presented by Keskin and Oturanc [50–52] to simplify the DTM calculations. The method is a reliable semi-analytical approach used to find solutions of many types of linear, non-linear, fractional non-fractional order partial differential equations (PDEs). There have been many application of RDTM [50–60]. Previously, the use of LFD with DTM and RDTM was introduced as local fractional DTM (LFDTM) [27] and local fractional reduced differential transform method (LFRDTM) [28]. Furthermore, some basic theorems and applications were given for these methods [27,28]. In addition to that, we now introduce the $(n + 1)$-dimensional case of RDTM with LFD for the first time. The basic definitions and theorems of $(n + 1)$-dimensional LFRDTM are given. Moreover, the presented method was applied to both $(n + 1)$-dimensional fractal homogeneous and inhomogeneous HLEs and WLEs. These equations have been studied by many researchers [60–64]. However, in fractal space, these problems are discussed using $(n + 1)$-dimensional LFRDTM for the first time.

In our present study, the basic definitions of local fractional calculus are given in Section 2. Two-dimensional LFRDTM and $(n + 1)$-dimensional LFRDTM with the basic definitions and theorems are presented in Section 3. In Section 4, the applications of the new method, graphics of the solutions and discussion are given and finally, we put forth our conclusions in Section 5.

2. Preliminaries

In this section, we give same basic definitions and important properties of LFD on fractal space [11,27].

Definition 1. *Let $C_\alpha(a,b)$ be a set of the non-differentiable functions with the fractal dimension $\alpha(\alpha \in (0,1])$. For $\psi(x) \in C_\alpha(a,b)$, the LFD operator of $\psi(x)$ of order $\alpha(\alpha \in (0,1])$ at the $x = x_0$ is defined as follows* [[11]]:

$$D^{(\alpha)}\psi(x_0) = \frac{d^\alpha \psi(x_0)}{dx^\alpha} = \lim_{x \to x_0} \frac{\Delta^\alpha(\psi(x) - \psi(x_0))}{(x - x_0)^\alpha}, \tag{1}$$

where:

$$\Delta^\alpha(\psi(x) - \psi(x_0)) \cong \Gamma(1+\alpha)[\psi(x) - \psi(x_0)] \tag{2}$$

Lemma 1 [11]. *In fractal space, let $\psi, \varphi \in C_\alpha(a,b)$ and $\alpha \in (0,1]$. Then:*

(i) $D^{(\alpha)}(\lambda\psi(x) \pm \gamma\varphi(x)) = \lambda D^{(\alpha)}\psi(x) \pm \gamma D^{(\alpha)}\varphi(x)$ *for* $\lambda, \gamma \in \mathbb{R}$,

(ii) $D^{(\alpha)}(\psi(x)\varphi(x)) = \left[D^{(\alpha)}\psi(x)\right]\varphi(x) + \psi(x)\left[D^{(\alpha)}\varphi(x)\right]$,

(iii) $D^{(\alpha)}\left(\frac{\psi(x)}{\varphi(x)}\right) = \frac{\left[D^{(\alpha)}\psi(x)\right]\varphi(x) - \psi(x)\left[D^{(\alpha)}\varphi(x)\right]}{\varphi^2(x)}$.

Some basic operations of LFD on fractal space are presented in Table 1 (see [26]).

Table 1. The some basic operations of LFD.

$\psi(x)$	$D^{(\alpha)}\psi(x)$	Special Functions
$\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$	$\frac{x^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)}$	
$E_\alpha(x^\alpha)$	$E_\alpha(x^\alpha)$	$E_\alpha(x^\alpha)=\sum\limits_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$
$E_\alpha(-x^\alpha)$	$-E_\alpha(-x^\alpha)$	$E_\alpha(x^\alpha)=\sum\limits_{k=0}^{\infty}\frac{(-1)^k x^{k\alpha}}{\Gamma(1+k\alpha)}$
$\sin_\alpha(x^\alpha)$	$\cos_\alpha(x^\alpha)$	$\sin_\alpha(x^\alpha)=\sum\limits_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}$
$\cos_\alpha(x^\alpha)$	$-\sin_\alpha(x^\alpha)$	$\cos_\alpha(x^\alpha)=\sum\limits_{k=0}^{\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(1+2k\alpha)}$
$\sinh_\alpha(x^\alpha)$	$\cosh_\alpha(x^\alpha)$	$\sinh_\alpha(x^\alpha)=\sum\limits_{k=0}^{\infty}\frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}$
$\cosh_\alpha(x^\alpha)$	$\sinh_\alpha(x^\alpha)$	$\cos_\alpha(x^\alpha)=\sum\limits_{k=0}^{\infty}\frac{x^{2k\alpha}}{\Gamma(1+2k\alpha)}$

Definition 2. *The local fractional partial derivative operator of $\psi(x,t)$ of order $\alpha(\alpha\in(0,1])$ with respect to t at the point (x,t_0) is defined as follows* [11,27]:

$$D_t^{(\alpha)}\psi(x,t_0)=\frac{\partial^\alpha\psi(x,t_0)}{\partial t^\alpha}=\lim_{t\to t_0}\frac{\Delta^\alpha(\psi(x,t)-\psi(x,t_0))}{(t-t_0)},\tag{3}$$

where:

$$\Delta^\alpha(\psi(x,t)-\psi(x,t_0))\cong\Gamma(1+\alpha)[\psi(x,t)-\psi(x,t_0)]\tag{4}$$

