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RESEARCH PAPER

NEW SPECTRAL TECHNIQUES FOR SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS USING FRACTIONAL-ORDER GENERALIZED LAGUERRE ORTHOGONAL FUNCTIONS

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Abstract

Fractional-order generalized Laguerre functions (FGLFs) are proposed depends on the definition of generalized Laguerre polynomials. In addition, we derive a new formula expressing explicitly any Caputo fractional-order derivatives of FGLFs in terms of FGLFs themselves. We also propose a fractional-order generalized Laguerre tau technique in conjunction with the derived fractional-order derivative formula of FGLFs for solving Caputo type fractional differential equations (FDEs) of order ν $(0 < \nu < 1)$. The fractional-order generalized Laguerre pseudo-spectral approximation is investigated for solving nonlinear initial value problem of fractional order ν. The extension of the fractional-order generalized Laguerre pseudo-spectral method is given to solve systems of FDEs. We present the advantages of using the spectral schemes based on FGLFs and compare them with other methods. Several numerical example are implemented for FDEs and systems of FDEs including linear and nonlinear terms. We demonstrate the high accuracy and the efficiency of the proposed techniques.

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1. Introduction

Fractional calculus is a topic that has challenged researchers in the first decade of this century and is certain to continue to do so in the next decades [17, 27, 6, 10, 20], due to their useful applications in many fields of science. Indeed, we may observe several applications in electrochemistry, viscoelasticity, electromagnetic, control, plasma physics, porous media, fluctuating environments, dynamical processes and so on. In consequence, fractional differential equations are gaining much attention from the researchers. For some recent developments on this subject, see [19, 26, 11, 35, 34, 31, 33].

In the recent years, there has been a great interest to present efficient numerical methods to find more accurate approximate solution of fractional differential equations. As is well known, one of the most accurate methods of discretization for solving numerous differential equations is spectral method. Spectral method employs linear combination from orthogonal polynomials as basis functions and so often provides accurate approximate solutions [9, 7, 13]. The spectral methods based on orthogonal systems like Jacobi polynomials and their special cases are only available for bounded domains for approximation of FDEs; see [14, 15, 8]. Indeed, several problems in finance, plasma physics, porous media, dynamical processes and engineering are set on unbounded domains.

In the last few years, there has been a growing interest in the use of spectral method for numerical treatments of FDEs in bounded domains. Doha et al. [16] introduced the fractional derivatives of Jacobi operational matrix which was applied in combination with the Jacobi tau scheme for solving linear multi-term FDEs. The authors of [28] presented a Legendre tau scheme combined with the operational matrix of Legendre polynomials for the numerical solution of multi-term FDEs. Recently, Kazem et al. [25] define a new orthogonal functions based on Legendre polynomials to obtain an efficient spectral technique for multi-term FDEs, the authors of [32] extended this definition and presented the operational matrix of fractional derivative and integration for such functions to construct a new tau technique for solving two-dimensional FDEs. Moreover, the authors of [1] adopted the operational matrix of fractional derivative for Legendre polynomials which is applied with tau method for solving a class of fuzzy FDEs. Indeed, with a few noticeable exceptions, a little work was done to use spectral methods in unbounded domains to solve such important classes of FDEs.

For fractional differential equations in unbounded domains. The operation matrices of fractional derivatives and fractional integrals of generalized Laguerre polynomials were investigated for solving multi-term FDEs on a semi-infinite interval, see [5, 2]. The generalized Laguerre spectral tau and collocation techniques were given in [2] to solve linear and nonlinear FDEs on the half line. These spectral techniques were developed and generalized by using the modified generalized Laguerre polynomials in [3, 4]. Indeed, the authors of [22, 22] presented a Caputo fractional extension of the classical Laguerre polynomials and proposed a new C-Laguerre functions.

The objective of this manuscript is to define new orthogonal functions on the half line namely, fractional-order generalized Laguerre functions (FGLFs) based on the definition of the generalized Laguerre polynomials and then the Caputo fractional-order derivatives of FGLFs in terms of FGLFs themselves is stated and proved. We, therefore, propose a direct solution technique for solving linear FDEs of fractional order ν $(0 < \nu < 1)$ using the fractional-order generalized Laguerre tau (FGLT) approximation.

We also aim to propose a new fractional-order generalized Laguerre collocation (FGLC) method, for solving fractional initial value problem of fractional order ν $(0 < \nu < 1)$ with nonlinear terms, in which the the nonlinear FDE is collocated at the N zeros of the new function which defined on the interval $\Lambda = (0, \infty)$. The resulting algebraic equations together with one algebraic equation resulted from treating the initial condition constitute $(N + 1)$ nonlinear algebraic equations which can then be solved by implementing Newton's iterative technique to find the unknown fractional-order generalized Laguerre functions coefficients. We extend the application of FGLC method based on these functions to solve a system of FDEs with fractional orders less than 1. Several illustrative examples are implemented to confirm the high accuracy and effectiveness of the present method for solving FDES of fractional order ν $(0 < \nu < 1)$.

What remains of this paper is organized as follows: We start by presenting some definitions of the fractional calculus. In Section **3**, we define the fractional-order generalized Laguerre functions. Section **4** is devoted to derive the main theorem of the paper which provides explicitly a new formula that expresses the fractional-order derivatives of the fractional-order generalized Laguerre functions in terms of themselves. In Section **5**, we apply the spectral methods based on FGLFs for solving FDEs and systems of FDEs including linear and nonlinear terms of fractional order less than 1. Several examples to illustrate the main ideas of this work are presented in Section **6**.

2. Preliminaries and notations

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article.

Definition **2.1**. The Riemann-Liouville fractional integral operator of order ν ($\nu \geq 0$) is defined as

$$\begin{aligned} J^{\nu} f(x) =& \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \qquad \nu > 0, \quad x > 0, \\ J^0 f(x) =& f(x). \end{aligned} \tag{2.1}$$

Definition **2.2**. The Riemann-Liouville fractional derivatives of order ν and the Caputo fractional derivatives of order ν are defined as

$$\begin{aligned} D^{\nu} f(x) =& J^{m-\nu} D^m f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} \frac{d^m}{dt^m} f(t)dt, \\ & m-1 < \nu \leq m, \ x > 0, \end{aligned} \tag{2.2}$$

where D^m is mth order differential operator.