In view of (1), the local fractional partial derivative operator of $\psi(x,t)$ of order $k\alpha(\alpha\in(0,1])$ is given by [11,27]:

$$D_t^{(k\alpha)}\psi(x,t)=\frac{\partial^{k\alpha}\psi(x,t)}{\partial t^{k\alpha}}=\underbrace{D_t^{(\alpha)}D_t^{(\alpha)}\cdots D_t^{(\alpha)}}_{k\text{ times}}\psi(x,t).\tag{5}$$

3. Main Results

In this section, we describe two-dimensional LFRDTM and $(n+1)$-dimensional LFRDTM.

3.1. Two-Dimensional LFRDTM

In this subsection, we recall and review briefly the local fractional Taylor theorems, and then, we extend two-dimensional LFRDTM.

Lemma 2 (Local fractional Taylor's theorem) [27,28]. *Suppose that $\frac{d^{(k+1)\alpha}}{dx^{(k+1)\alpha}}\psi(x)\in C_\alpha(a,b)$, for $a,b\in\mathbb{R}$, $k=0,1,2,...,n$ and $\alpha\in(0,1]$, we have:*

$$\psi(x)=\sum_{k=0}^{\infty}\frac{d^{k\alpha}}{dx^{k\alpha}}\psi(x_0)\frac{(x-x_0)^{\alpha k}}{\Gamma(1+k\alpha)}\tag{6}$$

where $a<x_0<x<b$, $\forall x\in(a,b)$.

Lemma 3 [27,28]. *Suppose that $\frac{d^{(k+1)\alpha}}{dx^{(k+1)\alpha}}\psi(x)\in C_\alpha(a,b)$, for $a,b\in\mathbb{R}$, $k=0,1,2,...,n$ and $\alpha\in(0,1]$, we have:*

$$\psi(x)=\sum_{k=0}^{\infty}\frac{d^{k\alpha}}{dx^{k\alpha}}\psi(0)\frac{x^{\alpha k}}{\Gamma(1+k\alpha)},\quad\forall x\in(a,b).\tag{7}$$

Definition 3. *The two-dimensional local fractional reduced differential transform (LFRDT) $\Psi_k(x)$ of the function $\psi(x,t)$ is defined by the following formula* [[27,28]]:

$$\Psi_k(x) = \frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}\psi(x,t)}{\partial t^{k\alpha}}\right]_{t=0} \tag{8}$$

where $k = 0,1,2,...,n$ and $\alpha \in (0,1]$.

Definition 4. *The two-dimensional local fractional reduced differential inverse transform of $\Psi_k(x)$ is defined by the following formula* [27,28]:

$$\psi(x,t) = \sum_{k=0}^{\infty}\Psi_k(x)t^{k\alpha} \tag{9}$$

where $\alpha \in (0,1]$.

Using Definitions 3 and 4, Based on results of [28], the fundamental mathematical operations of the two-dimensional LFRDTM are presented in Table 2:

Table 2. Basic operations of the two-dimensional LFRDTM.

Original Function	Transformed Function
$\psi(x,t)$	$\Psi_k(x) = \frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}\psi(x,t)}{\partial t^{k\alpha}}\right]_{t=0}$
$\psi(x,t) = a\pi(x,t) \pm b\varphi(x,t)$	$\Psi_k(x) = a\Pi_k(x) \pm b\Phi_k(x)$
$\psi(x,t) = \pi(x,t)\varphi(x,t)$	$\Psi_k(x) = \sum_{s=0}^{k}\Pi_s(x)\Phi_{k-s}(x) = \sum_{s=0}^{k}\Phi_s(x)\Pi_{k-s}(x)$
$\psi(x,t) = a\frac{\partial^{n\alpha}\varphi(x,t)}{\partial t^{n\alpha}}$	$\Psi_k(x) = \frac{\Gamma(1+k\alpha+n\alpha)}{\Gamma(1+k\alpha)}\Phi_{k+n}(x)$
$\psi(x,t) = \frac{x^{m\alpha}t^{n\alpha}}{\Gamma(1+m\alpha)\Gamma(1+n\alpha)}$	$\Psi_k(x) = \frac{x^{m\alpha}}{\Gamma(1+m\alpha)}\frac{\delta_\alpha(k-n)}{\Gamma(1+\alpha)}$, $\delta_\alpha(k-n) = \begin{cases} 1, k = n \\ 0, k \neq n \end{cases}$

In Table 2, the lowercase $\psi(x,t)$, $\pi(x,t)$ and $\varphi(x,t)$ represent the local fractional analytic original functions while the uppercase $\Psi_k(x)$, $\Pi_k(x)$ and $\Phi_k(x)$ stand for LFRDT functions. a and b are constants.

3.2. (n + 1)-Dimensional LFRDTM

In this subsection, the lowercase $\psi(\sigma,t)$ represents the local fractional analytic original function while the uppercase $\Psi_k(\sigma)$ stands for (n + 1)-dimensional LFRDT function. Here σ is used for $\sigma = (x_1, x_2, ..., x_n)$ through the study. The basic definitions of (n + 1)-dimensional LFRDTM are presented as follows.

Definition 5. *The (n + 1)-dimensional LFRDT $\Psi_k(\sigma)$ of the function $\psi(\sigma,t)$ is defined by the following formula*:

$$\Psi_k(\sigma) = \frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\psi(\sigma,t)\right]_{t=0} \tag{10}$$

where $k = 0,1,2,...,n$ and $\alpha \in (0,1]$.

Definition 6. *The (n + 1)-dimensional local fractional reduced differential inverse transform of* $\Psi_k(\sigma)$ *is defined by the following formula:*

$$\psi(\sigma,t)=\sum_{k=0}^{\infty}\Psi_k(\sigma)t^{k\alpha} \tag{11}$$

where $\alpha\in(0,1]$.

Using Definitions 5 and 6, the theorems of (n + 1)-dimensional LFRDTM, which is the more general form of the operations given in Table 2, are deduced as follows:

Theorem 1. *Let* λ *and* γ *be constants. If* $\psi(\sigma,t)=\lambda\pi(\sigma,t)\pm\gamma\varphi(\sigma,t)$, *then* $\Psi_k(\sigma)=\lambda\Pi_k(\sigma)\pm\gamma\Phi_k(\sigma)$.

Proof. (n + 1)-dimensional LFRDT of $\pi(\sigma,t)$ and $\varphi(\sigma,t)$ can be written as the following:

$$\left.\begin{aligned}\Psi_k(\sigma)&=\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\psi(\sigma,t)\right]_{t=0},\\ \Pi_k(\sigma)&=\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\pi(\sigma,t)\right]_{t=0},\\ \Phi_k(\sigma)&=\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\varphi(\sigma,t)\right]_{t=0}.\end{aligned}\right\} \tag{12}$$

From (12), $\Psi_k(\sigma)$ is obtained as:

$$\begin{aligned}\Psi_k(\sigma)&=\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\psi(\sigma,t)\right]_{t=0}\\ &=\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}(\lambda\pi(\sigma,t)\pm\gamma\varphi(\sigma,t))\right]_{t=0}\\ &=\frac{1}{\Gamma(1+k\alpha)}\left[\lambda\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\pi(\sigma,t)\pm\gamma\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\varphi(\sigma,t)\right]_{t=0}\\ &=\lambda\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\pi(\sigma,t)\right]_{t=0}\pm\gamma\frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\varphi(\sigma,t)\right]_{t=0}\\ &=\lambda\Pi_k(\sigma)\pm\gamma\Phi_k(\sigma).\end{aligned} \tag{13}$$

The proof is thus completed. □

Theorem 2. *If* $\psi(\sigma,t)=\pi(\sigma,t)\varphi(\sigma,t)$, *then* $\Psi_k(\sigma)=\sum_{s=0}^{k}\Pi_s(\sigma)\Phi_{k-s}(\sigma)$.

Proof. By the help of Definition 6, $\pi(\sigma,t)$ and $\varphi(\sigma,t)$ can be written that:

$$\left.\begin{aligned}\pi(\sigma,t)&=\sum_{k=0}^{\infty}\Pi_k(\sigma)t^{k\alpha},\\ \varphi(\sigma,t)&=\sum_{k=0}^{\infty}\Phi_k(\sigma)t^{k\alpha}.\end{aligned}\right\} \tag{14}$$

Then, from (14), $\psi(\sigma,t)$ is obtained as:

$$\begin{aligned}\psi(\sigma,t)&=\sum_{k=0}^{\infty}\Pi_k(\sigma)t^{k\alpha}\sum_{k=0}^{\infty}\Phi_k(\sigma)t^{k\alpha}\\ &=\left[\Pi_0(\sigma)+\Pi_1(\sigma)t^{\alpha}+\Pi_2(\sigma)t^{2\alpha}+\cdots+\Pi_n(\sigma)t^{n\alpha}+\cdots\right]\\ &\quad\times\left[\Phi_0(\sigma)+\Phi_1(\sigma)t^{\alpha}+\Phi_2(\sigma)t^{2\alpha}+\cdots+\Phi_n(\sigma)t^{n\alpha}+\cdots\right]\\ &=\Pi_0(\sigma)\Phi_0(\sigma)+[\Pi_0(\sigma)\Phi_1(\sigma)+\Pi_1(\sigma)\Phi_0(\sigma)]t^{\alpha}\\ &\quad+[\Pi_0(\sigma)\Phi_2(\sigma)+\Pi_1(\sigma)\Phi_1(\sigma)+\Pi_2(\sigma)\Phi_0(\sigma)]t^{2\alpha}+\cdots\\ &=\sum_{k=0}^{\infty}\sum_{s=0}^{k}\Pi_s(\sigma)\Phi_{k-s}(\sigma)t^{k\alpha}.\end{aligned} \tag{15}$$

Hence, $\Psi_k(\sigma)$ is found as:

$$\Psi_k(\sigma) = \sum_{s=0}^{k} \Pi_s(\sigma)\Phi_{k-s}(\sigma) \tag{16}$$

The proof is thus completed. □

Theorem 3. *If* $\psi(\sigma,t) = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}\pi(\sigma,t)$, *then* $\Psi_k(\sigma) = \frac{\Gamma(1+k\alpha+n\alpha)}{\Gamma(1+k\alpha)}\Pi_{k+n}(\sigma)$.