The Caputo fractional derivative operator satisfies

$$D^{\nu} C = 0, \qquad (C \text{ is a constant}), \tag{2.3}$$

$$D^{\nu} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} \, x^{\beta-\nu}, & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil \nu \rceil \\ & \text{or } \beta \notin N \text{ and } \beta > \lfloor \nu \rfloor, \end{cases} \tag{2.4}$$

where $\lceil \nu \rceil$ and $\lfloor \nu \rfloor$ are the ceiling and floor functions respectively, while $N = \{1, 2, \ldots\}$ and $N_0 = \{0, 1, 2, \ldots\}$.

The Caputo's fractional differentiation is a linear operation,

$$D^{\nu}(\lambda f(x) + \mu g(x)) = \lambda D^{\nu} f(x) + \mu D^{\nu} g(x), \tag{2.5}$$

where λ and μ are constants.

3. Fractional-order generalized Laguerre functions

We recall below some relevant properties of the generalized Laguerre polynomials (Szegö [29] and Funaro [18]). Let $\Lambda = (0, \infty)$ and $w^{(\alpha)}(x) = x^{\alpha} e^{-x}$ be a weight function on Λ in the usual sense. Define

$$L^2_{w^{(\alpha)}}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \ ||v||_{w^{(\alpha)}} < \infty\},$$

equipped with the following inner product and norm

$$(u,v)_{w^{(\alpha)}} = \int_{\Lambda} u(x)\ v(x)\ w^{(\alpha)}(x)\ dx, \qquad ||v||_{w^{(\alpha)}} = (u,v)^{\frac{1}{2}}_{w^{(\alpha)}}.$$

Next, let $L_i^{(\alpha)}(x)$ be the generalized Laguerre polynomials of degree i. We know from [29] that for $\alpha > -1$,

$$L_{i+1}^{(\alpha)}(x) = \frac{1}{i+1}[(2i+\alpha+1-x)L_i^{(\alpha)}(x) - (i+\alpha)L_{i-1}^{(\alpha)}(x)], \quad i = 1,2,\dots, \tag{3.1}$$

where $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = 1+\alpha-x$.

The set of generalized Laguerre polynomials is the $L^2_{w^{(\alpha)}}(\Lambda)$-orthogonal system, namely

$$\int_0^{\infty} L_j^{(\alpha)}(x) L_k^{(\alpha)}(x) w^{(\alpha)}(x) dx = h_k \delta_{jk}, \tag{3.2}$$

where δ_{jk} is the Kronecker symbol and $h_k = \dfrac{\Gamma(k+\alpha+1)}{k!}$.

The generalized Laguerre polynomials of degree i on the interval Λ, are given by [2]

$$L_i^{(\alpha)}(x) = \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{\Gamma(k+\alpha+1)\ (i-k)!\ k!}\ x^k, \quad i = 0,1,\dots \tag{3.3}$$

The special value

$$D^q L_i^{(\alpha)}(0) = (-1)^q \sum_{j=0}^{i-q} \frac{(i-j-1)!}{(q-1)!(i-j-q)!} L_j^{(\alpha)}(0), \quad i \geqslant q, \tag{3.4}$$

where $L_j^{(\alpha)}(0) = \dfrac{\Gamma(j+\alpha+1)}{\Gamma(\alpha+1)j!}$, will be of important use later.

We define a new "fractional" orthogonal functions based on the generalized Laguerre polynomials to obtain the solution of some FDEs more simply and efficiently.

The fractional-order generalized Laguerre functions (FGLFs) can be defined by introducing the change of variable $t = x^{\lambda}$ and $\lambda > 0$ on generalized Laguerre polynomials. Let the FGLFs $L_i^{(\alpha)}(x^{\lambda})$ be denoted by $L_i^{(\alpha,\lambda)}(x)$, by using (3.1) $L_i^{(\alpha,\lambda)}(x)$ may be obtained from the recurrence relation

$$L_{i+1}^{(\alpha,\lambda)}(x) = \frac{1}{i+1}[(2i+\alpha+1-x^{\lambda})L_i^{(\alpha,\lambda)}(x) - (i+\alpha)L_{i-1}^{(\alpha,\lambda)}(x)], \ i = 1,2,\dots, \tag{3.5}$$

with $L_0^{(\alpha,\lambda)}(x) = 1$ and $L_1^{(\alpha,\lambda)}(x) = 1+\alpha-x^{\lambda}$.

It is clear that the analytic form of $L_i^{(\alpha,\lambda)}(x)$ of fractional degree $i\lambda$ is:

$$L_i^{(\alpha,\lambda)}(x) = \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{\Gamma(k+\alpha+1)\ (i-k)!\ k!}\ x^{\lambda k}, \quad i = 0,1,\ldots \tag{3.6}$$

Lemma **3.1**. *The set of fractional-order generalized Laguerre functions is a $L^2_{w^{(\alpha,\lambda)}}(\Lambda)$-orthogonal system,*

$$\int_0^\infty L_j^{(\alpha,\lambda)}(x) L_k^{(\alpha,\lambda)}(x) w^{(\alpha,\lambda)}(x) dx = h_k, \tag{3.7}$$

where $w^{(\alpha,\lambda)}(x) = \lambda\ x^{(\alpha+1)\lambda-1} e^{-x^\lambda}$ *and* $h_k = \begin{cases} \dfrac{\Gamma(k+\alpha+1)}{k!}, & j = k, \\ 0, & j \neq k. \end{cases}$

P r o o f. The proof of this lemma can be accomplished directly by using the definition of FGLFs and the orthogonality condition of generalized Laguerre polynomials.

A function $u(x) \in L^2_{w^{(\alpha,\lambda)}}(\Lambda)$ may be expressed in terms of fractional-order generalized Laguerre functions as

$$u(x) = \sum_{j=0}^{\infty} a_j L_j^{(\alpha,\lambda)}(x), \tag{3.8}$$

and the coefficients a_j are obtained from

$$a_k = \frac{1}{h_k} \int_0^\infty u(x) L_k^{(\alpha,\lambda)}(x) w^{(\alpha,\lambda)}(x) dx, \quad k = 0,1,2,\cdots. \tag{3.9}$$

In particular applications, only the first $(N+1)$-terms fractional-order generalized Laguerre functions are considered. Then we have

$$u_N(x) = \sum_{j=0}^{N} a_j L_j^{(\alpha,\lambda)}(x). \tag{3.10}$$

Now, we construct the fractional-order generalized Laguerre-Gauss quadratures rule. We may have the privilege of using the generalized Laguerre-Gauss quadrature rule. We denote by $x^{(\alpha)}_{N,j}$, $0 \leqslant j \leqslant N$, the nodes of the generalized Laguerre-Gauss interpolation on the interval Λ. Their corresponding Christoffel numbers are $\omega^{(\alpha)}_{N,j}$, $0 \leqslant j \leqslant N$. The nodes of the fractional-order generalized Laguerre-Gauss interpolation on the interval Λ are the zeros of $L^{(\alpha,\lambda)}_{N+1}(x)$, which we denote by $x^{(\alpha,\lambda)}_{N,j}$, $0 \leqslant j \leqslant N$. Clearly $x^{(\alpha,\lambda)}_{N,j} = (x^{(\alpha)}_{N,j})^{\frac{1}{\lambda}}$, and their corresponding Christoffel numbers are $\omega^{(\alpha,\lambda)}_{N,j}$, $0 \leqslant j \leqslant N$,

$$\int_\Lambda \phi(x)w^{(\alpha,\lambda)}(x)dx = \int_\Lambda \phi(x^{\frac{1}{\lambda}})w^{(\alpha)}(x)dx = \sum_{j=0}^{N} \omega_{N,j}^{(\alpha)}\phi((x_{N,j}^{(\alpha)})^{\frac{1}{\lambda}}) \\ = \sum_{j=0}^{N} \omega_{N,j}^{(\alpha,\lambda)}\phi(x_{N,j}^{(\alpha,\lambda)}), \tag{3.11}$$

and from the previous relation, we have

$$\omega_{N,j}^{(\alpha,\lambda)} = -\frac{\Gamma(i+\alpha+1)}{(i+1)!L_i^{(\alpha,\lambda)}(x_{N,j}^{(\alpha,\lambda)})\partial_x L_{i+1}^{(\alpha,\lambda)}(x_{N,j}^{(\alpha,\lambda)})} \\ = \frac{\Gamma(i+\alpha+1)\ x_{N,j}^{(\alpha,\lambda)}}{(i+\alpha+1)(i+1)![L_i^{(\alpha,\lambda)}(x_{N,j}^{(\alpha,\lambda)})]^2}, \qquad 0 \le j \le i. \tag{3.12}$$

4. The fractional derivatives of FGLFs ($L_i^{(\alpha,\lambda)}(x)$)

The main objective of this section is to prove the following theorem for the fractional derivatives of the fractional-order generalized Laguerre functions. This theorem will be of fundamental importance in what follows.

THEOREM **4.1**. *The fractional derivative of order ν, $0 < \nu < 1$ in the Caputo sense for the fractional-order generalized Laguerre functions is given by*

$$D^\nu L_i^{(\alpha,\lambda)}(x) = \sum_{j=0}^{N} \Psi_\nu(i,j)\ L_j^{(\alpha,\lambda)}(x), \quad i = \lceil \nu \rceil, \cdots, N, \tag{4.1}$$

where

$$\Psi_\nu(i,j) = \sum_{k=1}^{i}\sum_{s=0}^{j} \frac{(-1)^{k+s}\ j!\Gamma(i+\alpha+1)\Gamma(\lambda k+1)\Gamma(k-\frac{\nu}{\lambda}+\alpha+s+1)}{s!k!(i-k)!(j-s)!\Gamma(\lambda k-\nu+1)\Gamma(k+\alpha+1)\Gamma(\alpha+s+1)}.$$

P r o o f. The analytic form of the fractional-order generalized Laguerre functions $L_i^{(\alpha,\lambda)}(x)$ of degree $i\lambda$ is given by (3.6). Using Eqs. (2.4)-(2.5) and (3.6), we have

$$D^\nu L_i^{(\alpha,\lambda)}(x) = \sum_{k=0}^{i} (-1)^k\ \frac{\Gamma(i+\alpha+1)}{(i-k)!\ k!\ \Gamma(k+\alpha+1)}\ D^\nu x^{\lambda k}$$

$$= \sum_{k=1}^{i} (-1)^k\ \frac{\Gamma(i+\alpha+1)\Gamma(\lambda k+1)}{(i-k)!\ k!\Gamma(\lambda k-\nu+1)\ \Gamma(k+\alpha+1)}\ x^{\lambda k-\nu}, \quad i = 1, \ldots, N. \tag{4.2}$$

The approximation of $x^{\lambda k-\nu}$ by $N+1$ terms of fractional-order generalized Laguerre series yields

$$x^{k-\nu} = \sum_{j=0}^{N} b_j L_j^{(\alpha,\lambda)}(x), \tag{4.3}$$

where b_j is given by

$$b_j = \sum_{s=0}^{j} (-1)^s \, \frac{j! \, \Gamma(k - \frac{\nu}{\lambda} + \alpha + s + 1)}{(j-s)! \, (s)! \, \Gamma(s+\alpha+1)} \, . \tag{4.4}$$

Thanks to (4.2)-(4.4) we can write

$$D^{\nu} L_i^{(\alpha,\lambda)}(x) = \sum_{j=0}^{N} \Psi_\nu(i,j) L_j^{(\alpha,\lambda)}(x), \quad i = \lceil \nu \rceil, \cdots, N, \tag{4.5}$$

where

$$\Psi_\nu(i,j) = \sum_{k=1}^{i}\sum_{s=0}^{j} \frac{(-1)^{k+s} \; j! \; \Gamma(i+\alpha+1) \; \Gamma(\lambda k+1) \; \Gamma(k - \frac{\nu}{\lambda} + \alpha + s + 1)}{s! k! (i-k)! (j-s)! \Gamma(\lambda k - \nu + 1) \Gamma(k+\alpha+1) \Gamma(\alpha+s+1)} . \tag{4.6}$$

5. Spectral methods for FDEs

In this section, we consider spectral tau and collocation methods based on the fractional derivative of FGLFs to solve numerically the linear and nonlinear FDEs of order ν.