Proof. By the help of Definition 6, (*n* + 1)-dimensional LFRDT of $\pi(\sigma,t)$ can be written that:

$$\Pi_k(\sigma,t) = \frac{1}{\Gamma(1+k\alpha)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}\pi(\sigma,t)\right]_{t=0}. \tag{17}$$

For $\psi(\sigma,t) = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}\pi(\sigma,t)$, by using (17), $\Psi_k(\sigma)$ can be obtained as:

$$\begin{aligned} \Psi_k(\sigma) &= \tfrac{1}{\Gamma(1+k\alpha)}\left[\tfrac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\tfrac{\partial^{n\alpha}}{\partial t^{n\alpha}}\pi(\sigma,t)\right)\right]_{t=0} \\ &= \tfrac{1}{\Gamma(1+k\alpha)}\left[\tfrac{\partial^{(n+k)\alpha}}{\partial t^{(n+k)\alpha}}\pi(\sigma,t)\right]_{t=0} \\ &= \tfrac{\Gamma(1+(n+k)\alpha)}{\Gamma(1+k\alpha)}\tfrac{1}{\Gamma(1+(n+k)\alpha)}\left[\tfrac{\partial^{(n+k)\alpha}}{\partial t^{(n+k)\alpha}}\pi(\sigma,t)\right]_{t=0} \\ &= \tfrac{\Gamma(1+k\alpha+n\alpha)}{\Gamma(1+k\alpha)}\Pi_{k+n}(\sigma) \end{aligned} \tag{18}$$

The proof is thus completed. □

Theorem 4. *If* $\psi(\sigma,t) = \frac{x_1^{q_1\alpha}}{\Gamma(1+q_1\alpha)}\frac{x_2^{q_2\alpha}}{\Gamma(1+q_2\alpha)}\cdots\frac{x_n^{q_n\alpha}}{\Gamma(1+q_n\alpha)}\frac{t^{m\alpha}}{\Gamma(1+m\alpha)}$, *then* $\Psi_k(\sigma) = \frac{x_1^{q_1\alpha}}{\Gamma(1+q_1\alpha)}\frac{x_2^{q_2\alpha}}{\Gamma(1+q_2\alpha)}\cdots\frac{x_n^{q_n\alpha}}{\Gamma(1+q_n\alpha)}\frac{\delta_\alpha(k-m)}{\Gamma(1+\alpha)}$, *where* $\delta_\alpha(k-m) = \begin{cases} 1 & if\ k = m \\ 0 & if\ k \neq m \end{cases}$.

Proof. From Definition 5, we have:

$$\begin{aligned} \Psi_k(\sigma) = &\ \tfrac{1}{\Gamma(1+k\alpha)}\left[\tfrac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\tfrac{x_1^{q_1\alpha}}{\Gamma(1+q_1\alpha)}\tfrac{x_2^{q_2\alpha}}{\Gamma(1+q_2\alpha)}\cdots\tfrac{x_n^{q_n\alpha}}{\Gamma(1+q_n\alpha)}\tfrac{t^{m\alpha}}{\Gamma(1+m\alpha)}\right)\right]_{t=0} \\ &= \tfrac{x_1^{q_1\alpha}}{\Gamma(1+q_1\alpha)}\tfrac{x_2^{q_2\alpha}}{\Gamma(1+q_2\alpha)}\cdots\tfrac{x_n^{q_n\alpha}}{\Gamma(1+q_n\alpha)}\tfrac{1}{\Gamma(1+k\alpha)}\left[\tfrac{\partial^{k\alpha}}{\partial t^{k\alpha}}\left(\tfrac{t^{m\alpha}}{\Gamma(1+m\alpha)}\right)\right]_{t=0} \\ &= \tfrac{x_1^{q_1\alpha}}{\Gamma(1+q_1\alpha)}\tfrac{x_2^{q_2\alpha}}{\Gamma(1+q_2\alpha)}\cdots\tfrac{x_n^{q_n\alpha}}{\Gamma(1+q_n\alpha)}\tfrac{\delta_\alpha(k-m)}{\Gamma(1+\alpha)}. \end{aligned} \tag{19}$$

The proof is thus completed. □

4. Applications of (*n* + 1)-Dimensional LFRDTM

Example 1. *Firstly, we consider (2 + 1)-dimensional local fractional homogeneous HLE on a Cantor set:*

$$\frac{\partial^\alpha}{\partial t^\alpha}\psi - \frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\psi - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\psi = 0, 0 < \alpha \leq 1, \tag{20}$$

with the initial condition (IC):

$$\psi(x,y,0) = \frac{y^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{21}$$

where $\psi = \psi(x,y,t)$.