5.1. Tau Method for Linear FDEs

We are interested in using the FGLT method to solve the linear multi-order FDE

$$D^{\nu} u(x) + \gamma u(x) = f(x), \quad \text{in } \Lambda, \tag{5.1}$$

with initial condition

$$u(0) = u_0, \tag{5.2}$$

where γ is constant and $0 < \nu \leq 1$. Moreover, $D^{\nu}u(x)$ denotes the Caputo fractional derivative of order ν for $u(x)$, and $f(x)$ is a source function. It is known that $\{L_i^{(\alpha,\lambda)}(x) : i \geq 0\}$ forms a complete orthogonal system in $L^2_{w^{(\alpha,\lambda)}}(\Lambda)$. Hence, if we define

$$S_N(\Lambda) = \text{span}\Big\{ L_0^{(\alpha,\lambda)}(x), L_1^{(\alpha,\lambda)}(x), \cdots, L_N^{(\alpha,\lambda)}(x) \Big\}, \tag{5.3}$$

then the standard fractional-order generalized Laguerre-tau approximation to (5.1) is to find $u_N \in S_N(\Lambda)$ such that

$$\begin{aligned}(D^\nu u_N, L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}} &+ \gamma(u_N, L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}} \\ &= ((f, L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}}, \quad k = 0, 1, \cdots, N-1, \\ u_N(0) = u_0.&\end{aligned} \tag{5.4}$$

Let us denote

$$\begin{aligned}u_N(x) &= \sum_{j=0}^{N} a_j L_j^{(\alpha,\lambda)}(x), \quad \mathbf{a} = (a_0, a_1, \cdots, a_N)^T, \\ f_k &= (f, L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}}, \quad k = 0, 1, \cdots, N-1, \\ \mathbf{f} &= (f_0, f_1, \cdots, f_{N-1}, d_0)^T.\end{aligned} \tag{5.5}$$

It is now clear that the variational formulation of Eq. (5.4) is equivalent to

$$\begin{aligned}\sum_{j=0}^{N} a_j \Big[(D^\nu L_j^{(\alpha,\lambda)}(x), L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}} &+ \gamma(L_j^{(\alpha,\lambda)}(x), L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}}\Big] \\ &= (f, L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}}, \quad k = 0, 1, \cdots, N-1, \\ \sum_{j=0}^{N} a_j D^{k-N} L_j^{(\alpha,\lambda)}(0) = \; u_{k-N}, \quad k = N.&\end{aligned} \tag{5.6}$$

Let us also denote

$$A = (a_{kj})_{0<k,j<N}, \quad B = (b_{kj})_{0<k,j<N}.$$

Then equation (5.6) is equivalent to the following matrix equation

$$(A + \gamma\, B)\, \mathbf{a} = \mathbf{f}, \tag{5.7}$$

where the nonzero elements of the matrices A, and B are given explicitly in the following theorem.

THEOREM **5.1**. *If we denote* $a_{kj} = (D^\nu L_j^{(\alpha,\lambda)}(x), L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}}$ $(0 \le k \le N-1,\ 0 \le j \le N)$, $a_{kj} = L_j^{(\alpha,\lambda)}(0)$ $(k = N,\ 0 \le j \le N)$ *and* $b_{kj} = (L_j^{(\alpha,\lambda)}(x), L_k^{(\alpha,\lambda)}(x))_{w^{(\alpha,\lambda)}}$ $(0 \le k \le N-1,\ 0 \le j \le N)$. *Then the nonzero elements of* a_{kj} *and* b_{kj} *are given as follows:*

$$a_{kj} = \begin{cases} h_k\ \Psi_\nu(j,k), & 0 \le k \le N-1,\ 1 \le j \le N, \\ \dfrac{\Gamma(j+\alpha+1)}{\Gamma(\alpha+1)\ j!}, & k = N,\ 0 \le j \le N. \end{cases}$$

$$b_{kj} = h_k, \qquad 0 \le k = j \le N-1.$$

5.2. Collocation Method for Nonlinear FDEs

We use the FGLC method to numerically solve the nonlinear FDE, namely

$$D^{\nu}u(x) = f\big(x, u(x)\big), \quad x \in \Lambda, \tag{5.8}$$

with initial conditions

$$u(0) = u_0, \tag{5.9}$$

where $0 < \nu \leq 1$.

Let

$$u_N(x) = \sum_{j=0}^{N} a_j L_j^{(\alpha,\lambda)}(x), \tag{5.10}$$

then, making use of formula (4.1) enables one to express explicitly the derivatives $D^{\nu}u(x)$, in terms of the expansion coefficients a_j. The criterion of spectral fractional-order generalized Laguerre collocation method for solving approximately (5.14)-(5.15) is to find $u_N(x) \in S_N(\Lambda)$ such that

$$D^{\nu}u_N(x) = F\big(x, u_N(x)\big), \tag{5.11}$$

is satisfied exactly at the collocation points $x_{N,k}^{(\alpha,\lambda)}$, $k = 0, 1, \cdots, N-1$. In other words, we have to collocate Eq. (5.17) at the N fractional-order generalized Laguerre roots $x_{N,k}^{(\alpha,\lambda)}$, which immediately yields

$$\sum_{j=0}^{N} a_j D^{\nu} L_j^{(\alpha,\lambda)}(x_{N,k}^{(\alpha,\lambda)}) = P\big(x_{N,k}^{(\alpha,\lambda)}, \sum_{j=0}^{N} a_j L_j^{(\alpha,\lambda)}(x_{N,k}^{(\alpha,\lambda)})\big), \tag{5.12}$$

with (5.15) written in the form

$$\sum_{j=0}^{N} a_j L_j^{(\alpha,\lambda)}(0) = u_0. \tag{5.13}$$

This constitute a system of $(N+1)$ nonlinear algebraic equations in the unknown expansion coefficients a_j $(j = 0, 1, \cdots, N)$, which can be solved by using any standard iteration technique, like Newton's iteration method.

5.3. FGLC method for solving systems of FDEs

We use the FGLC method to numerically solve the general form of systems of nonlinear FDE, namely

$$D^{\nu_i}u_i(x) = f_i\big(x, u_1(x), u_2(x), \ldots, u_n(x)\big), \quad x \in \Lambda, \quad i = 1, \ldots, n, \tag{5.14}$$

with initial conditions

$$u_i(0) = u_{i0}, \qquad i = 1, \ldots, n, \tag{5.15}$$

where $0 < \nu_i \leq 1$.

Let

$$u_{iN}(x) = \sum_{j=0}^{N} a_{ij} L_j^{(\alpha,\lambda)}(x). \tag{5.16}$$

The fractional derivatives $D^{\nu_i}u(x)$ can be expressed in terms of the expansion coefficients a_{ij} using (4.1). The implementation of fractional-order generalized Laguerre collocation method to solve (5.14)-(5.15) is to find $u_{iN}(x) \in S_N(\Lambda)$ such that

$$D^{\nu_i}u_{iN}(x) = F_i\big(x, u_{1N}(x), u_{2N}(x), ..., u_{nN}(x)\big), \quad x \in \Lambda, \tag{5.17}$$

is satisfied exactly at the collocation points $x_{i,N,k}^{(\alpha,\lambda)}$, $k = 0, 1, \cdots, N-1, i = 1, \cdots, n$, which immediately yields

$$\begin{aligned}\sum_{j=0}^{N} a_{ij} D^{\nu_i} L_j^{(\alpha,\lambda)}(x_{i,N,k}^{(\alpha,\lambda)}) = P_i\big(x_{i,N,k}^{(\alpha,\lambda)}, \sum_{j=0}^{N} a_{1j} L_j^{(\alpha,\lambda)}(x_{1,N,k}^{(\alpha,\lambda)}), \sum_{j=0}^{N} a_{2j} L_j^{(\alpha,\lambda)}(x_{2,N,k}^{(\alpha,\lambda)}),\\ \ldots, \sum_{j=0}^{N} a_{nj} L_j^{(\alpha,\lambda)}(x_{n,N,k}^{(\alpha,\lambda)})\big),\end{aligned} \tag{5.18}$$

with (5.15) written in the form

$$\sum_{j=0}^{N} a_{ij} L_j^{(\alpha,\lambda)}(0) = u_{i0}, \quad i = 1, \cdots, n. \tag{5.19}$$

This means the system (5.14) with its initial conditions has been reduced to a system of $n(N+1)$ nonlinear algebraic equations (5.18)-(5.19), which may be solved by using any standard iteration technique.

COROLLARY **5.1**. *In particular, the special case for fractional Laguerre polynomials may be obtained directly by taking* $\lambda = 0$*, which are used in* [2]. *However, the classical Laguerre polynomials may be achieved by replacing* $\lambda = 1$ *and* $\alpha = 0$*, which are used most frequently in practice for solving ordinary/partial differential equations and often denoted by* $L_i(x)$.

6. Applications and numerical results

In this section, we give some numerical results obtained by using the algorithms presented in the previous sections. Comparisons of our results with those obtained by other methods reveal that our methods is very effective and convenient.

N	η	$\zeta=\lambda$	$\alpha=0$	$\alpha=2$	$\eta=\zeta=\lambda$	$\alpha=0$	$\alpha=2$
8	0.9	0.5	$3.56.10^{-3}$	$9.34.10^{-3}$	0.75	$5.55.10^{-17}$	$4.37.10^{-16}$
16			$9.38.10^{-4}$	$2.42.10^{-3}$		$5.55.10^{-17}$	$4.44.10^{-16}$
24			$3.98.10^{-4}$	$1.31.10^{-3}$		$5.55.10^{-17}$	$4.51.10^{-16}$
32			$2.31.10^{-4}$	$7.35.10^{-4}$		$5.55.10^{-17}$	$4.44.10^{-16}$
40			$1.49.10^{-4}$	$4.38.10^{-4}$		$5.55.10^{-17}$	$4.44.10^{-16}$
48			$1.12.10^{-4}$	$2.69.10^{-4}$		$5.55.10^{-17}$	$4.51.10^{-16}$
8	0.8	0.6	$8.69.10^{-3}$	$2.30.10^{-2}$	0.999	$5.55.10^{-17}$	$3.33.10^{-16}$
16			$2.90.10^{-3}$	$9.69.10^{-3}$		$5.55.10^{-17}$	$3.33.10^{-16}$
24			$1.61.10^{-3}$	$5.85.10^{-3}$		$5.55.10^{-17}$	$3.12.10^{-16}$
32			$1.08.10^{-3}$	$4.11.10^{-3}$		$5.55.10^{-17}$	$3.33.10^{-16}$
40			$7.90.10^{-4}$	$3.09.10^{-3}$		$5.55.10^{-17}$	$3.33.10^{-16}$
48			$6.16.10^{-4}$	$2.37.10^{-3}$		$5.55.10^{-17}$	$3.12.10^{-16}$

TABLE 1. Maximum absolute error with various choices of η, ζ, ν and N in $x \in [0,1]$

EXAMPLE **6.1**. *Consider the equation, see* [5]

$$D^{\zeta}u(x)+u(x)=\frac{\Gamma(\eta+1)}{\Gamma(\eta-\zeta+1)}\,x^{\eta-\zeta}+x^{\eta},\quad 0<\zeta\leq\eta<1,\quad x\in\Lambda,$$

the exact solution is given by $u(x)=x^{\eta}$.