Now solve this problem by using (n + 1)-dimensional LFRDTM. By taking the (n + 1)-dimensional LFRDT of (20), it can be obtained that:

$$\frac{\Gamma(1+k\alpha+\alpha)}{\Gamma(1+k\alpha)}\Psi_{k+1}(x,y)=\frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\Psi_k+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\Psi_k \tag{22}$$

The (n + 1)-dimensional LFRDT of the IC in (21) is given by:

$$\Psi_0(x,y)=\frac{y^{2\alpha}}{\Gamma(1+2\alpha)} \tag{23}$$

By using (23) in (22), we can obtain the following $\Psi_k(x,y)$ values successively:

$$\left.\begin{array}{l} \Psi_1(x,y)=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)},\\ \Psi_2(x,y)=\frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+2\alpha)},\\ \vdots\\ \Psi_n(x,y)=\begin{cases}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+n\alpha)}\ if \quad n \text{ is even number},\\ \frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+n\alpha)}\ if \quad n \text{ is odd number.}\end{cases}\end{array}\right\} \tag{24}$$

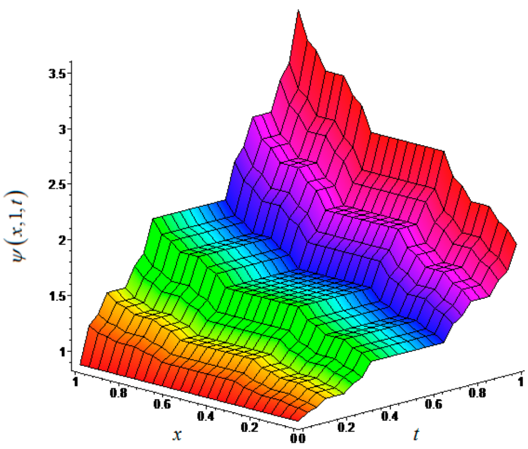
From (24), the $\{\Psi_k(x,y)\}_{k=0}^{n}$ values give the following approximation solution:

$$\widetilde{\psi}_n(x,y,t)=\sum_{k=0}^{n}\Psi_k(x,y,t)t^{k\alpha}=\sum_{k=0}^{n}\frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{t^{2k\alpha}}{\Gamma(1+2k\alpha)}+\sum_{k=0}^{n}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)} \tag{25}$$

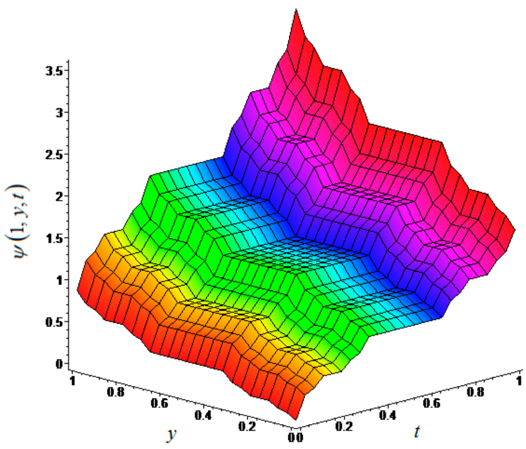
Hence, from (25), $\psi(x,y,t)$ is:

$$\psi(x,y,t)=\lim_{n\to\infty}\widetilde{\psi}_n(x,y,t)=\frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\cosh_\alpha(t^\alpha)+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sinh_\alpha(t^\alpha) \tag{26}$$

This finding is the exact solution of the (2 + 1)-dimensional local fractional homogeneous HLE (20) on the Cantor set. The graph of this solution is given in Figure 1 for $\alpha=\frac{\ln 2}{\ln 3}$.



(**a**)



(**b**)

Figure 1. (**a**) The exact solution of (2 + 1)-dimensional local fractional homogeneous HLE (20) in fractal space for $\psi(x,1,t)$ with α = ln2/ln3; (**b**) The exact solution of (2 + 1)-dimensional local fractional homogeneous HLE (20) in fractal space for $\psi(1,y,t)$ with α = ln2/ln3.

Example 2. *Secondly, consider the following (3 + 1)-dimensional local fractional inhomogeneous HLE on a Cantor set:*

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\psi - \frac{\Gamma(1+2\alpha)\Gamma(1+2\alpha)}{3\Gamma(1+4\alpha)}\left[\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\psi + \frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\psi + \frac{z^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial z^{2\alpha}}\psi\right] = \frac{(xyz)^{4\alpha}}{[\Gamma(1+4\alpha)]^4}, \tag{27}$$

subject to the IC:

$$\psi(x, y, z, 0) = 0, \tag{28}$$

here $0 < \alpha \leq 1$ *and* $\psi = \psi(x, y, z, t)$.

Using (n + 1)-dimensional LFRDTM, Equation (27) transforms to:

$$\frac{\Gamma(1+k\alpha+\alpha)}{\Gamma(1+k\alpha)}\Psi_{k+1}(x, y, z) = \frac{(xyz)^{4\alpha}}{[\Gamma(1+4\alpha)]^4}\delta(k) + \frac{\Gamma(1+2\alpha)}{3\Gamma(1+4\alpha)}\left[x^{2\alpha}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\Psi_k + z^{2\alpha}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\Psi_k + z^{2\alpha}\frac{\partial^{2\alpha}}{\partial z^{2\alpha}}\Psi_k\right]. \tag{29}$$