The solution of this problem is obtained by applying FGLT method. In Table 1, The maximum absolute errors of $u(x)-u_N(x)$ using FGLT method based on treating the right hand side of this problem by Gauss quadrature of fractional order generalized Laguerre functions, with various choices of η, ζ, α and N are compared with the results of the improved generalized Laguerre tau (GLT) method (see [5]) with various choices of η, ζ, α and N. From Table 1, we see that we can achieve a good approximation with the exact solution by using fractional-order generalized Laguerre functions and our method is more accurate than (GLT) [5]). Fig. 1 displays comparisons between the curves of exact solutions and approximate solutions at $N=10$, $\alpha=3$ and variable choices of η,ζ, and λ. Meanwhile, maximum absolute errors (MAE) for $N=10$ and different values of $\eta=\zeta=\lambda$ and α are shown in Fig. 2 and Fig. 3.

EXAMPLE **6.2**. *Consider the following nonlinear initial value problem*

$$D^{\nu}u(x)+u^2(x)=x+\Big(\frac{x^{\nu+1}}{\Gamma(\nu+2)}\Big)^2,\quad 0<\nu\leq 2,$$

whose exact solution is given by $u(x)=\dfrac{1}{\Gamma(\nu+2)}x^{\nu+1}$.

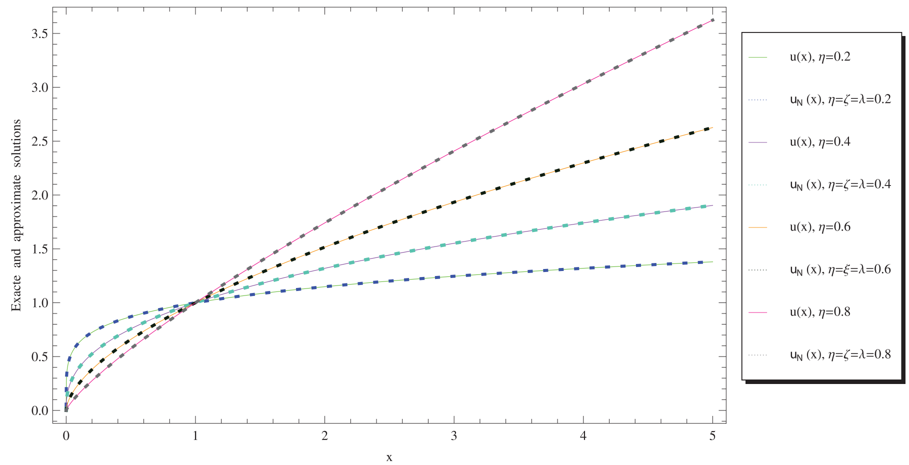

FIGURE 1. Comparing the exact and approximate solutions at $N = 10, \alpha = 3$ and $\eta = \zeta = \lambda = \{0.2, 0.4, 0.6, 0.8\}$.

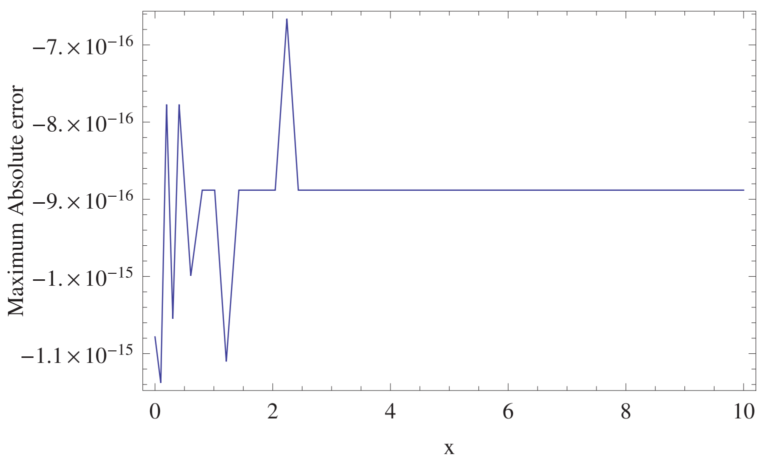

FIGURE 2. Graph of maximum absolute error functions at $N = 10, \alpha = 3$ and $\eta = \zeta = \lambda = 0.85$ for Example 6.1.

In Tables 2 and 3, we list maximum absolute errors using FGLC method in Section 5.2 with various choices of the fractional-order ν and λ for the fractional-order generalized Laguerre parameter $\alpha = \{1, 2\}$ and $N = 4, 8, 12, 16$ in the interval $[0, 10]$. Moreover, Fig. 4 displays comparisons between the curves of exact and approximate solutions at $\alpha = 0$ of proposed problem subject to $u(0) = 0$ in case of $N = 10$ and two different fractional orders $\nu = 0.5, 0.8$.

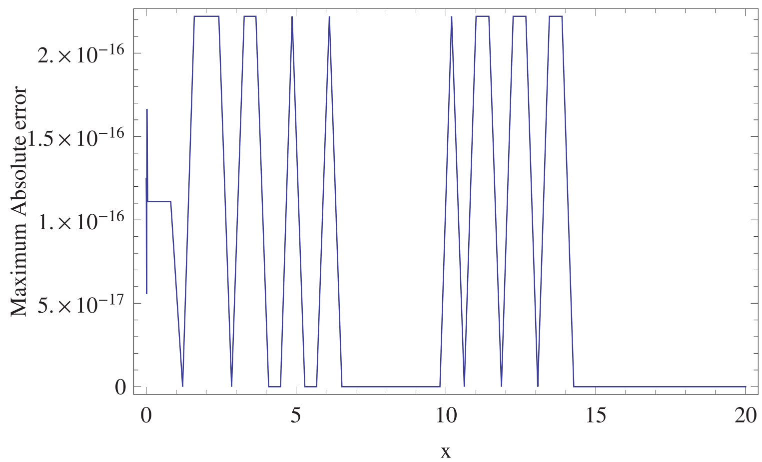

FIGURE 3. Graph of maximum absolute error functions at $N = 10, \alpha = 0$ and $\eta = \zeta = \lambda = 0.25$ for Example 6.1.

N	ν	λ	FGLC	ν	λ	FGLC	ν	λ	FGLC
4	0.8	0.8	$1.00.10^{-1}$	0.7	0.6	$2.83.10^{-2}$	0.5	0.5	$1.77.10^{-14}$
8			$6.08.10^{-3}$			$4.28.10^{-3}$			$1.42.10^{-14}$
12			$1.87.10^{-3}$			$1.74.10^{-3}$			$1.42.10^{-14}$
16			$9.51.10^{-4}$			$8.19.10^{-4}$			$1.06.10^{-14}$

TABLE 2. Maximum absolute error for $\alpha = \{1\}$ with various choices of ν, λ and N in $x \in [0, 10]$

N	ν	λ	FGLC	ν	λ	FGLC	ν	λ	FGLC
4	0.8	0.8	$3.50.10^{-2}$	0.7	0.6	$3.46.10^{-2}$	0.5	0.5	$7.10.10^{-15}$
8			$9.15.10^{-3}$			$7.99.10^{-3}$			$1.51.10^{-14}$
12			$4.73.10^{-3}$			$3.79.10^{-3}$			$2.84.10^{-14}$
16			$2.30.10^{-3}$			$2.07.10^{-3}$			$2.13.10^{-14}$

TABLE 3. Maximum absolute error for $\alpha = 2$ with various choices of ν, λ and N in $x \in [0, 10]$

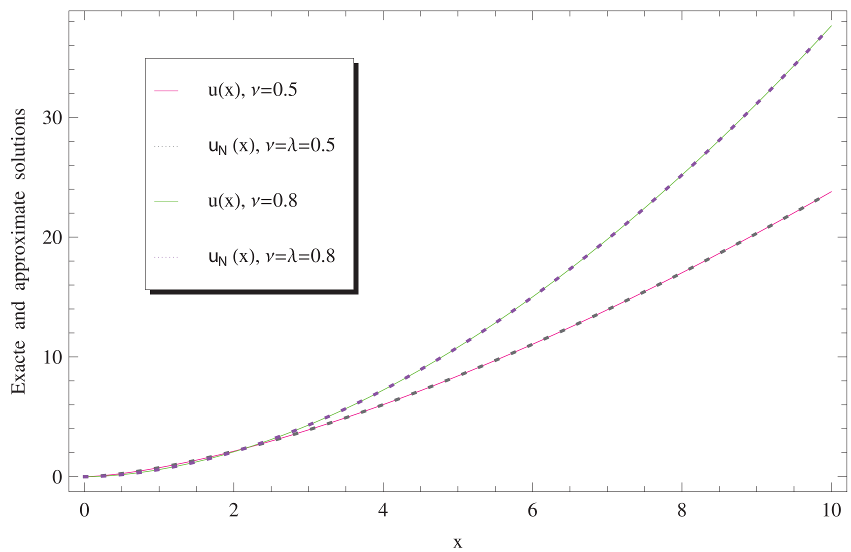

FIGURE 4. Comparing the exact and approximate solutions at $N = 15, \alpha = 0$ and $\nu = \lambda = 0.5$ for Example 6.2.

EXAMPLE **6.3**. *Consider the following system of fractional differential equations, see* [24]*:*

$$\begin{aligned} D^{\frac{1}{2}}X_1(x) &= X_1(x) - x^{\frac{1}{2}}X_3(x) + \frac{2}{\Gamma\left(\frac{1}{2}\right)}x^{\frac{1}{2}} \\ D^{\frac{1}{2}}X_2(x) &= -xX_1(x) + X_2(x) + \frac{3}{3\Gamma\left(\frac{1}{2}\right)}x^{\frac{3}{2}} \\ D^{\frac{1}{2}}X_3(x) &= xX_1(x) - X_2(x) + X_3(x) - x^{\frac{1}{2}} + \frac{\Gamma\left(\frac{1}{2}\right)}{2} \end{aligned} \tag{6.1}$$

with initial conditions

$$X_1(0) = 0, \quad X_2(0) = 0, \quad X_3(0) = 0, \qquad x \in [0, 10]. \tag{6.2}$$

The above system has a unique solution given by $X_1(x) = x, X_2(x) = x^2, X_3(x) = x^{\frac{1}{2}}$.

The solution of this problem is obtained by applying the FGLC method. The maximum absolute error for $N = 4$ and various choice of α is shown in Table 4. Moreover, Fig. 5 displays the maximum absolute error at $\alpha = \frac{1}{2}$ for $X_1(x)$, $X_2(x)$ *and* $X_3(x)$ in the interval $[0, 100]$.