From the IC (28), we write:

$$\Psi_0(x, y, z) = 0 \tag{30}$$

From (30) and (29), the following $\Psi_k(x, y, z)$ values can be obtained:

$$\left.\begin{array}{l} \Psi_1(x, y, z) = \frac{(xyz)^{4\alpha}}{[\Gamma(1+4\alpha)]^4}\frac{1}{\Gamma(1+\alpha)}, \\ \Psi_2(x, y, z) = \frac{(xyz)^{4\alpha}}{[\Gamma(1+4\alpha)]^4}\frac{1}{\Gamma(1+2\alpha)}, \\ \vdots \\ \Psi_n(x, y, z) = \frac{(xyz)^{4\alpha}}{[\Gamma(1+4\alpha)]^4}\frac{1}{\Gamma(1+n\alpha)}. \end{array}\right\} \tag{31}$$

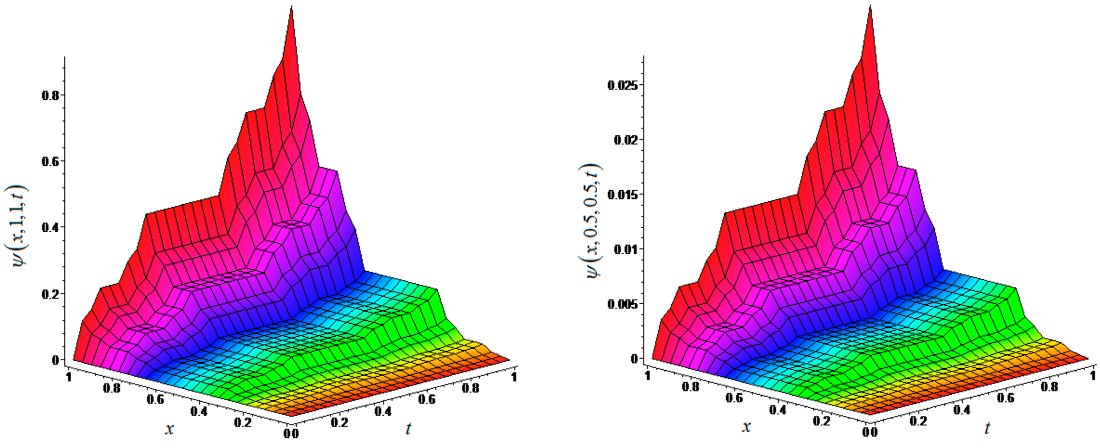
From (31), the following approximation solution can be written as:

$$\tilde{\psi}_n(x, y, z, t) = \sum_{k=0}^{n}\Psi_k(x, y, z, t)t^{k\alpha} = \sum_{k=1}^{n}\frac{(xyz)^{4\alpha}}{[\Gamma(1+4\alpha)]^4}\frac{t^{k\alpha}}{\Gamma(1+n\alpha)} \tag{32}$$

Hence, from (32), $\psi(x, y, z, t)$ is:

$$\psi(x, y, z, t) = \lim_{n\to\infty}\tilde{\psi}_n(x, y, z, t) = \frac{(xyz)^{4\alpha}}{[\Gamma(1+4\alpha)]^4}(E_{\alpha}(t^{\alpha}) - 1) \tag{33}$$

This result is the exact solution of the (3 + 1)-dimensional local fractional inhomogeneous HLE (27) on the Cantor set. The graph of this solution is given in Figure 2 for α = ln2/ln3.



(**a**) (**b**)

Figure 2. (**a**) The exact solution of (3 + 1)-dimensional local fractional inhomogeneous HLE (27) in fractal space for $\psi(x,1,1,t)$ with $\alpha = \ln 2/\ln 3$; (**b**) The exact solution of (3 + 1)-dimensional local fractional inhomogeneous HLE (27) in fractal space for $\psi(x,0.5,0.5,t)$ with $\alpha = \ln 2/\ln 3$.

Example 3. *Thirdly, we consider* (2 + 1)*-dimensional local fractional homogeneous WLE on a Cantor set:*

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\psi - \frac{\Gamma(1+2\alpha)\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}\left[\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\psi + \frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\psi\right] = 0, 0 < \alpha \le 1, \tag{34}$$

with the ICs:

$$\psi(x,y,0) = \frac{x^{4\alpha}}{\Gamma(1+4\alpha)}, \ \frac{\partial^{\alpha}}{\partial t^{\alpha}}\psi\bigg|_{t=0} = \frac{y^{4\alpha}}{\Gamma(1+4\alpha)}, \tag{35}$$

here $\psi = \psi(x,y,t)$.

We solve this problem by using $(n+1)$-dimensional LFRDTM. By taking $(n+1)$-dimensional LFRDT of (34), it can be obtained that:

$$\frac{\Gamma(1+k\alpha+2\alpha)}{\Gamma(1+k\alpha)}\Psi_{k+2}(x,y) = \frac{\Gamma(1+2\alpha)\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}\left[\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\Psi_k + \frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\Psi_k\right] \tag{36}$$

From the ICs (35), it can be written as follows:

$$\Psi_0(x,y) = \frac{x^{4\alpha}}{\Gamma(1+4\alpha)}, \quad \Psi_1(x,y) = \frac{y^{4\alpha}}{\Gamma(1+4\alpha)\Gamma(1+\alpha)} \tag{37}$$