α	$E\ for\ X_1(x)$	$E\ for\ X_2(x)$	$E\ for\ X_3(x)$
$-\frac{1}{2}$	$7.37.10^{-15}$	$5.02.10^{-14}$	$5.02.10^{-14}$
0	$1.13.10^{-14}$	$2.06.10^{-13}$	$2.06.10^{-13}$
$\frac{1}{2}$	$9.69.10^{-15}$	$4.74.10^{-14}$	$2.0.10^{-14}$
1	$6.98.10^{-14}$	$1.45.10^{-13}$	$1.45.10^{-13}$
2	$1.49.10^{-13}$	$1.99.10^{-13}$	$1.99.10^{-13}$

Table 4. Maximum absolute error for $N = 4$ with various choices of $\nu = \lambda = 0.5$, and α

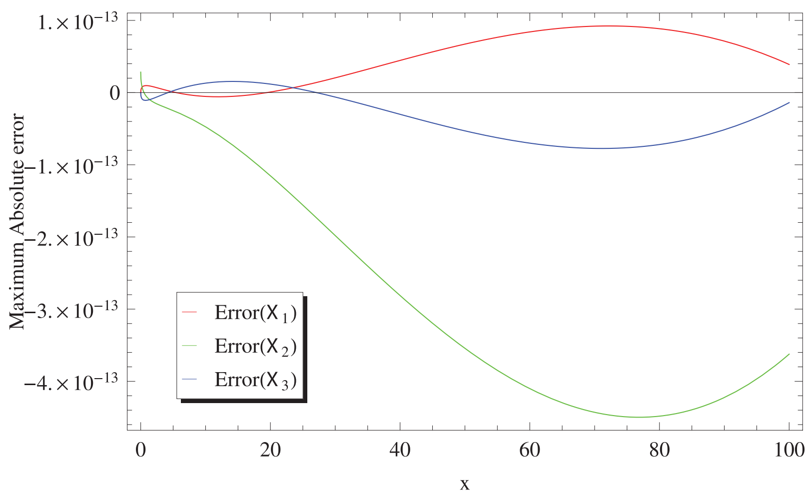

Figure 5. Graph of maximum absolute error functions at $N = 4$ *and* $\alpha = \frac{1}{2}$ for $X_1(x)$, $X_2(x)$ *and* $X_3(x)$ for Example 6.3.

Example **6.4**. *We next consider the following initial value problem for the inhomogeneous Bagley-Torvik equation* [12]

$$\frac{d^2 y(x)}{dx^2} + \frac{d^{\frac{3}{2}} y(x)}{dx^{\frac{3}{2}}} + y(x) = x + 1, \qquad y(0) = 1, \quad y'(0) = 1. \tag{6.3}$$

The exact solution is given by $y(x) = x + 1$.

Before coming to the description of our numerical scheme, we find it convenient to rewrite the original Bagley-Torvik equation (6.3) in the form of a system of fractional differential equations of order 1/2, that will later

α	E
$-\frac{1}{2}$	0
0	$2.22.10^{-16}$
$\frac{1}{2}$	$1.77.10^{-15}$
1	$8.88.10^{-16}$
2	0
3	$3.55.10^{-15}$

TABLE 5. MAE using FGLC method with various choices of α for Example 6.4.

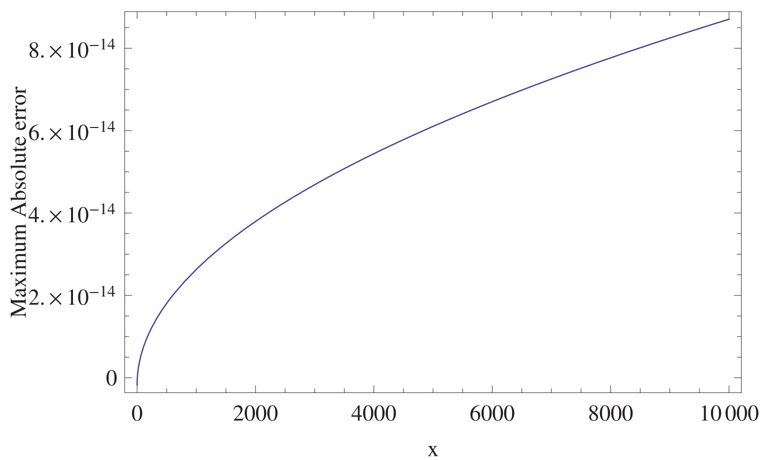

FIGURE 6. Graph of maximum absolute error functions at $\alpha = \frac{1}{2}$ in the interval $[0, 10000]$ for Example 6.4.

be solved numerically. In particular, let $y(x) = y_1(x)$, the system that we shall consider is of the form [12]

$$\begin{aligned} D^{\frac{1}{2}} y_1(x) &= y_2(x), \\ D^{\frac{1}{2}} y_2(x) &= y_3(x), \\ D^{\frac{1}{2}} y_3(x) &= y_4(x), \\ D^{\frac{1}{2}} y_4(x) &= -y_1(x) - y_4(x) + x + 1, \end{aligned} \tag{6.4}$$

with initial conditions

$$y_1(0) = y(0), \qquad qy_2(0) = 0, \qquad y_3(0) = y'(0), \qquad y_4(0) = 0. \tag{6.5}$$

The maximum absolute error for $y(x) = y_1(x)$ using FGLC method at various choices of α are shown in Table 5. Moreover, Fig. 6 plots the maximum absolute error at $\alpha = \frac{1}{2}$ in the interval $[0, 10000]$.

Conclusions

In this article, we have defined a fractional-order generalized Laguerre function depends on generalized Laguerre polynomials to obtain. In addition, we have stated and proved a new formula expressing explicitly any Caputo fractional-order derivatives of FGLFs in terms of FGLFs themselves. This formula was employed in the construction of spectral tau technique to obtain an accurate solution of FDEs with leading order ν $(0 < \nu < 1)$.

The Gauss quadrature rule for this new function was constructed. We have proposed the fractional-order generalized Laguerre pseudo-spectral approximation for solving nonlinear initial value problem of fractional order ν. Moreover, we have extended the application of the fractional-order generalized Laguerre pseudo-spectral method for solving systems of FDEs. The main advantages of using the spectral schemes based on FGLFs and compare them with other methods. Several numerical example are implemented for FDEs and systems of FDEs including linear and nonlinear terms. We demonstrate the high accuracy and the efficiency of the proposed techniques.

Numerical examples were given to test the applicability and validity of the proposed algorithms. During several numerical examples, it is observed that the proposed methods are simple and accurate. Indeed, while a limited number of fractional-order generalized Laguerre collocation nodes is utilized, very accurate numerical results are obtained.

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