By using (37) in (36), we can obtain the following $\Psi_k(x,y)$ values successively:

$$\left.\begin{aligned} \Psi_2(x,y) &= \tfrac{x^{4\alpha}}{\Gamma(1+4\alpha)\Gamma(1+2\alpha)}, \\ \Psi_3(x,y) &= \tfrac{y^{4\alpha}}{\Gamma(1+4\alpha)\Gamma(1+3\alpha)}, \\ &\vdots \\ \Psi_n(x,y) &= \begin{cases} \tfrac{x^{4\alpha}}{\Gamma(1+4\alpha)\Gamma(1+n\alpha)} \ if & n \ is \ even \text{ number}, \\ \tfrac{y^{4\alpha}}{\Gamma(1+4\alpha)\Gamma(1+n\alpha)} \ if & n \ is \text{ odd number}. \end{cases} \end{aligned}\right\} \tag{38}$$

From (38), the $\{\Psi_k(x,y)\}_{k=0}^{n}$ values give the following approximation solution:

$$\widetilde{\psi}_n(x,y,t) = \sum_{k=0}^{n} \Psi_k(x,y,t)t^{k\alpha} = \sum_{k=0}^{n} \frac{x^{4\alpha}}{\Gamma(1+4\alpha)} \frac{t^{2k\alpha}}{\Gamma(1+2k\alpha)} + \sum_{k=0}^{n} \frac{y^{4\alpha}}{\Gamma(1+4\alpha)} \frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)} \tag{39}$$

Hence, from (39), $\psi(x,y,t)$ is:

$$\psi(x,y,t) = \lim_{n\to\infty} \widetilde{\psi}_n(x,y,t) = \frac{x^{4\alpha}}{\Gamma(1+4\alpha)} \cosh_\alpha(t^\alpha) + \frac{y^{4\alpha}}{\Gamma(1+4\alpha)} \sinh_\alpha(t^\alpha) \tag{40}$$

This finding is the exact solution of the (2 + 1)-dimensional local fractional homogeneous WLE (34) on the Cantor set. The graph of this solution is given in Figure 3 for α = ln2/ln3.

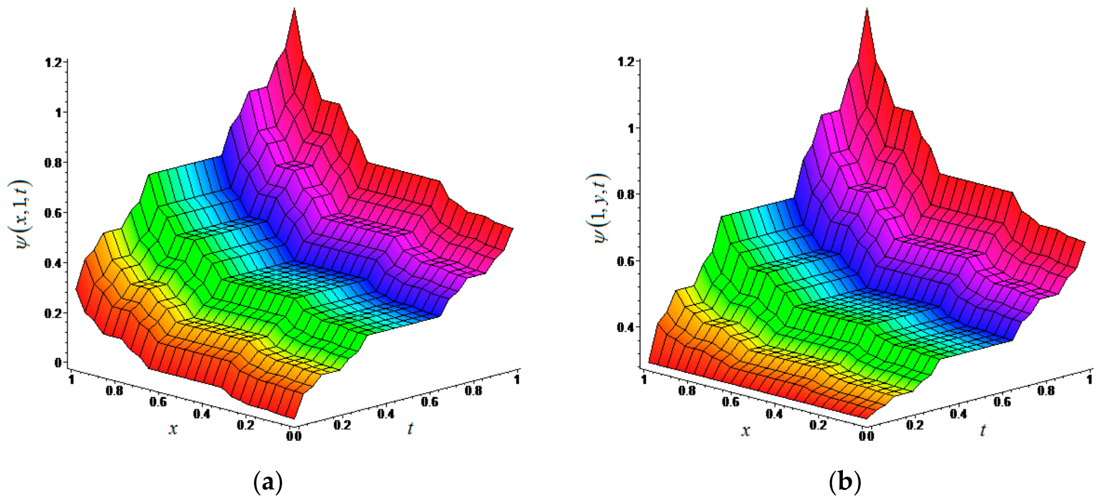


Figure 3. (**a**) The exact solution of (2 + 1)-dimensional local fractional homogeneous WLE (34) in fractal space for $\psi(x,1,t)$ with α = ln2/ln3; (**b**) The exact solution of (2 + 1)-dimensional local fractional homogeneous WLE (34) in fractal space for $\psi(1,x,t)$ with α = ln2/ln3.

Example 4. *Finally, consider the (3 + 1)-dimensional local fractional inhomogeneous WLE on a Cantor set:*

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\psi - \left[\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\psi + \frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\psi + \frac{z^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial z^{2\alpha}}\psi\right] = \frac{x^{2\alpha}+y^{2\alpha}+z^{2\alpha}}{\Gamma(1+2\alpha)}, 0 < \alpha \le 1, \tag{41}$$

subject to the ICs:

$$\psi(x,y,z,0) = 0, \ \frac{\partial^\alpha}{\partial t^\alpha}\psi\bigg|_{t=0} = \frac{x^{2\alpha}+y^{2\alpha}-z^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{42}$$

where $\psi = \psi(x,y,z,t)$.

According to $(n+1)$-dimensional LFRDTM, $(n+1)$-dimensional LFRDT of (41) can be written that:

$$\frac{\Gamma(1+k\alpha+2\alpha)}{\Gamma(1+k\alpha)}\Psi_{k+2}(x,y,z) = \left(\frac{x^{2\alpha}+y^{2\alpha}+z^{2\alpha}}{\Gamma(1+2\alpha)}\right)\delta(k) + \left[\begin{array}{c} \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\Psi_k + \frac{y^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\Psi_k \\ +\frac{z^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial z^{2\alpha}}\Psi_k \end{array}\right] \tag{43}$$

From the ICs (42) we write:

$$\Psi_0(x,y,z) = 0, \ \Psi_1(x,y,z) = \frac{x^{2\alpha}+y^{2\alpha}-z^{2\alpha}}{\Gamma(1+2\alpha)\Gamma(1+\alpha)} \tag{44}$$

According to (44) and (43), we can obtain the following $\Psi_k(x,y,z)$ values:

$$\left.\begin{array}{l} \Psi_2(x,y,z) = \frac{x^{2\alpha}+y^{2\alpha}+z^{2\alpha}}{\Gamma(1+2\alpha)\Gamma(1+2\alpha)}, \\ \Psi_3(x,y,z) = \frac{x^{2\alpha}+y^{2\alpha}-z^{2\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)}, \\ \vdots \\ \Psi_n(x,y,z) = \frac{x^{2\alpha}+y^{2\alpha}+(-1)^n z^{2\alpha}}{\Gamma(1+2\alpha)\Gamma(1+n\alpha)}. \end{array}\right\} \quad (45)$$

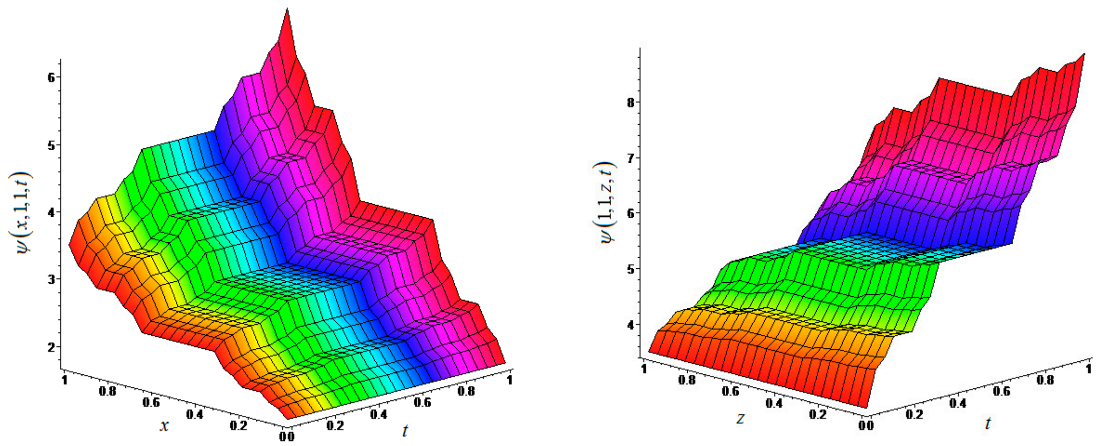
By using (38), the $\{\Psi_k(x,y,z)\}_{k=0}^{n}$ values give the following approximation solution:

$$\widetilde{\psi}_n(x,y,z,t) = \sum_{k=0}^{n} \Psi_k(x,y,z,t)t^{k\alpha} = \sum_{k=1}^{n} \left(\frac{x^{2\alpha}+y^{2\alpha}+(-1)^k z^{2\alpha}}{\Gamma(1+2\alpha)\Gamma(1+k\alpha)} \right) t^{k\alpha} \quad (46)$$

Hence, from (46), $\psi(x,y,z,t)$ is:

$$\psi(x,y,z,t) = \lim_{n\to\infty} \widetilde{\psi}_n(x,y,z,t) = \frac{x^{2\alpha}+y^{2\alpha}}{\Gamma(1+2\alpha)} E_\alpha(t^\alpha) + \frac{z^{2\alpha}}{\Gamma(1+2\alpha)} E_\alpha(-t^\alpha) - \left(\frac{x^{2\alpha}+y^{2\alpha}+z^{2\alpha}}{\Gamma(1+2\alpha)} \right) \quad (47)$$

This result obtained is the exact solution of the (3 + 1)-dimensional local fractional inhomogeneous WLE (41) on Cantor set. The graph of this solution is given in Figure 4 for α = ln2/ln3.



(**a**) (**b**)

Figure 4. (**a**) The exact solution of (3 + 1)-dimensional local fractional inhomogeneous WLE (41) in fractal space for $\psi(x,1,1,t)$ with α = ln2/ln3; (**b**) The exact solution of (3 + 1)-dimensional local fractional inhomogeneous WLE (41) in fractal space for $\psi(1,1,z,t)$ with α = ln2/ln3.

5. Conclusions

In this paper, a new technique, (n + 1)-dimensional local fractional reduced differential transform method (LFRDTM), was presented to find the analytical approximate solutions of local fractional PDEs. Then, the new method was applied to (n + 1)-dimensional fractal HLEs and WLEs. In the applications, our method directly gave us the exact solution for the problems without any transformation, discretization and any other restrictions. Physical behaviors of the solutions on fractal spaces were illustrated using 3D graphics. The results showed that presented method gives good outcomes for solutions of (n + 1)-dimensional local fractional PDEs. Hence, our results suggest that the new procedure (n + 1)-dimensional LFRDTM is reliable, useful and simplify for local fractional PDEs to solve many complicated fractal problems.

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